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LIMIT, CONTINUITY AND DIFFERENTIABILITY OF FUNCTION

LIMITS

INDETERMINATE FORM

Some times we come across with some functions which do not have definite value corresponding to some particular value of the variable.

For example for the function,

$$f(x)=\frac{x^2-4}{x-2},\ f(2)=\frac{4-4}{2-2}=\frac{0}{0}$$

which cannot be determined. Such a form is called an indeterminate form. Some other indeterminate forms are

$$0\times\infty,\ 0^0,\ 1^\infty,\ \infty-\infty,\ \frac{\infty}{\infty},\ \frac{0}{0},\ \infty^0$$

LIMITS OF A FUNCTION

Let $y=f(x)$ be a function of x and for some particular value of x say x = a, the value of y is indeterminate, then we consider the value of the function at the points which are very near to 'a'. If these values tend to a definite unique number ℓ as x tends to 'a' (either from left or from right) then this unique number *l* is called the limits of $f(x)$ at $x=a$ and we write it as

$$\lim_{x\to a} f(x)=\ell.$$

Meaning of '$x\to a$':

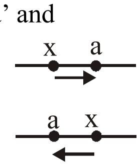
Let x be a variable and a be a constant. If x assumes values nearer and nearer to 'a' then we can say 'x tends to a' and we write '$x\to a$'.

It should be noted that as $x\to a$ we have $x\neq a$.

(i) $x\neq a$

(ii) x assumes values nearer and nearer to 'a' and

(iii) We are not specifying any manner in which x should approach to a. x may approach to a from left or right as shown in figure.



EVALUATION OF LEFT HAND AND RIGHT HAND LIMITS

The statement $x\to a^-$ means that x is tending to a from the left hand side, i.e. x is a number less than a but very close to a. Therefore $x\to a^-$ is equivalent to $x=a-h$ where $h>0$ such that $h\to 0$. Similarly $x\to a^+$ is equivalent to $x=a+h$ where $h\to 0$. Thus, we have the following algorithms for finding left hand and right hand limits at $x=a$.

Algorithm For Finding Left Hand Limit :

To evaluate LHL of $f(x)$ at $x=a$, i.e. $\lim_{x\to a^-} f(x)$ we proceed as

(i) Write $\lim_{x\to a^-} f(x)$

(ii) Put $x=a-h$ and replace $x\to a^-$ by $h\to 0$ to obtain $\lim_{h\to 0} f(a-h)$.

(iii) Simplify $\lim_{h\to 0} f(a-h)$ by using the formula for the given function.

(iv) The value obtain in step (iii) is the LHL of $f(x)$ at $x=a$.

Algorithm For Finding Right Hand Limit :

To evaluate RHL of $f(x)$ at $x=a$ i.e. $\lim_{x\to a^+} f(x)$ we proceed as follows :

(i) Write $\lim_{x\to a^+} f(x)$

(ii) Put $x=a+h$ and replace $x\to a^+$ by $h\to 0$ to obtain $\lim_{h\to 0} f(a+h)$

(iii) Simplify $\lim_{h\to 0} f(a+h)$ by using the formula for the given function.

(iv) The value obtained in step (iii) is the RHL of $f(x)$ at $x=a$.

EXISTENCE OF LIMIT

The limit of a function at some point exists only when its left-hand limit and right hand limit at that point exist and are equal. Thus $\lim_{x\to a} f(x)$ exists

$$\Rightarrow \lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=\ell$$

where ℓ is called the limit of the function

Example 1 :

If $f(x)=\begin{cases} x^2+2, & x\geq 1 \\ 2x+1, & x<1 \end{cases}$, then find the value of $\lim_{x\to 1} f(x)$

Sol. $\lim_{x\to 1-0} f(x)=\lim_{h\to 0}\ [2(1-h)+1]=3$

$\lim_{x\to 1+0} f(x)=\lim_{h\to 0}\ [(1+h)^2+2]=3$

$\therefore$ LHL = RHL, so $\lim_{x\to 1} f(x)=3$

Example 2 :

Find the value of $\lim_{x\to 3}\frac{x-3}{|x-3|}$

Sol. LHL $=\lim_{h\to 0}\frac{(3-h)-3}{|(3-h)-3|}=\lim_{h\to 0}\frac{-h}{|-h|}=-1$

RHL $=\lim_{h\to 0}\frac{(3+h)-3}{|(3+h)-3|}=\lim_{h\to 0}\frac{h}{|h|}=1$

LHL $\neq$ RHL, so limit does not exist

Example 3 :

Find the value of $\lim_{x\to 1}[x]$.

Sol. Left hand limit $=\lim_{x\to 1^-}f(x)=\lim_{x\to 1^-}[x]=0$

and right hand limit $=\lim_{x\to 1^+}f(x)=\lim_{x\to 1^+}[x]=1$

$\therefore \lim_{x\to 1^-}f(x)\neq\lim_{x\to 1^+}f(x)$ $\quad\therefore$ limit does not exist.

DIFFERENCE BETWEEN THE VALUE OF A FUNCTION AT A POINT AND THE LIMIT AT A POINT

Let f (x) be a function and let a be a point. Then, we have the following possibilities :

(i) $\lim_{x\to a}f(x)$ **exists but f (a) (the value of f (x) at x = a) does not exist**

Ex. Consider the function f (x) defined by $f(x)=\frac{x^2-9}{x-3}$

Clearly, this function is not defined at x = 3 i.e. f(3) does not exist, because it attains the form $\frac{0}{0}$. But, it can be easily seen that $\lim_{x\to 3^-}f(x)=\lim_{x\to 3^+}f(x)=6$ So, $\lim_{x\to 3}f(x)$ exists.

Thus, the $\lim_{x\to 3}f(x)$ exists but the value f(3) does not exist.

(ii) The value f(a) exists but $\lim_{x\to a}f(x)$ does not exists

Ex. Consider the function f (x) = [x]

Value of function at x = 2

$\Rightarrow f(2)=[2]=2$

LHL at x = 2

$\lim_{x\to 2^-}f(x)=\lim_{h\to 0}f(2-h)=\lim_{h\to 0}[2-h]=1$

RHL at x = 2

$=\lim_{x\to 2^+}f(x)=\lim_{h\to 0}f(2+h)=\lim_{h\to 0}[2+h]=2$

Clearly RHL $\neq$ LHL so limit of function does not exist but value of function exist.

(iii) $\lim_{x\to a}f(x)$ **and f (a) both exist but are unequal**

Ex. Consider the function f (x) defined by

$$f(x)=\begin{cases}\frac{x^2-4}{x-2}, & x\neq 2\\ 3\ , & x=2\end{cases}$$

It can be easily seen that $\lim_{x\to 2^-}f(x)=4=\lim_{x\to 2^+}f(x)$

So, $\lim_{x\to 2}f(x)$ exists and is equal to 4. Also, the value f(2) exists and is equal to 3.

Thus $\lim_{x\to 2}f(x)$ and f(2) both exist but are unequal.

(iv) $\lim_{x\to a}f(x)$ **and f (a) both exist and are equal**

Consider the function f (x) defined by

$$f(x)=\begin{cases}\frac{x^2-4}{x-2}, & x\neq 2\\ 4\ , & x=2\end{cases}$$

For this function, it can be easily seen that $\lim_{x\to 2}f(x)$ and f (2) both exist and are equal to 4.

THE ALGEBRA OF LIMITS

Let f and g be two real functions with domain D. We define four new functions f ± g, fg, f / g on domain D by setting

(f ± g) (x) = f (x) ± g(x),

(fg) (x) = f (x) g (x)

(f / g) (x) = f (x) / g (x), if g (x) ≠ 0 for any x ∈ D.

Following are some results concerning the limits of these functions.

Let $\lim_{x\to a}f(x)=\ell$ and $\lim_{x\to a}g(x)=m$. If ℓ and m exist.

(i) $\lim_{x\to a}(f\pm g)(x)=\lim_{x\to a}f(x)\pm\lim_{x\to a}g(x)=\ell\pm m$

(ii) $\lim_{x\to a}(fg)(x)=\lim_{x\to a}f(x).\lim_{x\to a}g(x)=\ell m$

(iii) $\lim_{x\to a}\left(\frac{f}{g}\right)(x)=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}=\frac{\ell}{m}$ provided $m\neq 0$

(iv) $\lim_{x\to a}k\,f(x)=k.\lim_{x\to a}f(x)$, where k is constant

(v) $\lim_{x\to a}[f(x)+k]=\lim_{x\to a}f(x)+k$ where k is a constant

(vi) $\lim_{x\to a}|f(x)|=|\lim_{x\to a}f(x)|=|\ell|$

(vii) $\lim_{x\to a}(f(x))^{g(x)}=\ell^{m}$

(viii) If $f(x)\leq g(x)$ for every x in the deleted nbd of a, then $\lim_{x\to a}f(x)\leq\lim_{x\to a}g(x)$

(ix) If $f(x) \le g(x) \le h(x)$ for every x in the neighbourhood of a and $\lim_{x\to a} f(x) = \ell = \lim_{x\to a} h(x)$ then $\lim_{x\to a} g(x) = \ell$

(x) $\lim_{x\to a} fog(x) = f(\lim_{x\to a} g(x)) = f(m)$

In particular

(a) $\lim_{x\to a} \log f(x) = \log\left(\lim_{x\to a} f(x)\right) = \log \ell$

(b) $\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)} = e^{\ell}$

(xi) If $\lim_{x\to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x\to a} \frac{1}{f(x)} = 0$

METHODS OF EVALUATION OF LIMITS

1. When $x \to \infty$: In this case expression should be expressed as a function 1/x and then after removing indeterminate form, (If it is there) replace 1/x by 0.

Example 4 :

Find the value of $\lim_{x\to\infty} \frac{2x^2+3x}{3x^2+4}$

Sol. $\lim_{x\to\infty} \frac{2x^2+3x}{3x^2+4} = \lim_{x\to\infty} \frac{2+(3/x)}{3+(4/x^2)} = \frac{2}{3}$

Example 5 :

Find the value of $\lim_{x\to\infty} \frac{ax^2+bx+c}{dx^2+ex+f}$

Sol. $\lim_{x\to\infty} \frac{ax^2+bx+c}{dx^2+ex+f} = \lim_{x\to\infty} \frac{a+\frac{b}{x}+\frac{c}{x^2}}{d+\frac{e}{x}+\frac{f}{x^2}} = \frac{a}{d}$

Example 6 :

Find the value of $\lim_{x\to\infty} \frac{1^2+2^2+3^2+......+x^2}{x^3}$

Sol. $\lim_{x\to\infty} \frac{1^2+2^2+3^2+......+x^2}{x^3}$

$= \lim_{x\to\infty} \frac{x(x+1)(2x+1)}{6x^3} = \lim_{x\to\infty} \frac{(1+1/x)(2+1/x)}{6} = \frac{1}{3}$

Example 7 :

Find the value of $\lim_{n\to\infty} \frac{3^{n+1}+4^{n+1}}{3^n+4^n}$

Sol. Given limit $= \lim_{n\to\infty} \frac{3^{n+1}+4^{n+1}}{3^n+4^n} = \lim_{n\to\infty} \frac{4+3(3/4)^n}{1+(3/4)^n} = 4$

2. When $x \to a$, $a \in R$.

(i) Factorisation method :

If f (x) is of the form $\frac{h(x)}{g(x)}$ and of indeterminant form then this form is removed by factorising g (x) and h (x) and cancel the common factors, then put the value of x.

Example 8 :

Find the value of $\lim_{x\to -1} \left[\frac{x^2-1}{x^2+3x+2}\right]$

Sol. Limit $= \lim_{x\to -1} \frac{(x-1)(x+1)}{(x+2)(x+1)} = \frac{-1-1}{-1+2} = -2$

Example 9 :

Find the value of $\lim_{x\to 64} \left(\frac{\sqrt{x}-8}{\sqrt[3]{x}-4}\right)$

Sol. $\lim_{x\to 64} \left(\frac{\sqrt{x}-8}{\sqrt[3]{x}-4}\right) \Rightarrow \lim_{y\to 2} \frac{y^3-8}{y^2-4}$ Here $y^6 = x$

$\Rightarrow \lim_{y\to 2} \frac{(y-2)(y^2+2y+4)}{(y-2)(y+2)} = \lim_{y\to 2} \frac{(y^2+2y+4)}{y+2}$

$= \frac{4+4+4}{2+2} = 3$

Example 10 :

Find the value of $\lim_{x\to 2} \frac{x^2-3x+2}{x^2+x-6}$

Sol. $\lim_{x\to 2} \frac{x^2-3x+2}{x^2+x-6} = \lim_{x\to 2} \frac{(x-2)(x-1)}{(x-2)(x+3)}$

$= \frac{2-1}{2+3} = \frac{1}{5}$

(ii) Substitution Method : For evaluating $\lim_{x\to a} \frac{g(x)}{h(x)}$, we follow the following steps-

(a) Put $x = a + h$, where h is small $(\neq 0)$ as $x \to a$, $h \to 0$

(b) Simplify numerator and denominator and cancle h throughout $(h \neq 0)$

(c) Put $h = 0$, we get the required limit.

Example 11 :

Evaluate $\lim_{x\to 1} \frac{x^2-1}{x-1}$

Sol. $= \lim_{h\to 0} \frac{(1+h)^2-1}{(1+h)-1} = \lim_{h\to 0} \frac{h^2+2h}{h} = \lim_{h\to 0} (h+2) = 2$

Example 12 :

$\lim_{x\to 0} \frac{\sqrt{1+x}+\sqrt{1-x}}{1+x}$

Sol. From direct substitution, $\frac{\sqrt{1+0}+\sqrt{1-0}}{1+0} = \frac{2}{1} = 2$

(iii) Rationalisation Method : In this method we rationalise the factor containing the square root and simplify and we put the value of x.

Example 13 :

Evaluate $\lim_{x\to 0} \frac{x}{\sqrt{1+x}-\sqrt{1-x}}$

Sol. Limit $= \lim_{x\to 0} \frac{x(\sqrt{1+x}+\sqrt{1-x})}{(1+x)-(1-x)} = \lim_{x\to 0} \frac{\sqrt{1+x}+\sqrt{1-x}}{2} = 1$

Example 14 :

Evaluate $\lim_{x\to 5} \frac{1-\sqrt{x-4}}{x-5}$

Sol. $\lim_{x\to 5} \frac{1-\sqrt{x-4}}{x-5} \Rightarrow \lim_{x\to 5} \frac{1-\sqrt{x-4}}{x-5} \cdot \frac{1+\sqrt{x-4}}{1+\sqrt{x-4}}$

$\Rightarrow \lim_{x\to 5} \frac{1-x+4}{(x-5)(1+\sqrt{x-4})} = \lim_{x\to 5} \frac{-(x-5)}{(x-5)(1+\sqrt{x-4})}$

$\Rightarrow \lim_{x\to 5} \frac{-1}{(1+\sqrt{x-4})} = \frac{-1}{(1+\sqrt{5-4})} = \frac{-1}{2}$

Example 15 :

Evaluate $\lim_{x\to 0} \frac{\sqrt{1+x+x^2}-1}{x}$

Sol. By rationalisation of numerator

$= \lim_{x\to 0} \frac{\sqrt{1+x+x^2}-1}{x} \cdot \frac{\sqrt{1+x+x^2}+1}{\sqrt{1+x+x^2}+1}$

$= \lim_{x\to 0} \frac{1+x+x^2-1}{x\left(\sqrt{1+x+x^2}+1\right)} = \lim_{x\to 0} \frac{x(1+x)}{x\left(\sqrt{1+x+x^2}+1\right)}$

$\lim_{x\to 0} \frac{1+x}{\sqrt{1+x+x^2}+1} = \frac{1}{2}$

(iv) Expansion Method :

If $x \to 0$ and there is atleast one function in the given expression which can be expanded then we express numerator and Denominator in the ascending of x and remove the common factor there. The following expansions of some standard functions are given.

(i) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

(ii) $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + ...$

(iii) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ....$

(iv) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - ......$

(v) $a^x = 1 + (x\log a) + \frac{(x\log a)^2}{2!} + \frac{(x\log a)^3}{3!} + ...$

(vi) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ......$

(vii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - .....$

(viii) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + ...$

(ix) $\sin^{-1} x = x + \frac{x^3}{3!} + \frac{9x^5}{5!} + ...$

(x) $\cos^{-1} x = \frac{\pi}{2} - \left(x + \frac{x^3}{3!} + \frac{9x^5}{5!} + ...\right)$

(xi) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

(xii) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ...$

Example 16 :

Find the value of $\lim_{x\to 0}\left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right]$

Sol. Limit $= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^2 \cdot \sin^2 x} = \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + ...\right)^2 - x^2}{x^2\left(x - \frac{x^3}{3!} + ...\right)^2}$

$= \lim_{x\to 0} \frac{x^2 - \frac{1}{3}x^4 + ... - x^2}{x^4\left(1 - \frac{x^2}{3!} + ...\right)^2} = -1/3$

Example 17 :

Evaluate $\lim_{x\to 0}\dfrac{e^x+\log\{(1-x)/e\}}{\tan x - x}$

Sol. Limit $=\lim_{x\to 0}\dfrac{e^x+\log(1-x)-1}{\tan x - x}$

$$=\lim_{x\to 0}\frac{\left(1+x+\frac{x^2}{2!}+.....\right)-x-\frac{x^2}{2}-\frac{x^3}{3}....-1}{x+\frac{x^3}{3}+\frac{2x^5}{15}+....-x}=-1/2$$

Example 18 :

Find the value of $\lim_{x\to 0}\dfrac{1-\cos x}{\sin x}$.

Sol. $\lim_{x\to 0}\dfrac{1-\cos x}{\sin x}\Rightarrow\lim_{x\to 0}\dfrac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-......\right)}{x-\frac{x^3}{3!}+\frac{x^5}{5!}-......}$

$$\Rightarrow\lim_{x\to 0}\frac{x\left(\frac{1}{2!}+\frac{x^2}{4!}-......\right)}{1-\frac{x^2}{3!}+......}=0$$

3. Evaluation Of Limits By Using De'L' Hospital's Rule :

If f (x) and g(x) be two functions of x such that

(i) $\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=0$

(ii) both are continuous at x = a,

(iii) both are differential able at x = a,

(iv) $f'(x)$ and $g'(x)$ are continuous at the point x = a , then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}\text{ provided that } g(a)\neq 0.$$

Note : The above rule is also applicable if $\lim_{x\to a}f(x)=\infty$ and $\lim_{x\to a}g(x)=\infty$.

Generalisation : If $\lim_{x\to a}\dfrac{f'(x)}{g'(x)}$ assumes the indeterminate form $\dfrac{0}{0}$ and $f'(x), g'(x)$ satisfy all the conditions embodied in De' L' Hospitals rule, we can repeat the application of this rule on $\dfrac{f'(x)}{g'(x)}$ to get $\lim_{x\to a}\dfrac{f(x)}{g(x)}=\lim_{x\to a}\dfrac{f''(x)}{g''(x)}$

Sometimes it may be necessary to repeat this process a number of times till our goal of evaluating limit is achieved.

Example 19 :

Evaluate $\lim_{x\to 0}\dfrac{a^x-b^x}{x}$

Sol. $\lim_{x\to 0}\dfrac{a^x-b^x}{x}\left(\dfrac{0}{0}\text{ form}\right)$

$\Rightarrow\lim_{x\to 0}\dfrac{a^x\log a-b^x\log b}{1}$ (by 'L' Hospital rule)

$\Rightarrow\log(a/b)$

Example 20 :

Find the value of $\lim_{x\to 0}\dfrac{x^2-2x}{2\sin x}$

Sol. $\lim_{x\to 0}\dfrac{x^2-2x}{2\sin x}\Rightarrow\lim_{x\to 0}\dfrac{2x-2}{2\cos x}=-\dfrac{2}{2}=-1$

(by 'L' Hospital rule)

Example 21 :

Evaluate $\lim_{x\to 0}\dfrac{\sin x+\log(1-x)}{x^2}$

Sol. It is in $\dfrac{0}{0}$ form, so using Hospital rule, we have

$$\text{Limit}=\lim_{x\to 0}\frac{\cos x-\frac{1}{1-x}}{2x}\quad\left(\frac{0}{0}\text{form}\right)$$

$$=\lim_{x\to 0}\frac{-\sin x-\frac{1}{(1-x)^2}}{2}=-1/2$$

4. SOME STANDARD LIMITS

(i) $\lim_{x\to 0}\dfrac{\sin x}{x}=\lim_{x\to 0}\dfrac{x}{\sin x}=1;\ \lim_{x\to 0}\sin x=0$

(ii) $\lim_{x\to 0}\cos x=\lim_{x\to 0}\left(\dfrac{1}{\cos x}\right)=1$

(iii) $\lim_{x\to 0}\dfrac{\tan x}{x}=\lim_{x\to 0}\dfrac{x}{\tan x}=1;\ \lim_{x\to 0}\tan x=0$

(iv) $\lim_{x\to 0}\dfrac{\sin^{-1}x}{x}=\lim_{x\to 0}\dfrac{x}{\sin^{-1}x}=1$

(v) $\lim_{x\to 0}\dfrac{\tan^{-1}x}{x}=\lim_{x\to 0}\dfrac{x}{\tan^{-1}x}=1$

(vi) $\lim_{x\to\infty}\left(1+\dfrac{a}{x}\right)^x=\lim_{x\to 0}(1+ax)^{1/x}=e^a$

(vii) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a \quad (a>0)$

(viii) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(ix) $\lim_{x\to a} \frac{x^n - a^n}{x-a} = n\,a^{n-1}$

(x) $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

(xi) $\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = n$

(xii) $\lim_{x\to\infty} \frac{\sin x}{x} = \lim_{x\to\infty} \frac{\cos x}{x} = 0$

(xiii) $\lim_{x\to\infty} \frac{\sin 1/x}{1/x} = 1$

(xiv) $\lim_{x\to\infty} 1/x = 0$

(xv) $\lim_{x\to\infty} \frac{1}{|x|} = \infty$

(xvi) $\lim_{x\to\infty} a^x = \begin{cases} 0, & \text{if } |a|<1 \\ 1, & \text{if } a=1 \\ \infty & \text{if } a>1 \\ \text{does not exist} & \text{if } a\le -1 \end{cases}$

(xvii) $\lim_{x\to a} [f(x)]^{g(x)} = e^{\lim_{x\to a} g(x)[f(x)-1]}$

Example 22 :

Evaluate $\lim_{x\to 0} \frac{\sin^2 2x}{x^2}$

Sol. $\lim_{x\to 0} \frac{\sin^2 2x}{x^2}$

$\Rightarrow \lim_{x\to 0} \frac{(2\sin x\cos x)^2}{x^2} = 4 \lim_{x\to 0} \frac{\sin^2 x}{x^2} . \cos^2 x = 4$

Example 23 :

Evaluate $\lim_{x\to 0} \frac{x(e^x - 1)}{1-\cos x}$

Sol. $\lim_{x\to 0} \frac{x(e^x-1)}{1-\cos x} = \lim_{x\to 0} \frac{2x^2(e^x-1)}{4\sin^2 \frac{x}{2}}$

$= 2 \lim_{x\to 0} \left[\frac{(x/2)^2}{\sin^2(x/2)}\right]\left(\frac{e^x-1}{x}\right) = 2$

Example 24 :

Evaluate $\lim_{x\to 0} \frac{x^3 \cot x}{1-\cos x}$

Sol. $\lim_{x\to 0} \frac{x^3\cot x}{1-\cos x} = \lim_{x\to 0} \left(\frac{x^3 \cot x}{1-\cos x} \times \frac{1+\cos x}{1+\cos x}\right)$

$= \lim_{x\to 0} \left(\frac{x}{\sin x}\right)^3 \times \lim_{x\to 0} \cos x \times \lim_{x\to 0} (1+\cos x) = 2$

SOME LIMITS WHICH DO NOT EXIST

(i) $\lim_{x\to 0} \left(\frac{1}{x}\right)$ (ii) $\lim_{x\to 0} x^{1/x}$

(iii) $\lim_{x\to 0} \frac{|x|}{x}$ (iv) $\lim_{x\to a} \frac{|x-a|}{x-a}$

(v) $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ (vi) $\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$

(vii) $\lim_{x\to 0} e^{1/x}$ (viii) $\lim_{x\to\infty} \sin x$

(ix) $\lim_{x\to\infty} \cos x$

5. LIMITS OF THE FORM $\lim_{x\to a} (f(x))^{g(x)}$:

Form : $0^0, \infty^0$

Let $L = \lim_{x\to a} (f(x))^{g(x)}$

$\Rightarrow \log_e L = \log_e \left[\lim_{x\to a} (f(x))^{g(x)}\right]$

$= \lim_{x\to a} g(x) \log_e [f(x)]$

Form : 1^∞

(i) $\lim_{x\to 0} (1+x)^{1/x} = e$ or $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$

(ii) $L = \lim_{x\to a} f(x)^{g(x)}$ if $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$

Then $L = \lim_{x\to a} f(x)^{g(x)}$

$= \lim_{x\to a} (1+f(x)-1)^{\frac{1}{f(x)-1}(f(x)-1)\times g(x)}$,

$= \left(\lim_{x\to a} (1+f(x)-1)^{\frac{1}{f(x)-1}}\right)^{\lim_{x\to a} (f(x)-1)\times g(x)}$

$= e^{\lim_{x\to a} (f(x)-1)\times g(x)}$

Example 25 :

Evaluate $\lim\limits_{x\to 0}(1+x)^{\operatorname{cosec} x}$

Sol. $\lim\limits_{x\to 0}(1+x)^{\operatorname{cosec} x} = \lim\limits_{x\to 0}\left[(1+x)^{1/x}\right]^{\frac{x}{\sin x}}$

$$= \left[\lim_{x\to 0}\left[(1+x)^{1/x}\right]\right]^{\lim\limits_{x\to 0}\frac{x}{\sin x}} = e^1 = e$$

Example 26 :

Evaluate $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\left(\frac{\sin x}{x-\sin x}\right)}$

Sol. Since, $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$ and

$$\lim_{x\to 0}\frac{\sin x}{x-\sin x} = \lim_{x\to 0}\frac{1}{\left(\frac{x}{\sin x}-1\right)} = \frac{1}{1-1} = \infty$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\left(\frac{\sin x}{x-\sin x}\right)} = e^{\lim\limits_{x\to 0}\left(\frac{\sin x}{x}-1\right)\left(\frac{\sin x}{x-\sin x}\right)}$$

$$= e^{-\lim\limits_{x\to 0}\frac{\sin x}{x}} = e^{-1} = \frac{1}{e}$$

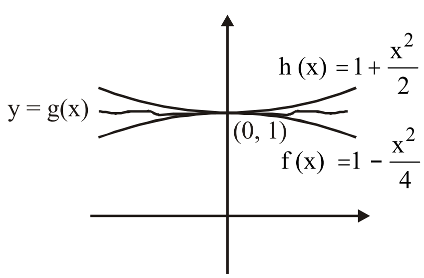
6. SANDWICH THEOREM

(Squeeze play theorem for evaluating limits) :

General : The squeeze principle is used on limit problems where the usual algebraic methods (factorisation or algebraic manipulation etc.) are not effective. However it requires to "squeeze" our problem in between two other simpler function whose limits can be easily computed and equal.

Statement : If f, g and h are 3 functions such that $f(x) \le g(x) \le h(x)$ for all x in some interval containing the point $x = c$, and if

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L \Rightarrow \lim_{x\to c} g(x) = L$$



From the figure note that $\lim\limits_{x\to 0} g(x) = 1$

Note : The quantity c may be a finite number, $+\infty$ or $-\infty$. Similarly, L may be a finite number, $+\infty$ or $-\infty$.

Example 27 :

Evaluate $\lim\limits_{x\to\infty}\dfrac{x+7\sin x}{-2x+13}$ using Sandwich theorem.

Sol. We know that $-1 \le \sin x \le 1$ for all x.

$\Rightarrow\ -7 \le 7\sin x \le 7$

$\Rightarrow\ x-7 \le x+7\sin x \le x+7$

Dividing throughout by $-2x + 13$, we get

$$\frac{x-7}{-2x+13} \ge \frac{x+7\sin x}{-2x+13} \ge \frac{x+7}{-2x+13}$$ for all x that are large.

Now, $$\lim_{x\to\infty}\frac{x-7}{-2x+13} = \lim_{x\to\infty}\frac{1-\frac{7}{x}}{-2+\frac{13}{x}} = \frac{1-0}{-2+0} = -\frac{1}{2}$$

and $$\lim_{x\to\infty}\frac{x+7}{-2x+13} = \lim_{x\to\infty}\frac{1+\frac{7}{x}}{-2+\frac{13}{x}} = \frac{1+0}{-2+0} = -\frac{1}{2}$$

$$\therefore \lim_{x\to\infty}\frac{x+7\sin x}{-2x+13} = \frac{-1}{2}$$

7. LIMITS OF FUNCTIONS HAVING BUILT IN LIMIT WITH THEM

Examples :

$$\lim_{n\to\infty} a^n = \begin{cases} 0, & 0<a<1 \\ 1, & a=1 \\ \infty, & a>1 \end{cases}, \quad \lim_{n\to\infty} a^n = \begin{cases} \infty, & 0<a<1 \\ 1, & a=1 \\ 0, & a>1 \end{cases}$$

Example 28 :

$f(x) = \lim\limits_{n\to\infty}\dfrac{\tan \pi x^2 + (x+1)^n \sin x}{x^2+(x+1)^n}$, find $\lim\limits_{x\to 0} f(x)$.

Sol. $$f(x) = \begin{cases} \sin x, & x>0 \\ \dfrac{\tan \pi x^2}{x^2}, & x<0 \end{cases}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sin x = 0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{\tan \pi x^2}{x^2} = \pi$$

$\Rightarrow$ LHL $\ne$ RHL $\Rightarrow \lim\limits_{x\to 0} f(x) = \text{DNE}$

8. ONE SIDED LIMITS

Example 29 :

Evaluate $\lim_{x\to 0}(1+\tan^2\sqrt{x})^{1/x}$

Sol. Let $(1+\tan^2\sqrt{x})^{1/x}$, $\ell = \lim_{x\to 0^+}\frac{1}{x}\ln(1+\tan^2\sqrt{x})$

$$\ell = e^{\lim_{x\to 0^+}\frac{1}{x}\left(\tan^2\sqrt{x}-\frac{(\tan^2\sqrt{x})^2}{2}+\ldots\right)} = e^1 = e$$

$$\ell = e^{\lim_{x\to 0^-}\frac{1}{x}\ln(1+\tan^2\sqrt{x})}$$

So, here left hand limit has no significance as $\sqrt{x}$ is not defined for $x < 0$.

TRY IT YOURSELF-1

Q.1 Evaluate the following limits ([.], { . } denotes greatest integer function and fractional part respectively)

(i) $\lim_{x\to 0}\frac{|x|}{x}$ (ii) $\lim_{x\to 0}\frac{1}{\ln|x|}$

(iii) $\lim_{x\to 0}[x]+\sqrt{\{x\}}$ (iv) $\lim_{x\to 0}\sin^{-1}[\sec x]$

Q.2 Evaluate the left and right-hand limits of the function

$$f(x)=\begin{cases}\dfrac{|x-4|}{x-4}, & x\neq 4\\ 0, & x=4\end{cases}$$

Q.3 Evaluate $\lim_{x\to 1}\frac{x^2+x\log_e x-\log_e x-1}{(x^2-1)}$

Q.4 Evaluate $\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}$

Q.5 Evaluate $\lim_{x\to 2}\frac{x^{10}-1024}{x^5-32}$

Q.6 Evaluate $\lim_{x\to\infty}\frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3}$

Q.7 Evaluate $\lim_{x\to\infty}(\sqrt{25x^2-3x}+5x)$

Q.8 $\lim_{x\to 0}\frac{(\cos x)^{1/3}-(\cos x)^{1/2}}{\sin^2 x}=$

(A) 1/12 (B) 1/6
(C) 1/3 (D) 1/2

Q.9 $\lim_{x\to 0}\frac{\ln(\sin 2x)}{\ln(\sin x)}=$

(A) 0 (B) 1
(C) 2 (D) non-existent

Q.10 $\lim_{x\to 0}\left(1^{\text{cosec}^2 x}+2^{\text{cosec}^2 x}+3^{\text{cosec}^2 x}+\ldots\ldots+100^{\text{cosec}^2 x}\right)^{\sin^2 x}$

Q.11 Evaluate $\lim_{x\to 0}\frac{1}{x}\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Q.12 Evaluate $\lim_{x\to\infty}2^{x-1}\tan\left(\frac{a}{2^x}\right)$

Q.13 Evaluate $\lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}$

Q.14 Evaluate $\lim_{x\to a}\frac{\log x-\log a}{x-a}$

Q.15 Evaluate $\lim_{x\to 0}\log_{\tan^2 x}(\tan^2 2x)$

Q.16 Evaluate $\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3}$

Q.17 $\lim_{x\to\tan^{-1}3}\frac{[\tan^2 x]-2[\tan x]-3}{[\tan^2 x]-4[\tan x]+3}$ (where [x] is the greatest integer function of x)

(A) is 1/3 (B) is 2
(C) is 3 (D) does not exist.

Q.18 Evaluate $\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}$

ANSWERS

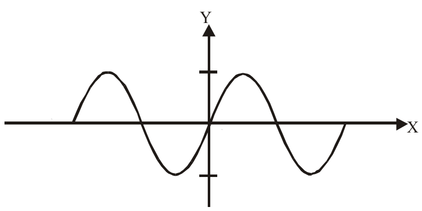
(1) (i) does not exist, (ii) exist (iii) exist (iv) exist
(2) LHL = –1, RHL = 1 **(3)** 1
(4) $1/2\sqrt{2}$ **(5)** 64
(6) $\frac{\sqrt{3}-\sqrt{2}}{4}$ **(7)** 3/10 **(8)** (A)
(9) (B) **(10)** 100 **(11)** 2
(12) a/2 **(13)** 2 log 2 **(14)** 1/a
(15) 1 **(16)** 1/2 **(17)** 1/3
(18) –e/2

CONTINUITY

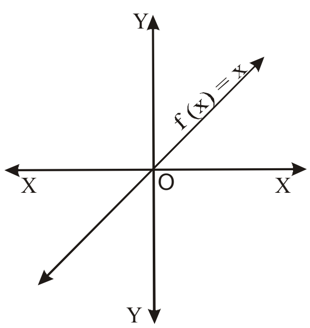
INTRODUCTION

The word 'Continuous' means without any break or gap. If the graph of a function has no break or hole or gap or jump, then it is said to be Continuous. A function which is not continuous is called a discontinuous function.

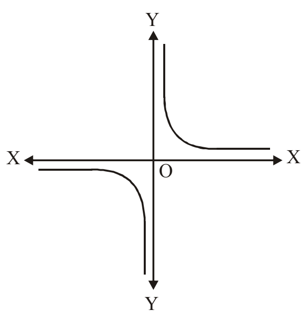
In other words, if there is slight (finite) change in the value of a function by slightly changing the value of x then function is continuous, otherwise discontinuous, while studying graphs of functions, we see that graphs of function sin x, x, cos x, e^x etc. are continuous but greatest integer function [x] has break at every integral point, so it is not continuous, Similarly tan x, cot x, sec x, 1/x etc. are also discontinuous function. Ex.



Y = sinx



(Continuous Function)



Y= 1/x
Discontinuous at x = 0

For examining continuity of a function at a point , we find its limit and value at that point, If these two exist and are equal, then function is continuous at that point..

CONTINUITY OF A FUNCTION AT A POINT

A function f (x) is said to be continuous at a point x = a if

(i) f (a) exists

(ii) $\lim\limits_{x \to a} f(x)$ exists and finite

so $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x)$

(iii) $\lim\limits_{x \to a} f(x) = f(a)$

or function f (x) is continuous at x = a

If $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x) = f(a)$

i.e. If right hand limit at 'a' = left hand limit at 'a' = value of the function at 'a'

Example 30 :

Test the continuity of the function f (x) at the origin :

$$f(x) = \begin{cases} \dfrac{|x|}{x}; & x \neq 0 \\ 1; & x = 0 \end{cases}$$

Sol. We have (LHL at x = 0)

$$= \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} (-1) = -1$$

and, (RHL at x = 0)

$$= \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

Thus, we have $\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$

Hence f (x) is not continuous at the origin

ALITER We have,

$$f(x) = \begin{cases} \dfrac{|x|}{x}; & x \neq 0 \\ 1; & x = 0 \end{cases} = \begin{cases} \dfrac{x}{x} = 1, & x > 0 \\ \dfrac{-x}{x} = -1, & x < 0 \\ 1, & x = 0 \end{cases}$$

$$\left[\because |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}\right]$$

$$\therefore \text{(LHL at } x = 0) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0} -1 = -1$$

$$\left[\begin{array}{l} \because f(x) = -1 \text{ for } x < 0 \text{ and } x \to 0^- \\ \text{means that } x < 0 \text{ s.t. } x \to 0 \end{array}\right]$$

$$\text{(RHL at } x = 0) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} 1 = 1$$

$$\left[\begin{array}{l} \because f(x) = 1 \text{ for } x > 0 \text{ and } x \to 0^+ \\ \text{means that } x > 0 \text{ s.t. } x \to 0 \end{array}\right]$$

$$\therefore \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x).$$

Hence f (x) is not continuous at the origin.

Example 31 :

Examine the continuity of the function

$$f(x)=\begin{cases}\dfrac{x^2-9}{x-3}, & \text{when } x\neq 3\\ 6, & \text{when } x=3\end{cases};\ \text{at } x=3$$

Sol. f(3) = 6 (given)

$$\lim_{x\to 3} f(x)=\lim_{x\to 3}\frac{(x-3)(x+3)}{(x-3)}=6$$

$\because \lim_{x\to 3} f(x)=f(3)$

$\therefore$ f(x) is continuous at x = 3

Example 32 :

If $f(x)=\begin{cases}x^k\cos(1/x), & x\neq 0\\ 0, & x=0\end{cases}$ is continuous at x = 0, then

(1) k < 0 (2) k > 0
(3) k = 0 (4) k ≥ 0

Sol. (2). Since f(x) is continuous at x = 0

$\therefore \lim_{x\to 0} f(x)=f(0)$

but f(0) = 0 (given)

$\therefore \lim_{x\to 0} f(x)=\lim_{x\to 0} x^k\cos(1/x)$

$= 0$, if $k>0$

CONTINUITY FROM LEFT AND RIGHT

Function f(x) is said to be

(i) Left continuous at x = a if $\lim_{x\to a-0} f(x)=f(a)$

(ii) Right continuous at x = a if $\lim_{x\to a+0} f(x)=f(a)$

Thus a function f(x) is continuous at a point x = a if it is left continuous as well as right continuous at x = a.

CONTINUITY OF A FUNCTION

Continuity On An Open Interval:

A function f(x) is said to be continuous on an open interval (a,b) iff it is continuous at every point on the interval (a,b).

Continuity On A Closed Interval :

A function f(x) is said to be continuous on a closed interval [a,b] iff

(i) f is continuous on the open interval (a,b)

(ii) $\lim_{x\to a^+} f(x)=f(a)$, and

(iii) $\lim_{x\to b^-} f(x)=f(b)$

In other words, f(x) is continuous on [a,b] iff it is continuous on (a,b) and it is continuous at a from the right and at b from the left.

Example 33 :

Check the continuity of the function

$$f(x)=\begin{cases}x+2, & x>3\\ 5, & x=3\\ 8-x, & x<3\end{cases}\ \text{at } x=3$$

Sol. f(3) = 5

Left hand limit $\lim_{x\to 3^+}(3+h)+2$

$\lim_{h\to 0}(5+h)=5$

Right hand limit $\lim_{x\to 3^-} 8-(3-h)$

$\lim_{h\to 0} 5+h=5=\text{LHL}$

$\because$ f(3) = RHL = LHL

$\therefore$ function is continuous

Example 34 :

If the function $f(x)=\begin{cases}1, & x\le 2\\ ax+b, & 2<x<4\\ 7, & x\ge 4\end{cases}$

is continuous at x = 2 and 4, then find the values of a and b.

Sol. Since f(x) is continuous at x = 2

$\therefore f(2)=\lim_{x\to 2^+} f(x)$

$\Rightarrow 1=\lim_{x\to 2^+}(ax+b)$

$\therefore 1=2a+b$ ..... (1)

Again f(x) is continuous at x = 4,

$\therefore f(4)=\lim_{x\to 4^-} f(x)$

$\Rightarrow 7=\lim_{x\to 4^-}(ax+b)$

$\therefore 7=4a+b$ .... (2)

Solving (1) and (2), we get a = 3, b = – 5

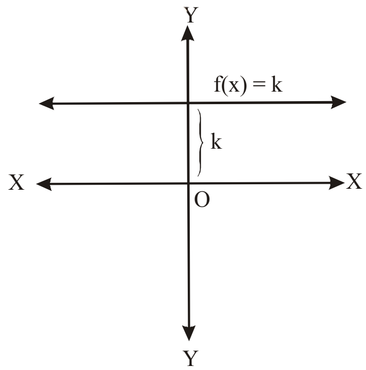
CONTINUOUS FUNCTIONS

A function f(x) is said to be continuous if it is continuous at each point of its domain.

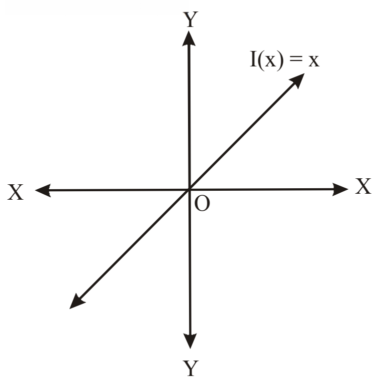
Everywhere Continuous Function : A function f(x) is said to be everywhere continuous if it is continuous on the entire real line (–∞, ∞).

Listed below are some common type of functions that are continuous in their domain.

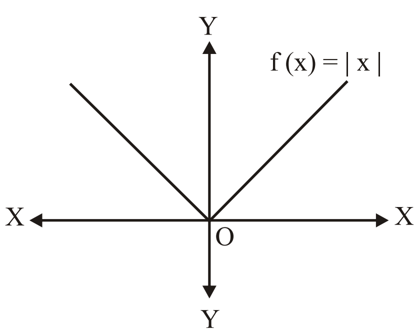
(i) Constant Function : Every constant function is everywhere continuous. Let f(x) = k, where k is a constant, be a constant function. Its graph is shown. It is evident from the graph that this function is everywhere continuous.



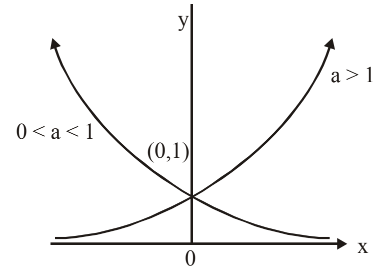
(ii) Identity Function : The identity function I(x) is defined by I(x) = x for all $x \in R$. This function is everywhere continuous as is evident from its graph.



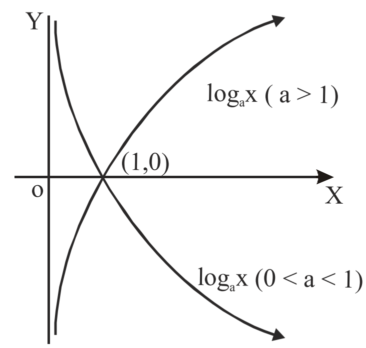
(iii) Modulus Function : The modulus function f (x) is defined as $f(x) = |x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{if } x < 0 \end{cases}$ Clearly, the domain of f (x) is R and this function is everywhere continuous.



(iv) Exponential Function : If a is a positive real number, other than 1, then the function f (x) defined by $f(x) = a^x$ for all $x \in R$, is called the exponential function.
The domain of this function is R . It is evident from its graph that it is everywhere continuous.



(v) Logarithmic Function : If a is positive real number other than unity, then a function defined by $f(x) = \log_a x$ is called the logarithmic function. Clearly its domain is the set of all positive real numbers and it is continuous on its domain. It should be noted that it is not everywhere continuous .



(vi) Polynomial Function : A function of the form $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$, where $a_0, a_1, a_2, ..., a_n \in R$ is called a polynomial function. This function is everywhere continuous.

(vii) Rational Function : If p(x) and q(x) are two polynomials, then a function f (x) of the form $f(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$ is called a rational function. This function is continuous on its domain, i.e. it is everywhere continuous except at points where $q(x) = 0$.

(viii) Trigonometric Functions :
(a) The sine & cosine functions are everywhere continuous.
(b) The tangent, cot, sec and cosec functions are continuous in their domain.

(ix) Inverse Trigonometric Functions :
We know that the graph of the inverse of a function is the mirror image of the graph of the given function in the line y = x. Therefore, graphs of $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$ and $\text{cosec}^{-1}x$ are the mirror images of those of the corresponding trigonometric functions in the line y = x. Since trigonometric function are continuous in their domains, Therefore each of the inverse trigonometric functions is continuous in its domain.

Properties of continuous functions:

(A) If f and g are two continuous functions on their common domain D, then
(i) f + g is continuous on D
(ii) f – g is continuous on D

(iii) fg is continuous on D

(iv) αf is continuous on D where α is any real number.

(v) $\dfrac{f}{g}$ is continuous on $D - \{x ; g(x) \neq 0\}$

(vi) $\dfrac{1}{f}$ is continuous on $D - \{x ; f(x) \neq 0\}$

(B) The composition of two continuous functions is a continuous function.

(C) If f is continuous on its domain D, then $|f|$ is also continuous on D.

For examples :

(i) $e^{2x} + \sin x$ is a continuous function because it is the sum of two continuous function e^{2x} and $\sin x$.

(ii) $\sin(x^2 + 2)$ is a continuous function because it is the composite of two continuous functions $\sin x$ and $x^2 + 2$.

Note: The product of one continuous and one discontinuous function may or may not be continuous.

For example :

(i) $f(x) = x$ is continuous and $g(x) = \cos 1/x$ is discontinuous whereas their product $x \cos 1/x$ is continuous

(ii) $f(x) = C$ is continuous and $g(x) = \sin 1/x$ is discontinuous whereas their product $C \sin 1/x$ is discontinuous.

Continuity of functions involving limit $\lim\limits_{n\to\infty} a^n$:

We know that $\lim\limits_{n\to\infty} a^n = \begin{cases} 0, & 0 \le a < 1 \\ 1, & a = 1 \\ \infty, & a > 1 \end{cases}$

Example 35 :

Discuss the continuity of $f(x) = \lim\limits_{n\to\infty} \dfrac{x^{2n} - 1}{x^{2n} + 1}$

Sol. $f(x) = \lim\limits_{n\to\infty} \dfrac{(x^2)^n - 1}{(x^2)^n + 1} = \lim\limits_{n\to\infty} \dfrac{1 - \dfrac{1}{(x^2)^n}}{1 + \dfrac{1}{(x^2)^n}}$

$$= \begin{cases} -1, & 0 \le x^2 < 1 \\ 0, & x^2 = 1 \\ 1, & x^2 > 1 \end{cases} = \begin{cases} 1, & x < -1 \\ 0, & x = -1 \\ -1, & -1 < x < 1 \\ 0, & x = 1 \\ 1, & x > 1 \end{cases}$$

Thus, $f(x)$ is discontinuous at $x = \pm 1$

Continuity of functions in which f (x) is defined differently for rational and irrational values of x :

Example 36 :

Discuss the continuity of the following function :

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

Sol. For any $x = a$,

L.H.L. $= \lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^-} f(a - h) = 0$ or 1

[as $\lim\limits_{h\to 0}(a - h)$ can be rational or irrational.

Similarly, R.H.L. $= \lim\limits_{x\to a^+} f(x) = \lim\limits_{h\to 0} f(a + h) = 0$ or 1

$f(x)$ oscillates between 0 and 1 as for all values of a.

$\therefore$ L.H.L. and R.HL. do not exist.

$\Rightarrow f(x)$ is discontinuous at a point $x = a$ for all values of a.

Continuity of composite functions

If f is continuous at $x = a$ and g is continuous at $x = f(a)$ then the composite $g[f(x)]$ is continuous at $x = a$.

Example 37 :

If $f(x) = \dfrac{x+1}{x-1}$ and $g(x) = \dfrac{1}{x-2}$, then discuss the continuity of $f(x)$, $g(x)$ and $fog(x)$.

Sol. (a) $f(x) = \dfrac{x+1}{x-1}$

$\therefore$ f is not defined at $x = 1$.

$\therefore$ f is discontinuous at $x = 1$.

(b) $g(x) = \dfrac{1}{x-2}$

$g(x)$ is not defined at $x = 2$

$\therefore$ g is discontinuous at $x = 2$.

(c) Now, fog will be discontinuous at $x = 2$ [point of discontinuity of $g(x)$]

$g(x) = 1$ [when $g(x) =$ point of discontinuity of $f(0)$]

If $g(x) = 1 \Rightarrow \dfrac{1}{x-2} = 1 \Rightarrow x = 3$

$\therefore$ $fog(x)$ is discontinuous at $x = 2$ and $x = 3$.

Also, $fog(x) = \dfrac{\dfrac{1}{x-2} + 1}{\dfrac{1}{x-2} - 1}$

Here, $fog(2)$ is not defined.

$$\lim_{x\to 2} fog(x) = \lim_{x\to 2} \dfrac{\dfrac{1}{x-2} + 1}{\dfrac{1}{x-2} - 1} = \lim_{x\to 2} \dfrac{1 + x - 2}{1 - x + 2} = 1$$

∴ fog (x) is discontinuous at x = 2 and it has a removable discontinuity at x = 2.
For continuity at x = 3

$$\lim_{x\to 3^+} \text{fog}(x) = \lim_{x\to 3} \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = \infty$$

$$\lim_{x\to 3^-} \text{fog}(x) = \lim_{x\to 3} \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = -\infty$$

∴ fog (x) is discontinuous at x = 3 and it is a non-removable discontinuity at x = 3.

DISCONTINUOUS FUNCTIONS

A function is said to be a discontinuous function if it is discontinuous at at least one point in its domain.
The discontinuity may arise due to any of the following situations:

(i) $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ of both may not exist

(ii) $\lim_{x\to a^+} f(x)$ as well as $\lim_{x\to a^-} f(x)$ may exist, but are unequal

(iii) $\lim_{x\to a^+} f(x)$ as well as $\lim_{x\to a^-} f(x)$ both may exist, but either of the two or both may not be equal to f (a).

We classify the points of discontinuity according to various situations discussed above.

Removable Discontinuity: A function f is said to have removable discontinuity at x = a if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ but their common value is not equal to f (a).
Such a discontinuity can be removed by assigning a suitable value to the function f at x = a.

Discontinuity of the first kind:
A function f is said to have a discontinuity of the first kind at x = a if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist but are not equal.
f is said to have a discontinuity of the first kind from the left at x = a if $\lim_{x\to a^-} f(x)$ exists but not equal to f (a). Discontinuity of the first kind from the right is similarly defined.

Discontinuity of second kind : A function f is said to have a discontinuity of the second kind at x = a if neither $\lim_{x\to a^-} f(x)$ nor $\lim_{x\to a^+} f(x)$ exists.
f is said to have a discontinuity of the second kind from the left at x = a if $\lim_{x\to a^-} f(x)$ does not exist.

Similarly, if $\lim_{x\to a^+} f(x)$ does not exist, then f is said to have discontinuity of the second kind from the right at x = a.
Following are examples of some discontinuous function -

(i) $f(x) = 1/x$ at $x = 0$
(ii) $f(x) = e^{1/x}$ at $x = 0$
(iii) $f(x) = \sin 1/x$, $f(x) = \cos 1/x$ at $x = 0$
(iv) $f(x) = [x]$ at every integer
(v) $f(x) = x - [x]$ at every integer
(vi) $f(x) = \tan x$, $f(x) = \sec x$, when $x = (2n+1)\pi/2$, $n \in z$.
(vii) $f(x) = \cot x$, $f(x) = \text{cosec}\, x$, when $x = n\pi$, $n \in z$.

Example 38 :

Find the points of discontinuity of $f(x) = \frac{1}{2\sin x - 1}$

Sol. $f(x) = \frac{1}{2\sin x - 1}$

f (x) is discontinuous when 2 sin x – 1 = 0

$\Rightarrow \sin x = \frac{1}{2} \Rightarrow x = 2n\pi + \frac{\pi}{6}$ or $x = 2n\pi + \frac{5\pi}{6}$, $n \in Z$

TRY IT YOURSELF-2

Q.1 Find the points of discontinuity of $f(x) = \frac{1}{1 - e^{\frac{x+1}{x-2}}}$

Q.2 $f(x) = \begin{cases} (\cos x)^{\cot^2 x} & \text{if } x \neq 0 \\ e^{-1/2} & \text{if } x = 0 \end{cases}$ find whether the f (x) is continuous at x = 0 or not.

Q.3 $f(x) = \begin{cases} \frac{1-\cos 4x}{x^2} & \text{if } x < 0 \\ a & \text{if } x = 0 \\ \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}} - 4} & \text{if } x > 0 \end{cases}$

Determine 'a' if possible so that the function is continuous at x = 0.

Q.4 Let $f(x) = \begin{cases} \frac{(e^{2x}+1)-(x+1)(e^x+e^{-x})}{x(e^x-1)} & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$

If f (x) is continuous at x = 0 then k is equal to –
(A) 1/2 (B) 1
(C) 3/2 (D) 2

Q.5 Let $f(x) = \dfrac{\sqrt{x^2+kx+1}}{x^2-k}$. The interval (s) of all possible values of k for which f is continuous for every $x \in R$, is

(A) $(-\infty, -2]$ (B) $[-2, 0)$
(C) $R-(-2, 2)$ (D) $(-2, 2)$

Q.6 If the function $f(x) = \left[\dfrac{(x-2)^3}{a}\right]\sin(x-2) + a\cos(x-2)$.
[.] denotes the greatest integer function which is continuous in [4, 6], then find the value of a.

Q.7 If $f(x) = \text{sgn}(2\sin x + a)$ is continuous for all x, then find the possible values of a.

Q.8 Find the value of k so that the function

$f(x) = \begin{cases} kx+1 & \text{if } x \le \pi \\ \cos x & \text{if } x > \pi \end{cases}$, is continuous at at $x = \pi$

Q.9 Find the values of a and b such that the function defined

by $f(x) = \begin{cases} 5 & \text{if } x \le 2 \\ ax+b & \text{if } 2 < x < 10 \\ 21 & \text{if } x \ge 10 \end{cases}$ is a continuous function.

Q.10 Find all the points of discontinuity of f defined by $f(x) = |x| - |x+1|$.

ANSWERS

(1) $x = 2$ and $x = 1$	**(2)** continuous	**(3)** 8
(4) (B)	**(5)** (B)	**(6)** $a > 64$
(7) $a < -2$ or $a > 2$	**(8)** $-2/11$	**(9)** $a = 1, b = 1$
(10) no point		

DIFFERENTIABILITY

DIFFERENTIABILITY OF A FUNCTION

A function f (x) is said to be differentiable at a point of its domain if it has a finite derivative at that point. Thus f (x) is differentiable at x = a

$\Rightarrow \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists finitely

$\Rightarrow \lim\limits_{x\to a^-} \dfrac{f(x)-f(a)}{x-a} = \lim\limits_{x\to a^+} \dfrac{f(x)-f(a)}{x-a}$

$\Rightarrow \lim\limits_{h\to 0} \dfrac{f(a-h)-f(a)}{-h} = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$

$\Rightarrow f'(a-0) = f'(a+0)$

$\Rightarrow$ Left - hand derivative = Right - hand derivative

Generally derivative of f (x) at x = a is denoted by f ′ (a)

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

Example 39 :

Show that $f(x) = x^2$ is differentiable at x = 1 and find $f'(1)$.

Sol. (LHD at x = 1)

$= \lim\limits_{x\to 1^-} \dfrac{f(x)-f(1)}{x-1} = \lim\limits_{h\to 0} \dfrac{f(1-h)-f(1)}{1-h-1}$

$= \lim\limits_{h\to 0} \dfrac{(1-h)^2-1^2}{-h} = \lim\limits_{h\to 0} \dfrac{-2h+h^2}{-h} = \lim\limits_{h\to 0} 2-h = 2$

(RHD at x = 1)

$= \lim\limits_{x\to 1^+} \dfrac{f(x)-f(1)}{x-1} = \lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{1+h-1}$

$= \lim\limits_{h\to 0} \dfrac{(1+h)^2-1^2}{h} = \lim\limits_{h\to 0} \dfrac{2h+h^2}{h} = \lim\limits_{h\to 0} 2+h = 2$

$\therefore$ (LHD at x = 1) = (RHD at x = 1) = 2.

So, f (x) is differentiable at x = 1 and $f'(1) = 2$

Example 40 :

For what choice of a and b is the function

$f(x) = \begin{cases} x^2 & , x \le c \\ ax+b, & x < c \end{cases}$ is differentiable at x = c.

Sol. It is given that f (x) is differentiable at x = c and every differentiable function is continuous. So, f (x) is continuous at x = c.

$\therefore \lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x) = f(c)$

$\Rightarrow \lim\limits_{x\to c^-} x^2 = \lim\limits_{x\to c^+} (ax+b) = c^2$ [Using def. of f (x)]

$\Rightarrow c^2 = ac + b$ .....(i)

Now, f (x) is differentiable at x = c

$\Rightarrow$ (LHD at x = c) = (RHD at x = c)

$\Rightarrow \lim\limits_{x\to c^-} \dfrac{f(x)-f(c)}{x-c} = \lim\limits_{x\to c^+} \dfrac{f(x)-f(c)}{x-c}$

$\Rightarrow \lim\limits_{x\to c} \dfrac{x^2-c^2}{x-c} = \lim\limits_{x\to c} \dfrac{(ax+b)-c^2}{x-c}$ [Using def. of f (x)]

$\Rightarrow \lim\limits_{x\to c} \dfrac{x^2-c^2}{x-c} = \lim\limits_{x\to c} \dfrac{ax+b-(ac+b)}{x-c}$ [Using (i)]

$\Rightarrow \lim\limits_{x\to c} (x+c) = \lim\limits_{x\to c} a$

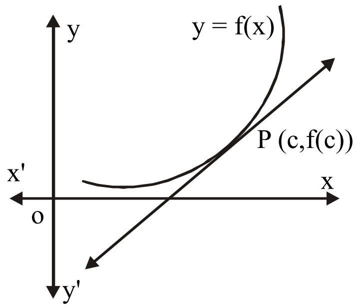
$\Rightarrow 2c = a$ .....(ii)

From (i) and (ii), we get

$c^2 = 2c^2 + b \Rightarrow b = -c^2$.

Hence, a = 2c and $b = -c^2$.

GEOMETRICAL MEANING OF DIFFERENTIABILITY



Thus, f (x) is differentiable at point P, iff there exists a unique tangent at point P. In other words, f (x) is differentiable at a point P iff the curve does not have P as a corner point.

DIFFERENTIABILITY FROM LEFT AND RIGHT

A function f (x) is said to be

(i) Left differentiability at $x = a$ if $\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a}$ exists.

(ii) Right differentiable at $x = a$ if $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a}$ exists.

DIFFERENTIABILITY OF A FUNCTION IN A SET

Differentiability on an open interval : A function f (x) defined on an open interval (a, b) is said to be differentiable or derivable in open interval (a, b) if it is differentiable at each point of (a, b).

Differentiability on a closed interval : A function f (x) defined on [a, b] is said to be differentiable or derivable at the end points a and b if it is differentiable from the right at a and from the left at b.

In other words $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a}$ and $\lim_{x\to b^-} \frac{f(x)-f(b)}{x-b}$ both exist.

If f is derivable in the open interval (a, b) and also at the end points a and b, then f is said to be derivable in the closed interval [a, b]

(i) Differentiable at every point of interval (a, b)

(ii) Right derivative exists at $x = a$

(iii) Left derivative exists at $x = b$.

Example 41 :

$$\text{Function } f(x) = \begin{cases} -x & \text{if} \quad x < 0 \\ x^2 & \text{if} \quad 0 \le x \le 1 \\ x^3 - x + 1 & \text{if} \quad x > 1 \end{cases}$$

is differentiable at –

(1) $x = 0$ but not at $x = 1$ (2) $x = 1$ but not at $x = 0$

(3) $x = 0$ and $x = 1$ both (4) neither $x = 0$ nor $x = 1$

Sol. (2). Differentiability at $x = 0$

$$R[f'(0)] = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{(0+h)^2 - 0}{h} = \lim_{h\to 0} h = 0$$

$$L[f'(0)] = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{-(0-h)-0}{-h} = -1$$

$\therefore$ $R[f'(0)] \ne L[f'(0)]$

$\therefore$ f (x) is not differentiable at $x = 0$

Differentiability at $x = 1$

$$R[f'(1)] = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{(1+h)^3 - (1+h) + 1 - 1}{h} = \lim_{h\to 0} \frac{2h + 3h^2 + h^3}{h}$$

$$L[f'(1)] = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0} \frac{f(1-h)-1}{-h}$$

$$= \lim_{h\to 0} \frac{-2h + h^2}{-h} = 2$$

Thus $R[f'(1)] = L f'(1)]$

$\because$ Function f (x) is differentiable at $x = 1$

Example 42 :

Show that the function f (x) defined by

$$f(x) = \begin{cases} x & \text{for} \quad x < 1 \\ 2 - x & \text{for} \quad 1 \le x \le 2 \\ -2 + 3x - x^2 & \text{for} \quad x > 2 \end{cases}$$

is differentiable at = 2 but not at $x = 1$

Sol. Differentiability at $x = 2$

$$(\text{LHD at } x = 2) = \left(\frac{d}{dx}(2-x)\right)_{x=2} = -1$$

$$(\text{RHD at } x = 2) = \left(\frac{d}{dx}(-2 + 3x - x^2)\right)_{x=2}$$

$$= (3 - 2x)_{x=2} = 3 - 4 \quad = -1$$

$\therefore$ (LHD at $x = 2$) = (RHD at $x = 2$)

So, f (x) is differentiable at $x = 2$.

Differentiability at $x = 1$

$$(\text{LHD at } x = 1) = \left(\frac{d}{dx}(x)\right)_{x=1} = 1$$

$$(\text{LHD at } x = 1) = \left(\frac{d}{dx}(2-x)\right)_{x=1} = -1$$

Clearly, (LHD at $x = 1$) $\ne$ (RHD at $x = 1$)

So, f (x) is not differentiable at $x = 1$.

DIFFERENTIABLE FUNCTION

A function f is said to be a differentiable function if it is differentiable at every point of its domain.

Every where differentiable function :

If a function is differentiable at each $x \in R$, then it is said to be every where differentiable.

SOME STANDARD RESULTS ON DIFFERENTIABILITY :

(i) Every polynomial function is differentiable at each $x \in R$.

(ii) The exponential function a^x, $a>0$ is differentiable at each $x \in R$.

(iii) Every constant function is differentiable at each $x \in R$.

(iv) The logarithmic function is differentiable at each point in its domain.

(v) Trigonometric and inverse-trigonometric functions are differentiable in their domains.

(vi) The sum, difference, product and quotient of two differentiable functions is differentiable.

(vii) The composition of differentiable function is a differentiable function.

(viii) If a function is not differentiable but is continuous at a point, it geometrically implies there is a sharp corner or a kink at that point.

(ix) If $f(x)$ and $g(x)$ both are not differentiable at a point, then the sum function $f(x)+g(x)$ and the product function $f(x).g(x)$ can still be differentiable at that point.

RELATION BETWEEN CONTINUITY AND DIFFERENTIABILITY

(i) If a function $f(x)$ is differentiable at a point $x = a$ then it is continuous at $x = a$.

(ii) If $f(x)$ is only continuous at a point $x = a$, there is no guarantee that $f(x)$ is differentiable there.

(iii) If $f(x)$ is not differentiable at $x = a$ then it may or may not be continuous at $x = a$.

(iv) If $f(x)$ is not continuous at $x = a$, then it is not differentiable at $x = a$.

(v) If left hand derivative and right hand derivative of $f(x)$ at $x = a$ are finite (they may or may not be equal) then $f(x)$ is continuous at $x = a$.

REMEMBER

For a function f :

Differentiability $\Rightarrow$ Continuity ;

Continuity $\not\Rightarrow$ derivability ;

Non derivibality $\not\Rightarrow$ discontinuous

But discontinuity $\Rightarrow$ Non derivability

Example 43 :

Prove the function $f(x) = |x|$ is continuous at $x = 0$ but it is not differentiable at $x = 0$

Sol. Checking of continuity at $x = 0$

[LHL at $x = 0$]

$$\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} |x| = 0$$

[RHL at $x = 0$]

$$\Rightarrow \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} |x| = 0 \quad \text{[Value of function at } x=0]$$

$f(0) = |0| = 0$; LHL = RHL = f(0)

So function $f(x)$ is continuous at $x = 0$

Checking of differentiability at $x = 0$

(LHD at $x = 0$)

$$= \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{0-h-0}$$

$$= \lim_{h\to 0} \frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{|-h|-|0|}{-h} = \lim_{h\to 0} \frac{|-h|}{-h},$$

$$= \lim_{h\to 0} \frac{h}{-h} = \lim_{h\to 0}(-1) = -1$$

(RHD at $x = 0$)

$$= \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{x\to 0} \frac{|h|-|0|}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

$\therefore$ (LHD at $x = 0$) $\neq$ (RHD at $x = 0$)

So, $f(x)$ is not differentiable at $x = 0$

TRY IT YOURSELF-3

Q.1 If $f(x) = \begin{cases} x^m \cdot \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

is continuous but not differentiable at $x = 0$ then find m.

Q.2 Let f be defined as follows:

$$f(x) = \begin{cases} \sin x & \text{if } x < \pi \\ mx + n & \text{if } x \geq \pi \end{cases}$$

where m and n are constants. Determine m and n such that f is derivable on set of real numbers.

Q.3 If $f(x)$ is differentiable at $x = a$ and $f'(a) = 1/4$. Find

$$\lim_{h\to 0} \frac{f(a+2h^2)-f(a-2h^2)}{h^2}$$

Q.4 Prove that the function f given by $f(x) = |x-1|$, $x \in R$ is not differentiable at $x = 1$.

Q.5 Prove that the greatest integer function defined by $f(x) = [x]$, $0 < x < 3$ is not differentiable at $x = 1$ and $x = 2$.

Q.6 Discuss the differentiability of $f(x) = |x| + |x-1|$.

ANSWERS

(1) $m \in [0, 1]$ **(2)** $m = -1, n = \pi$ **(3)** 1

(6) $f(x)$ is non-differentiable at $x = 0$ and $x = 1$.

MEAN VALUE THEOREM

Rolle's theorem

Let f (x) be a function of x subject to the following conditions

(i) f (x) is a continuous function of x in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

(iii) $f(a) = f(b)$.

Then there exists at least one point $x = c$ such that $a<c<b$ where $f'(c) = 0$.

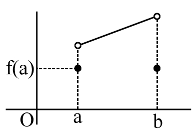
Alternative statement: Rolles theorem states that between ay two real zeroes of a differentiable real function f, lies at least one critical point of f (x).

Note :

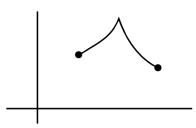
1. Converse of Rolle's theorem is **Not true** i.e. f ' (x) may vanish at a point within (a, b) without satisfying all the three conditions of Rolle's Theorem.

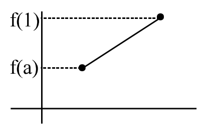
2. The three conditions are sufficient but not necessary for f ' (x) = 0 for some x in (a, b)

3. If the function y = f (x) defined over [a, b] does not satisfy even one of the 3 conditions then Rolle's Theorem fails i.e. there may or may not exist point in (a, b) where f ' (x) = 0.



f is not continuous in [a, b] and Rolles theorem failing

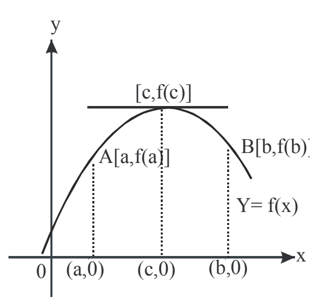

f is not derivable in open (a, b)



$f(a) \ne f(b)$

Geometrical interpretation of rolle's theorem :

Let f(x) be a real valued function defined on [a,b] such that the curve y = f (x) is a continuous curve between points A(a,f(a) and B(b,f (b)) and it is possible to draw a unique tangent at every point on the curve y = f (x) between points A and B. Also, the ordinates at the end points of the interval



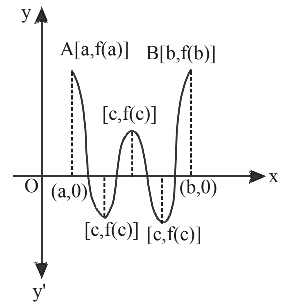
[a,b] are equal. Then there exists at least one point (c,f(c)) lying between A and B on the curve y = f(x) where tangent is parallel to x-axis.

Algebraic interpretation of rolle's theorem :

Let f (x) be a polynomial with a and b as its roots.

Since a polynomial function is every where continuous and differentiable and a and b are roots of f(x), therefore $f(a) = f(b) = 0$. So f(x) satisfies conditions of Rolle's theorem. Consequently there exists $c \in (a,b)$ such that $f'(c) = 0$ i.e. $f'(x) = 0$ at $x = c$. In other words $x = c$ is a root of $f'(x)$.

Thus, algebraically Rolle's theorem can be interpreted as follows :



Between any two roots of a polynomial f(x), there is always a root of its derivative $f'(x)$. On Rolle's theorem generally two types of problems are formulated.

(a) To check the applicability of Rolle's theorem to a given function on a given interval.

(b) To verify Rolle's theorem for a given function on a given interval. In both types of problems we first check whether f(x) satisfies conditions of Rolle's theorem or not.

The following result are very helpful in doing so.

(i) A polynomial function is everywhere continuous and differentiable.

(ii) The exponential function, sine and cosine functions are everywhere continuous and differentiable.

(iii) Logarithmic function is continuous and differentiable in its domain

(iv) tan x is not continuous at $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2$ .....

(v) | x | is not differentiable at x = 0.

(vi) If $f'(x)$ tends to $\pm\infty$ as $x \to k$, then f(x) is not differentiable at x = k. For example, if $f(x) = (2x-1)^{1/2}$,

then $f'(x) = \dfrac{1}{\sqrt{2x-1}}$ is such that $\lim\limits_{x \to (1/2)^+} f'(x) = \infty$.

So, f (x) is not differentiable at x = 1/2.

(vii) The sum, difference, Product and quotient of continuous (differentiable) functions is continuous (differentiable).

Example 44 :

Discuss the applicability of Rolle's theorem for the following function on the indicated intervals $f(x) = |x|$ on $[-1, 1]$

Sol. We have, $f(x) = |x| = \begin{cases} -x, & \text{when } -1 \le x < 0 \\ x, & \text{when } 0 \le x \le 1 \end{cases}$

Since a polynomial function is everywhere continuous and differentiable. Therefore, f (x) is continuous and differentiable for all x < 0 and for all x > 0 except possibly at x = 0. So, consider the point x = 0.

At x = 0, we have $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} x = 0$ and,

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0} x = 0$

$\therefore \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = f(0) \Rightarrow f(x)$ is continuous at x = 0.

Hence, f (x) is continuous on [–1, 1]

Now, $(\text{LHD at } x=0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{-x-0}{x}$

$[\because f(x) = -x \text{ for } x<0 \text{ and } f(0)=0]$

$= \lim_{x\to 0} \frac{-x}{x} = \lim_{x\to 0} -1 = -1$ (RHD at $x=0$)

$= \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x-0}{x} = \lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1$

$\therefore$ (LHD at $x=0$) $\neq$ RHD at $x=0 \Rightarrow f(x)$ is not differentiable at $x=0 \in (-1, 1)$.

Thus, the condition of derivability at each point of $(-1,1)$ is not satisfied. Hence, Rolle's theorem is not applicable to $f(x) = |x|$ on $[-1, 1]$.

Example 45 :

Discuss the applicability of Rolle's theorem for the following function on the indicated intervals $f(x) = \tan x$ on $[0, \pi]$

Sol. We have, $f'(x) = \tan x, x \in [0,\pi]$.

Since $\frac{\pi}{2} \in [0, \pi]$ and $f(x)$ is not continuous at $x = \frac{\pi}{2}$

So, the condition of continuity at each point of $[0, \pi]$ is not satisfied.

Hence, Rolle's theorem is not applicable to $f(x) = \tan x$ on the interval $[0, \pi]$

Example 46 :

Discuss the applicability of Rolle's theorem on the function

$$f(x) = \begin{cases} x^2+1, & \text{when } 0 \le x \le 1 \\ 3-x & \text{when } 1 < x \le 2 \end{cases}$$

Sol. Since a polynomial function is everywhere continuous and differentiable. Therefore, $f(x)$ is continuous and differentiable at all points except possibly at $x = 1$.

Now, we consider the differentiability of $f(x)$ at $x = 1$.

We have, (LHD at $x = 1$)

$= \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{(x^2+1)-(1+1)}{x-1}$

$[\because f(x) = x^2+1 \text{ for } 0 \le x \le 1]$

$= \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} (x+1) = 2$ and, (RHD at $x = 1$)

$= \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$

$= \lim_{x\to 1} \frac{(3-x)-(1+1)}{x-1} = \lim_{x\to 1} \frac{-(x-1)}{x-1} = -1$

$\therefore$ (LHD at $x = 1$) $\neq$ (RHD at $x = 1$).

So, $f(x)$ is not differentiable at $x = 1$,

Thus, the condition of differentiability at each point of the given interval is not satisfied.

Hence, Rolle's theorem is not applicable to the given function on the interval $[0, 2]$

Example 47 :

If $2a + 3b + 6c = 0$, then show that the equation $ax^2 + bx + c = 0$ has at least one real root between 0 and 1.

Sol. Let $f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx$ be a polynomial.

Then $f(0) = 0$ and $f(1) = \frac{a}{3} + \frac{b}{2} + c = \frac{2a+3b+6c}{6} = 0$

$[\because 2a + 3b + 6c = 0 \text{ (given)}]$

This shows that 0 and 1 are roots of $f(x) = 0$

Hence, by the algebraic interpretation of Rolle's Theorem $f'(x) = 0$ i.e. $ax^2 + bx + c = 0$ has a real root between 0 and 1.

Example 48 :

Verify Rolle's theorem for function on indicated intervals $f(x) = \sin x + \cos x - 1$ on $[0, \pi/2]$

Sol. Since sin x and cos x are everywhere continuous and differentiable. Therefore, $f(x) = \sin x + \cos x - 1$ is continuous on $[0, \pi/2]$ and differentiable on $(0, \pi/2)$

Also $f(0) = \sin 0 + \cos 0 - 1$

and $f(\pi/2) = \sin \pi/2 + \cos \pi/2 - 1 = 1 - 1 = 0$

$\therefore f(0) = f(\pi/2)$. Thus, $f(x)$ satisfies conditions of Rolle's theorem on $[0, \pi/2]$, Therefore, there exists $c \in (0, \pi/2)$ such that $f'(c) = 0$

Now, $f(x) = \sin x + \cos x - 1 \Rightarrow f'(x) = \cos x - \sin x$

$\therefore f'(x) = 0$

$\Rightarrow \cos x - \sin x = 0 \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \pi/4$

Thus, $c = \pi/4 \in (0,\pi/2)$ such that $f'(c) = 0$,

Hence, Rolle's theorem is verified.

Example 49:

Find the point on the curve $y = \cos x - 1$, $x \in [\pi/2, 3\pi/2]$ at which the tangent is parallel to the x-axis.

Sol. Let $f(x) = \cos x - 1$

Clearly, $f(x)$ is continuous on $[\pi/2, 3\pi/2]$ and differentiable on $(\pi/2, 3\pi/2)$.

Also, $f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} - 1 = -1 = f\left(\frac{3\pi}{2}\right)$.

Thus, all the conditions of Rolle's theorem are satisfied.

Consequently, there exists atleast one point $c \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

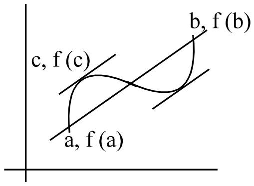
for which $f'(c) = 0$. But. $f'(c) = 0 \Rightarrow -\sin c = 0 \Rightarrow c = \pi$

$\therefore f(c) \Rightarrow \cos \pi - 1 = -2$

Lagrange's theorem about finite increments of a function (Mean value theorem, LMVT)

Let f (x) be a function of x subject to the following conditions

(i) $f(x)$ is a continuous function of x in the closed interval of $a \le x \le b$.

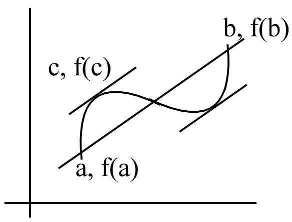


(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

(iii) $f(a) \neq f(b)$. Then there exists at least one point $x = c$ such that $a < c < b$, where $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Geometrically meaning of LMVT :

The slope of the secant line joining the curve at $x = a$ & $x = b$ is equal to the slope of the tangent line drawn to the curve at $x = c$.



Now $[f(b) - f(a)]$ is the change in the value of function f as x changes from a to b so that $[f(b) - f(a)] / (b - a)$ is the **average rate of change** of the function over the interval [a, b]. Also f '(c) is the instantaneous rate of change of the function at x = c. Thus, the theorem states that the average rate of change of a function over an interval is also the actual rate of change of the function at some point of the interval. In particular, for instance, the average velocity of a particle over an interval of time is equal to the velocity at some instant belonging to the interval.

This interpretation of the theorem justifies the name "Mean Value" for the theorem.

Rolles theorem is a special case of LMVT since

$$f(a) = f(b) \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a} = 0.$$

Alternative form of LMVT :

Another form of statement of Lagrange's Mean Value Theorem. If a function f is continuous in a closed interval [a, a + h] and derivable in the open interval] a, a + h [, then there exists at least one number $\theta \in (0, 1)$ such that

$f(a + h) = f(a) + h f'(a + \theta h)$. $\theta \in (0, 1)$

We write b – a = h so that h denotes the length of the interval [a, b] which may now be rewritten as [a, a + h]. The number, 'c' which lies between a and a + h, is greater than a and less than a + h so that we may write $c = a + \theta h$, where θ is some number between 0 and 1.

Thus the equation (i) becomes

$$\frac{f(a + h) - f(a)}{h} = f'(a + \theta h)$$

$$\Rightarrow \quad f(a + h) = f(a) + h f'(a + \theta h)$$

Example 50 :

State the mean value theorem in the equation $f(b) - f(a) = (b - a) f'(c)$ determine c lying between a and b, if $f(x) = (x - 1)(x - 2)(x - 3)$ and a = 0, b = 4.

Sol. $f(x) = (x - 1)(x - 2)(x - 3)$

$f(0) = (0 - 1)(0 - 2)(0 - 3) = -6$

$f(4) = (4 - 1)(4 - 2)(4 - 3) = 6$

$f'(x) = (x - 2)(x - 3) + (x - 1)(x - 3) + (x - 1)(x - 2)$

$= (x^2 - 5x + 6) + (x^2 - 4x + 3) + (x^2 - 3x + 2)$

$= 3x^2 - 12x + 11$

$f'(c) = 3c^2 - 12c + 11$ .... (1)

By Mean value theorem, we have

$f(b) - f(a) = (b - a) f'(c)$

$f(4) - f(0) = (4 - 0) f'(c)$

$6 - (-6) = 4 f'(c)$; $3 = f'(c)$

Putting the value of f ' (c) in (1), we get

$3 = 3c^2 - 12c + 11$

$3c^2 - 12c + 8 = 0$

$$c = \frac{12 \pm \sqrt{144 - 96}}{6}$$

$$c = \frac{12 \pm \sqrt{48}}{6} = \frac{12 \pm 4\sqrt{3}}{6} = 2 \pm \frac{2}{3}\sqrt{3} = 2 \pm \frac{2}{\sqrt{3}}$$

lies between 0 and 4.

Example 51 :

Using LMVT prove that $|\cos a - \cos b| \leq |a - b|$

Sol. Consider $f(x) = \cos x$ in [a, b]

$$\left|\frac{\cos b - \cos a}{b - a}\right| = |-\sin c| \leq 1$$

$\Rightarrow \quad |\cos b - \cos a| \leq |b - a|$

$|\cos a - \cos b| \leq |a - b|$

Example 52 :

Using Lagrange's mean value theorem, find a point on the curve $y = \sqrt{x - 2}$, defined on the interval [2, 3], where the tangent is parallel to the chord joining the end points of the curve.

Sol. Let $f(x) = \sqrt{x - 2}$. Since for each $x \in [2, 3]$, the function f (x) attains a unique definite value.

So, f (x) is continuous on [2, 3]

Also, $f'(x) = \dfrac{1}{2\sqrt{x - 2}}$ exists for all $x \in (2, 3)$. So, f (x) is differentiable on (2, 3).

Thus both the conditions of Lagrange's mean value theorem are satisfied. Consequently, there must exist some $c \in (2, 3)$ such that $f'(c) = \dfrac{f(3) - f(2)}{3 - 2}$

Now, $f(x) = \sqrt{x - 2} \Rightarrow f'(x) = \dfrac{1}{2\sqrt{x - 2}}$, $f(3) = 1$ & $f(2) = 0$

$$\therefore f'(x) = \frac{f(3) - f(2)}{3 - 2} \Rightarrow \frac{1}{2\sqrt{x - 2}} = \frac{1 - 0}{3 - 2}$$

$\Rightarrow \dfrac{1}{2\sqrt{x-2}} = 1 \Rightarrow 4(x-2) = 1 \Rightarrow x - 2 = 1/4 \Rightarrow x = 9/4$

Thus, $c = \dfrac{9}{4} \in (2, 3)$ such that $f'(c) = \dfrac{f(3)-f(2)}{3-2}$

Now, $f(c) = \sqrt{\dfrac{9}{4} - 2} = \dfrac{1}{2}$

Thus, (c, f(c)) = (9/4, 1/2) is a point on the curve $y = \sqrt{x-2}$ such that the tangent at it is parallel to the chord joining the end points of the curve.

TRY IT YOURSELF-4

Q.1 Verify Rolle's theorem for the function $f(x) = x^3 - 6x^2 + 11x - 6$ in the interval [1, 3].

Q.2 Discuss the applicability of Rolle's theorem in the interval [–1, 1] to the function $f(x) = |x|$.

Q.3 Verify Lagrange's Mean-value theorem for $f(x) = \log_e x$ in the interval [1, 2].

Q.4 Applying Mean-value theorem, show that

$$x > \log_e(1+x) > x - \frac{1}{2}x^2 \text{ for } x > 0.$$

Q.5 Using Lagrange's Mean value theorem, show that $\sin x < x$ for $x > 0$.

ADDITIONAL EXAMPLES

Example 1 :

If $f(x) = \begin{cases} x^2+1, & x \ge 1 \\ 3x-1, & x < 1 \end{cases}$, then find the value of $\lim_{x\to 1} f(x)$.

Sol. Left hand limit $= \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3x-1) = 3.1 - 1 = 2$

and Right hand limit $= \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2+1) = 1^2 + 1 = 2$

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 2$. So $\lim_{x\to 1} f(x) = 2$

Example 2 :

Evaluate $\lim_{\theta\to -\frac{\pi}{4}} \dfrac{\cos\theta + \sin\theta}{\theta + \frac{\pi}{4}}$

Sol. Put $\theta + \dfrac{\pi}{4} = h$ or $\theta = -\dfrac{\pi}{4} + h$

$$\text{Limit} = \lim_{h\to 0} \frac{\cos\left(\frac{\pi}{4} - h\right) - \sin\left(\frac{\pi}{4} - h\right)}{h}$$

$$= \lim_{h\to 0} \frac{\cos\left(\frac{\pi}{4} - h\right) - \cos\left(\frac{\pi}{4} + h\right)}{h}$$

$$= \lim_{h\to 0} \frac{2\sin\frac{\pi}{4}.\sinh}{h} = \sqrt{2}$$

Example 3 :

Evaluate $\lim_{x\to 2} \dfrac{\sin(e^{x-2} - 1)}{\log(x-1)}$

Sol. Given limit $= \lim_{h\to 0} \dfrac{\sin(e^h - 1)}{\log(1+h)} = \lim_{h\to 0} \dfrac{\sin\left(h + \frac{h^2}{2} + ...\right)}{h - \frac{h^2}{2} + ...}$

$$= \lim_{h\to 0} \frac{\left(h + \frac{h^2}{2} + ...\right) - \frac{1}{3!}\left(h + \frac{h^2}{2} + ...\right)^3 + ...}{h - \frac{h^2}{2} + ...} = 1$$

Example 4 :

Evaluate $\lim_{x\to 0} \dfrac{1}{x} - \dfrac{1}{x^2}\log(1+x)$

Sol. $\lim_{x\to 0} \dfrac{1}{x} - \dfrac{1}{x^2}\log(1+x)$

$$= \lim_{x\to 0} \frac{x - \log(1+x)}{x^2} = \lim_{x\to 0} \frac{x - \left\{x - \frac{x^2}{2} + \frac{x^3}{3}....\right\}}{x^2} = \frac{1}{2}$$

Example 5 :

If $f(x) = \dfrac{x + |x|}{x}$, then find $\lim_{x\to 0} f(x)$

Sol. LHL $= \lim_{h\to 0} \dfrac{-h + |h|}{-h} = \lim_{h\to 0} (0) = 0$

RHL $= \lim_{h\to 0} \dfrac{h + |h|}{h} = 2$; LHL $\ne$ RHL $\Rightarrow$ does not exist

Example 6 :

Evaluate $\lim_{x\to\infty} \dfrac{\sin x}{x}$

Sol. $\lim_{x\to\infty} \dfrac{\sin x}{x} = \lim_{x\to\infty} \dfrac{1}{x}.\lim_{x\to\infty} \sin x$

= 0 × (a finite number between – 1 and 1) = 0

Example 7 :

If f(9) = 9 and f′(9) = 4, then find the value of $\lim_{x\to 9} \dfrac{\sqrt{f(x)} - 3}{\sqrt{x} - 3}$

Sol. Given limit is in 0/0 form, so using Hospital rule, we get

$$\text{Limit} = \lim_{x\to 9} \frac{\frac{1}{2\sqrt{f(x)}}.f'(x)}{\frac{1}{2\sqrt{x}}} = \frac{f'(9).\sqrt{9}}{\sqrt{f(9)}} = \frac{4.3}{3} = 4$$

Example 8 :

Evaluate $\lim_{x\to 0}\frac{x(2^x-1)}{1-\cos x}$

Sol. $L=\lim_{x\to 0}\frac{2^x-1}{x}\cdot\frac{x^2}{1-\cos x}=\lim_{x\to 0}\frac{2^x-1}{x}\cdot\lim_{x\to 0}\frac{x^2}{2\sin^2\frac{x}{2}}$

$=\log 2.\ 2\lim_{x\to 0}\left(\frac{x/2}{\sin(x/2)}\right)^2=2\log 2$

Example 9 :

Evaluate : $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$

Sol. $\text{Limit}=\lim_{x\to 0}\frac{\left(x+\frac{x^3}{3}+\frac{2x^5}{15}+.....\right)-\left(x-\frac{x^3}{3!}+......\right)}{x^3}$

$=\lim_{x\to 0}\frac{x^3\left[\left(\frac{1}{3}+\frac{1}{6}\right)+(-)x^2+.....\right]}{x^3}=1/2$

Example 10 :

If $f(x)=\begin{cases}\frac{\sin[x]}{[x]} & ,[x]\neq 0\\ 0 & ,[x]=0\end{cases}$ then find $\lim_{x\to 0}f(x)$

Sol. When $-1\le x<0$, then $f(x)=\frac{\sin(-1)}{-1}=\sin 1$

and when $0\le x<1$, then $f(x)=0\ [\because\ [x]=0\Rightarrow f(x)=0]$

$\therefore\ f(0-0)=\lim_{h\to 0}\sin 1=\sin 1$

$\therefore\ f(0+0)=\lim_{h\to 0}(0)=0$

$\therefore\ f(0-0)\neq f(0+0)\ \ \therefore\ \lim_{x\to 0}f(x)$ does not exist.

Example 11 :

If $\lim_{x\to 5}\frac{x^k-5^k}{x-5}=500$, then find the value of k.

Sol. It is in 0/0 form, so by L' Hospital rule, we have

$\lim_{x\to 5}\frac{kx^{k-1}-0}{1}=500\Rightarrow k\,5^{k-1}=500=4\times 5^3$

$\therefore\ k=4$

Example 12 :

Evaluate $\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}$

Sol. Limit is in 0/0 form, so by Hospital Rule

$\text{Limit}=\lim_{x\to 0}\frac{\cos(\pi\cos^2 x)2\pi\cos x(-\sin x)}{2x}$

$=\lim_{x\to 0}\{-\pi.\cos(\pi\cos^2 x).\cos x\}\ .\ \lim_{x\to 0}\left(\frac{\sin x}{x}\right)$

$=\pi.1=\pi$

Example 13 :

If f (x), g(x) be differentiable functions and f(1) = g(1) = 2

then find the value of $\lim_{x\to 1}\frac{f(1)g(x)-f(x)g(1)-f(1)+g(1)}{g(x)-f(x)}$.

Sol. $\lim_{x\to 1}\frac{f(1)g(x)-f(x)g(1)-f(1)+g(1)}{g(x)-f(x)}\quad\left(\frac{0}{0}\text{form}\right)$

$=\lim_{x\to 1}\frac{f(1)g'(x)-f'(x)g(1)}{g'(x)-f'(x)}=2\lim_{x\to 1}\frac{g'(x)-f'(x)}{g'(x)-f'(x)}=2$

Example 14 :

Let f (x) be a twice differentiable function and f '' (0) = 5,

then find the value of $\lim_{x\to 0}\frac{3f(x)-4f(3x)+f(9x)}{x^2}$.

Sol. $\lim_{x\to 0}\frac{3f(x)-4f(3x)+f(9x)}{x^2}\quad\left(\frac{0}{0}\text{form}\right)$

$=\lim_{x\to 0}\frac{3f'(x)-12f'(3x)+9f'(9x)}{2x}\quad\left(\frac{0}{0}\text{form}\right)$

$=\lim_{x\to 0}\frac{3f''(x)-36f''(3x)+81f''(9x)}{2}$

$=\frac{3f''(0)-36f''(0)+81f''(0)}{2}=24f''(0)=24.5=120$

Example 15 :

Find the value of $\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}$

Sol. $\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}=e^{\lim_{x\to\frac{\pi}{2}}\tan x(\sin x-1)}$

$=e^{\lim_{x\to\frac{\pi}{2}}\frac{\sin^2 x-\sin x}{\cos x}}=e^{\lim_{x\to\frac{\pi}{2}}\frac{2\sin x\cos x-\cos x}{-\sin x}}=e^0=1$

Example 16 :

Check the continuity and differentiability of

$$f(x)=\begin{cases} x\sin 1/x & , \quad x\neq 0 \\ 0 & , \quad x=0 \end{cases} \text{ at } x=0$$

Sol. For function to be continuous

$f(0+h)=f(0-h)=f(0)$

$f(0+h)=\lim_{h\to 0} h\sin 1/h = 0\times(\text{a finite quantity})=0$

$f(0-h)=\lim_{h\to 0} -h\sin(1/-h)$

$=0\times(\text{a finite quantity})=0$

Also, $\lim_{x\to 0} x\sin 1/x \;=0\times(\text{a finite quantity})=0$

$\Rightarrow$ function is continuous at x = 0

For function to be differentiable

$f'(0+h)=f'(0-h)$

$$f'(0+h)=\frac{f(0+h)-f(0)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{h\sin\frac{1}{h}-0}{h} \Rightarrow \lim_{h\to 0}\sin\left(\frac{1}{h}\right)$$

which does not exist.

$$f'(0-h)=\lim_{h\to 0}\frac{(-h)\sin\left(-\frac{1}{h}\right)-0}{-h}=\lim_{h\to 0}\sin\left(-\frac{1}{h}\right)$$

which does not exist. So function is not differentiable at x=0

Example 17 :

$$\text{If } f(x)=\begin{cases} \dfrac{\log(1+ax)-\log(1-bx)}{x} & , \quad x\neq 0 \\ k & , \quad x=0 \end{cases}$$

is continuous at x = 0, then find the value of k.

Sol. $\because$ f (x) is continuous at x = 0, so

$f(0)=\lim_{x\to 0} f(x)$

$$\Rightarrow k=\lim_{x\to 0}\frac{\log(1+ax)-\log(1-bx)}{x} \quad \left(\frac{0}{0}\text{form}\right)$$

$$=\lim_{x\to 0}\left(\frac{a}{1+ax}+\frac{b}{1-bx}\right)=a+b$$

Example 18 :

Let $f(x)=\dfrac{\tan\left(\frac{\pi}{4}-x\right)}{\cot 2x}, x\neq\dfrac{\pi}{4}$. Find the value which should be assigned to f at x = π/4, so that it is continuous everywhere

Sol. For f to be continuous at x = π/4, we must have

$$\lim_{x\to\frac{\pi}{4}} f(x)=f\left(\frac{\pi}{4}\right)$$

$$\text{But } \lim_{x\to\frac{\pi}{4}} f(x)=\lim_{x\to\frac{\pi}{4}}\frac{\tan\left(\frac{\pi}{4}-x\right)}{\cot 2x}$$

$$=\lim_{x\to\frac{\pi}{4}}\tan 2x\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{2\tan x}{1-\tan^2 x}\cdot\frac{1-\tan x}{1+\tan x}=\lim_{x\to\frac{\pi}{4}}\frac{2\tan x}{(1+\tan x)^2}$$

$$=\frac{2(1)}{(1+1)^2}=\frac{2}{4}=\frac{1}{2} \quad ; \quad f\left(\frac{\pi}{4}\right)=\frac{1}{2}$$

Example 19 :

$$\text{Let } f(x)=\begin{cases} ax(x-1)+b & , \quad x<1 \\ x-1 & , \quad 1\leq x\leq 3 \\ px^2+qx+2 & , \quad x>3 \end{cases}$$

Find the values of the constants a, b, p and q so that

(i) f (x) is continuous for all x

(ii) f (x) is not differentiable at x = 1

(iii) f ′ (x) is continuous at x = 3

Sol. It is given that f(x) is everywhere continuous.

So, it is continuous at x = 1 and x = 3

$\therefore \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=f(1)$

and $\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-} f(x)=f(3)$

$\Rightarrow \lim_{x\to 1} ax(x-1)+b=\lim_{x\to 1} x-1$

and, $\lim_{x\to 3} x-1=\lim_{x\to 3} px^2+qx+2$

$\Rightarrow$ b = 0 and 2 = 9p + 3q + 2

$\Rightarrow$ b = 0 and 9p + 3q = 0

$\Rightarrow$ b = 0 and 3p + q = 0 ....(i)

Now, f (x) is not differentiable at x = 1

$\Rightarrow$ (LHD at x = 1) ≠ (RHD at x = 1)

$$\Rightarrow \left[\frac{d}{dx}(ax(x-1)+b)\right]_{x=1}\neq\left[\frac{d}{dx}(x-1)\right]_{x=1}$$

$\Rightarrow [a(2x-1)]_{x=1}\neq 1 \Rightarrow a\neq 1$

It is given that f′ (x) is continuous at x = 3

$\therefore \lim_{x\to 3^-} 1=\lim_{x\to 3^+}(2px+q)$

$\Rightarrow$ 1 = 6p + q .....(ii)

Solving (i) and (ii), we get p = 1/3 and q = – 1

Hence, a ≠ 1, b = 0, p = 1/3 and q = – 1

QUESTION BANK

CHAPTER 4 : LIMITS, CONTINUITY AND DIFFERENTIABILITY

EXERCISE - 1 [LEVEL-1]

PART - 1 : LIMITS

Evaluate the following limits (Q.1-Q.10)

Q.1 $\lim_{x\to 3} x+3$

(A) 2 (B) 4
(C) 6 (D) 8

Q.2 $\lim_{x\to 0} \frac{\sin ax}{\sin bx}$, $a, b \neq 0$

(A) a/b (B) 1/b
(C) b/a (D) 1/a

Q.3 $\lim_{x\to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$

(A) π (B) $2/\pi$
(C) $1/\pi$ (D) 2π

Q.4 $\lim_{x\to 0} \frac{\cos x}{\pi - x}$

(A) π (B) $2/\pi$
(C) $1/\pi$ (D) 2π

Q.5 $\lim_{x\to 0} \frac{\cos 2x - 1}{\cos x - 1}$

(A) 2 (B) 4
(C) 6 (D) 8

Q.6 $\lim_{x\to 0} \frac{ax + x\cos x}{b\sin x}$

(A) $\frac{a+1}{b}$ (B) $\frac{b+1}{a}$
(C) $\frac{a-1}{b}$ (D) $\frac{b-1}{a}$

Q.7 $\lim_{x\to 0} x\sec x$

(A) 0 (B) 1
(C) 2 (D) 3

Q.8 $\lim_{x\to 0} \frac{\sin ax + bx}{ax + \sin bx}$, $a, b, a+b \neq 0$

(A) 0 (B) 1
(C) 2 (D) 3

Q.9 $\lim_{x\to 0} (\operatorname{cosec} x - \cot x)$

(A) 0 (B) 1
(C) 2 (D) 3

Q.10 $\lim_{x\to \pi/2} \frac{\tan 2x}{x - \frac{\pi}{2}}$

(A) 0 (B) 1
(C) 2 (D) 3

Q.11 Find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$, where

$$f(x) = \begin{cases} 2x+3, & x \le 0 \\ 3(x+1), & x > 0 \end{cases}$$

(A) 1, 2 (B) 3, 6
(C) 2, 3 (D) 2, 4

Q.12 Find $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} x^2 - 1, & x \le 1 \\ -x^2 - 1, & x > 1 \end{cases}$

(A) 0 (B) –1
(C) –2 (D) does not exist

Q.13 Evaluate $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(A) 0 (B) 1
(C) 2 (D) does not exist

Q.14 If $f(x) = \frac{x + |x|}{x}$, then $\lim_{x\to 0} f(x)$ equals-

(A) 2 (B) 0
(C) 1 (D) Does not exist

Q.15 $\lim_{x\to \infty} \frac{\sin x}{x}$ equals-

(A) 1 (B) 0
(C) ∞ (D) Does not exist

Q.16 $\text{Lim}_{x\to 0} \frac{1 + e^{-1/x}}{1 - e^{-1/x}}$ is equal to-

(A) 1 (B) –1
(C) 0 (D) Does not exist

Q.17 If $f(x) = \sqrt{\frac{x - \sin x}{x + \cos^2 x}}$, then $\lim_{x\to \infty} f(x)$ equals-

(A) 0 (B) ∞
(C) 1 (D) None of these

Q.18 If $G(x) = -\sqrt{25 - x^2}$, then $\lim_{x\to 1} \frac{G(x) - G(1)}{x - 1}$ equals-

(A) 1/24 (B) 1/5
(C) $-\sqrt{24}$ (D) None of these

Q.19 If $f(9) = 9$ and $f'(9) = 4$, then $\lim_{x\to 9} \frac{\sqrt{f(x)} - 3}{\sqrt{x} - 3}$ is equal to-

(A) 1 (B) 3
(C) 4 (D) 9

Q.20 $\lim_{x \to 0} \frac{\sin x^{o}}{x}$ is equal to -

(A) 1 (B) π
(C) x (D) $\pi/180$

Q.21 If $f(x) = \begin{cases} x-1, & x<0 \\ 1/4, & x=0 \\ x^2, & x>0 \end{cases}$ then $\lim_{x \to 0} f(x)$ equals-

(A) 0 (B) 1
(C) –1 (D) Does not exist

Q.22 $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x =$

(A) 1 (B) e
(C) e^a (D) None of these

Q.23 $\lim_{x\to 0} \frac{\sqrt{3+x}-\sqrt{3-x}}{x} =$

(A) –1 (B) 0
(C) $\sqrt{3}$ (D) $\frac{1}{\sqrt{3}}$

Q.24 If $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ -x, & \text{if } x \text{ is irrational} \end{cases}$, then $\lim_{x\to 0} f(x)$ is

(A) Equal to 0 (B) Equal to 1
(C) Equal to –1 (D) Indeterminate

Q.25 The value of $\lim_{x\to 2} \frac{3^{x/2}-3}{3^x-9}$ is

(A) 0 (B) 1/3
(C) 1/6 (D) ln 3

PART - 2 - CONTINUITY

Q.26 If function $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$,

then the number of points at which f (x) is continuous, is

(A) ∞ (B) 1
(C) 0 (D) 2

Q.27 If $f(x) = \text{sgn}(x)$ and $g(x) = x(1-x^2)$, then the number of points of discontinuity of function f (g (x)) is –

(A) exact two (B) exact three
(C) finite and more than 3 (D) infinitely many

Q.28 Examine the continuity of the function

$$f(x) = \begin{cases} \frac{x^2-9}{x-3}, & \text{when } x \neq 3 \\ 6, & \text{when } x = 3 \end{cases}; \text{ at } x=3$$

(A) f (x) is continuous at x = 3
(B) f (x) is continuous at x=2
(C) f (x) is continuous at x = 1
(D) f (x) is continuous at x = 4

Q.29 If $f(x) = |x-2|$, then

(A) $\lim_{x\to 2+} f(x) \neq 0$ (B) $\lim_{x\to 2-} f(x) \neq 0$

(C) $\lim_{x\to 2+} f(x) \neq \lim_{x\to 2-} f(x)$ (D) f (x) is continuous at x= 2

Q.30 If $f(x) = \begin{cases} (x^2/a)-a, & \text{when } x<a \\ 0, & \text{when } x=a, \\ a-(x^2/a), & \text{when } x>a \end{cases}$ then

(A) $\lim_{x\to a} f(x) = a$
(B) f (x) is continuous at x = a
(C) f (x) is discontinuous at x = a
(D) $\lim_{x\to a} f(x) = 1$

Q.31 If $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x=0 \end{cases}$, then

(A) $f(0^+) = 2$ (B) $f(0^-) = 1$
(C) f is continuous at x = 0 (D) $f(0^+) = -1$

Q.32 If $f(x) = \begin{cases} -x^2, & \text{when } x \le 0 \\ 5x-4, & \text{when } 0<x\le 1 \\ 4x^2-3x, & \text{when } 1<x<2 \\ 3x+4, & \text{when } x \ge 2 \end{cases}$, then

(A) f (x) is continuous at x = 0
(B) f (x) is continuous x = 2
(C) f (x) is continuous at x = 1
(D) Both (B) and (C)

Q.33 If $f(x) = \begin{cases} \frac{5}{2}-x, & \text{when } x<2 \\ 1, & \text{when } x=2 \\ x-\frac{3}{2}, & \text{when } x>2 \end{cases}$, then

(A) f (x) is continuous at x = 2
(B) f (x) is discontinuous at x = 2
(C) $\lim_{x\to 2} f(x) = 1$
(D) $\lim_{x\to 2} f(x) = 1/3$

Q.34 If $f(x) = \begin{cases} \frac{1-\cos 4x}{x^2}, & \text{when } x<0 \\ a, & \text{when } x=0, \\ \frac{\sqrt{x}}{\sqrt{(16+\sqrt{x})}-4}, & \text{when } x>0 \end{cases}$

is continuous at x = 0, then the value of ‘a’ will be

(A) 8 (B) – 8
(C) 4 (D) 2

Q.35 Let $f(x)=\begin{cases}\dfrac{x-4}{|x-4|}+a, & x<4\\ a+b, & x=4\\ \dfrac{x-4}{|x-4|}+b, & x>4\end{cases}$.

Then $f(x)$ is continuous at x = 4 when

(A) a = 0, b = 0 (B) a = 1, b = 1
(C) a = –1, b = 1 (D) a = 1, b = –1

Q.36 If the function $f(x)=\begin{cases}(\cos x)^{1/x}, & x\neq 0\\ k, & x=0\end{cases}$

is continuous at x = 0, then the value of k is

(A) 1 (B) –1
(C) 0 (D) e

Q.37 If $f(x)=\begin{cases}\dfrac{\sin[x]}{[x]+1}, & \text{for } x>0\\ \dfrac{\cos\frac{\pi}{2}[x]}{[x]}, & \text{for } x<0\\ k, & \text{at } x=0\end{cases}$; where [x] denotes the greatest integer less than or equal to x, then in order that f be continuous at x = 0, the value of k is

(A) Equal to 0 (B) Equal to 1
(C) Equal to –1 (D) Indeterminate

Q.38 If $f(x)=\dfrac{x^2-10x+25}{x^2-7x+10}$ for $x\neq 5$ and f is continuous at $x=5$, then $f(5)=$

(A) 0 (B) 5
(C) 10 (D) 25

Q.39 If $f(x)=\begin{cases}\dfrac{1-\sin x}{\pi-2x}, & x\neq\dfrac{\pi}{2}\\ \lambda, & x=\dfrac{\pi}{2}\end{cases}$, be continuous at $x=\pi/2$ then value of λ is

(A) –1 (B) 1
(C) 0 (D) 2

Q.40 The function $f(x)=\dfrac{1-\sin x+\cos x}{1+\sin x+\cos x}$ is not defined at $x=\pi$. The value of $f(\pi)$ so that f (x) is continuous at $x=\pi$, is

(A) –1/2 (B) 1/2
(C) – 1 (D) 1

Q.41 The function f (x) = [x], where [x] denotes the greatest integer not greater than x, is –

(A) continuous for all nonintegral values of x.
(B) continuous only at positive integral values of x.
(C) continuous for all real values of x.
(D) continuous only at rational values of x.

Q.42 If $f(x)=\begin{cases}\dfrac{\log x}{x-1} & \text{if } x\neq 1\\ k & \text{if } x=1\end{cases}$ is continuous at x = 1, then the value of k is –

(A) e (B) 1
(C) –1 (D) 0

Q.43 If $f(x)=\begin{cases}\dfrac{3\sin\pi x}{5x}, & x\neq 0\\ 2K, & x=0\end{cases}$ is continuous at x = 0, then the value of K is

(A) $\pi/10$ (B) $3\pi/10$
(C) $3\pi/2$ (D) $3\pi/5$

Q.44 $f(x)=\begin{cases}3x-8 & \text{if } x\le 5\\ 2k & \text{if } x>5\end{cases}$ is continuous, find k

(A) 2/7 (B) 3/7
(C) 4/7 (D) 7/2

PART - 3 -DIFFERENTIABILITY

Q.45 Which of the following is not true

(A) Every differentiable function is continuous.
(B) If derivative of a function is zero at all points, then the function is constant.
(C) If a function has maximum or minima at a point, then the function is differentiable at that point and its derivative is zero.
(D) If a function is constant, then its derivative is zero at all points.

Q.46 Which of the following statements is true

(A) A continuous function is an increasing function
(B) An increasing function is continuous
(C) A continuous function is differentiable
(D) A differentiable function is continuous

Q.47 The function $f(x)=|x|$ at x = 0 is

(A) Continuous but non-differentiable
(B) Discontinuous and differentiable
(C) Discontinuous and non-differentiable
(D) Continuous and differentiable

Q.48 The left-hand derivative of $f(x)=[x]\sin(\pi x)$ at x = k, k is an integer and [x] = greatest integer $\le$ x is

(A) $(-1)^k(k-1)\pi$ (B) $(-1)^{k-1}(k-1)\pi$
(C) $(-1)^k k\pi$ (D) $(-1)^{k-1}k\pi$

Q.49 Let $f(x+y)=f(x)f(y)$ and $f(x)=1+\sin(3x)g(x)$ where g (x) is continuous then f ' (x) is

(A) f(x) g (0) (B) 3 g (0)
(C) f (x) cos 3x (D) 3 f (x)g (0)

Q.50 If $f(x) = x^2 - 2x + 4$ and $\frac{f(5) - f(1)}{5 - 1} = f'(c)$ then value of c will be
(A) 0 (B) 1
(C) 2 (D) 3

Q.51 If $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2x - 1, & 1 < x \end{cases}$, then
(A) f is discontinuous at x = 1
(B) f is differentiable at x = 1
(C) f is continuous but not differentiable at x = 1
(D) None of these

Q.52 If $f(x) = \begin{cases} ax^2 + b; & x \le 0 \\ x^2; & x > 0 \end{cases}$ possesses derivative at x = 0, then
(A) a = 0, b = 0 (B) a > 0, b = 0
(C) $a \in R$, b = 0 (D) a = 1, b = 1

Q.53 If $f(x) = x(\sqrt{x} - \sqrt{x+1})$, then
(A) f (x) is continuous but non- differentiable at x = 0
(B) f (x) is differentiable at x = 0
(C) f (x) is not continuous at x = 0
(D) Both (B) and (C)

Q.54 The function $f(x) = |x - 2| + x$ is –
(A) differentiable at both x = 2 and x = 0.
(B) differentiable at x = 2 but not at x = 0.
(C) continuous at x = 2 but not at x = 0.
(D) continuous at both x = 2 and x = 0.

PART - 4 -MEAN VALUE THEOREM

Q.55 Consider the function $f(x) = e^{-2x}$ sin 2x over the interval $(0, \pi/2)$. A real number $c \in (0, \pi/2)$, as guaranteed by Rolle's theorem, such that f ' (c) = 0 is
(A) $\pi/8$ (B) $\pi/6$
(C) $\pi/4$ (D) $\pi/3$

Q.56 If the function $f(x) = x^3 - 6x^2 + ax + b$ satisfies Rolle's theorem in the interval [1, 3] and $f'\left(\frac{2\sqrt{3}+1}{\sqrt{3}}\right) = 0$, then
(A) a = – 11 (B) a = – 6
(C) a = 6 (D) a = 11

Q.57 In the mean value theorem, $f(b) - f(a) = (b - a)f'(c)$ if a = 4, b = 9 and $f(x) = \sqrt{x}$ then the value of c is
(A) 8.00 (B) 5.25
(C) 4.00 (D) 6.25

Q.58 If the function $f(x) = x^3 - 6ax^2 + 5x$ satisfies the conditions of Lagrange's mean value theorem for the interval [1, 2] and the tangent to the curve y = f (x) at x = 7/4 is parallel to the chord that joins the points of intersection of the curve with the ordinates x = 1 and x = 2. Then the value of a is
(A) 35/16 (B) 35/48
(C) 7/16 (D) 5/16

Q.59 Let f (x) satisfy the requirements of Lagrange's Mean Value Theorem in [0, 2]. If f(0) = 0 and $|f'(x)| \le \frac{1}{2}$ for all x in [0, 2], then -
(A) $f(x) \le 2$ (B) $|f(x)| \le 1$
(C) f (x) = 2x (D) None of these

Q.60 If for $f(x) = 2x - x^2$, Lagrange's theorem satisfies in [0, 1], then the value of $c \in [0, 1]$ is
(A) c = 0 (B) c = 1/2
(C) c = 1/4 (D) c = 1

Q.61 If $f(x) = x^3$ and $g(x) = x^3 - 4x$ in $-2 \le x \le 2$, then consider the statements:
(a) f (x) and g (x) satisfy mean value theorem.
(b) f (x) and g (x) both satisfy Rolle's theorem.
(c) Only g (x) satisfies Rolle's theorem.
Of these statements
(A) (a) and (b) are correct (B) (a) alone is correct
(C) None is correct (D) (a) and (c) are correct

PART - 5 -MISCELLANEOUS

Q.62 If $f(x) = \frac{e^{2x} - (1 + 4x)^{1/2}}{\ln(1 - x^2)}$ for $x \ne 0$, then f has –
(A) an irremovable discontinuity at x = 0
(B) a removable discontinuity at x = 0 and f (0) = – 4
(C) a removable discontinuity at x = 0 and f (0) = – 1/4
(D) a removable discontinuity at x = 0 and f (0) = 4

Q.63 Given $f(x) = \begin{cases} \sqrt{10 - x^2} & \text{if } -3 < x < 3 \\ 2 - e^{x-3} & \text{if } x \ge 3 \end{cases}$
The graph of f (x) is –
(A) continuous and differentiable at x = 3
(B) continuous but not differentiable at x = 3
(C) differentiable but not continuous at x = 3
(D) neither differentiable nor continuous at x = 3

Q.64 If $f(x) = \lim_{n \to \infty} \frac{\log(2 + x) - x^{2n} \sin x}{1 + x^{2n}}$, then –
(A) f is continuous at x = 1 (B) $\lim_{x \to 1^+} f(x) \ne f(1)$
(C) $\lim_{x \to 1^+} f(x)$ does not exist (D) $\lim_{x \to 1^-} f(x)$ does not exist

Q.65 Function $f(x) = \begin{cases} -x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 1 \\ x^3 - x + 1 & \text{if } x > 1 \end{cases}$ is differentiable at -
(A) x = 0 but not at x = 1 (B) x = 1 but not at x = 0
(C) x = 0 and x = 1 both (D) neither x = 0 nor x = 1

Q.66 If $f(x)=\dfrac{x-a}{\sqrt{x}-\sqrt{a}}$ is continuous at x = 1, then f(1) equals

(A) $\sqrt{a}$ (B) $2\sqrt{a}$
(C) a (D) 2a

Q.67 Let $f(x)=\lim\limits_{x\to\infty}\sin^{2n}x$, then number of point(s) where f(x) is discontinuous is –

(A) 0 (B) 1
(C) 2 (D) infinitely many

Q.68 If $f(x)=(x+1)^{\cot x}$ is continuous at x = 0, then f(0) =

(A) 0 (B) 1
(C) 1/e (D) e

Q.69 Let f(x) be defined as follows :

$$f(x)=\begin{cases}(\cos x-\sin x)^{\operatorname{cosec} x}, & -\dfrac{\pi}{2}<x<0\\ a, & x=0\\ \dfrac{e^{1/x}+e^{2/x}+e^{3/x}}{ae^{2/x}+be^{3/x}}, & 0<x<\dfrac{\pi}{2}\end{cases}$$

If f(x) is continuous at x = 0, then ordered pair (a, b) is equal to

(A) (e, 1/e) (B) (1/e, e)
(C) (e, e) (D) (e^{-1}, e^{-1})

Q.70 The values of a, b and c which make the function

$$f(x)=\begin{cases}\dfrac{\sin(a+1)x+\sin x}{x}, & x<0\\ c, & x=0\\ \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}}, & x>0\end{cases}$$ continuous at x = 0 are –

(A) $a=\dfrac{-3}{2}, c=\dfrac{1}{2}, b=0$ (B) $a=\dfrac{3}{2}, c=\dfrac{1}{2}, b\neq 0$
(C) $a=\dfrac{-3}{2}, c=\dfrac{1}{2}, b\neq 0$ (D) None of these

Q.71 If function $f(x)=\begin{cases}1+\sin \pi x/2, & -\infty<x\le 1\\ ax+b, & 1<x<3\\ 6\tan \pi x/12, & 3\le x<6\end{cases}$

is continuous in (– ∞ , 6), then values of a and b are

(A) 0, 2 (B) 1, 1
(C) 2, 0 (D) 2, 1

Q.72 If $f(x)=[x](\sin kx)^p$ is continuous for real x, then–

(A) $k\in\{n\pi, n\in I\}, p>0$
(B) $k\in\{2n\pi, n\in I\}, p>0$
(C) $k\in\{n\pi, n\in I\}, p\in R-\{0\}$
(D) $k\in\{n\pi, n\in I, n\neq 0\}, p\in R-\{0\}$

EXERCISE - 2 [LEVEL - 2]

Q.1 Choose the correct statements –

(A) If $f'(a^+)$ and $f'(a^-)$ exist finitely at a point then f is continuous at x = a.
(B) The function f(x) = 3 tan 5x – 7 is differentiable at all points in its domain.
(C) The existence of $\lim\limits_{x\to c}(f(x)+g(x))$ does not imply of existence of $\lim\limits_{x\to c}f(x)$ and $\lim\limits_{x\to c}g(x)$
(D) All of these

Q.2 If $f(x)=\begin{cases}\dfrac{e^{e/x}-e^{-e/x}}{e^{1/x}+e^{-1/x}}, & x\neq 0\\ k, & x=0\end{cases}$

(A) f is continuous at x, when k = 0
(B) f is not continuous at x = 0 for any real k.
(C) limit does not exist at x = 0
(D) Both (B) and (C)

Q.3 In the function $f(x)=\dfrac{2x-\sin^{-1}x}{2x+\tan^{-1}x}, (x\neq 0)$ is continuous at each point of its domain, then the value of f(0) is

(A) 2 (B) 1/3
(C) 2/3 (D) –1/3

Q.4 If $f(x)=\begin{cases}\dfrac{1-(x)}{1+x}, & x\neq -1\\ 1, & x=-1\end{cases}$, then the value of f(|2k|) will be (where [ ] shows the greatest integer function)

(A) Continuous at x = –1 (B) Continuous at x = 0
(C) Discontinuous at x = 1/2 (D) All of these

Q.5 Function $f(x)=\dfrac{1-\cos 4x}{8x^2}$, where x ≠ 0 and f(x) = k where x = 0 is a continous function at x = 0 then the value of k will be

(A) k = 0 (B) k = 1
(C) k = –1 (D) k = –2

Q.6 The function defined by

$$f(x)=\begin{cases}|x-3|; & x\ge 1\\ \dfrac{1}{4}x^2-\dfrac{3}{2}x+\dfrac{13}{4}; & x<1\end{cases}$$ is

(A) Continuous at x = 1 (B) Continuous at x = 3
(C) Differentiable at x = 1 (D) All the above

Q.7 If $f(x)=\begin{cases}e^x+ax, & x<0\\ b(x-1)^2, & x\ge 0\end{cases}$ is differentiable at x = 0, then (a, b) is

(A) (–3, –1) (B) (–3, 1)
(C) (3, 1) (D) (3, –1)

Q.8 The function $f(x)=\begin{cases} e^{2x}-1 & , x\le 0 \\ ax+\dfrac{bx^2}{2}-1 & , x>0\end{cases}$

is continuous and differentiable for

(A) $a=1, b=2$ (B) $a=2, b=4$
(C) $a=2$, any b (D) any a, $b=4$

Q.9 The function $f(x)=x^2\sin\frac{1}{x}, x\ne 0, f(0)=0$ at $x=0$

(A) Is continuous but not differentiable
(B) Is discontinuous
(C) Is having continuous derivative
(D) Is continuous and differentiable

Q.10 Let $f(x)=\begin{cases}\sin x, & \text{for } x\ge 0\\ 1-\cos x, & \text{for } x\le 0\end{cases}$ and $g(x)=e^x$. Then $(gof)'(0)$ is

(A) 1 (B) –1
(C) 0 (D) –2

Q.11 Function $y=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is not differentiable for

(A) $|x|<1$ (B) $x=1,-1$
(C) $|x|>1$ (D) $x=2,-2$

Q.12 If $f(x)=\begin{cases}3+|x-k| & , x\le k\\ a^2-2+\dfrac{\sin(x-k)}{(x-k)} & , x>k\end{cases}$

has minimum at $x=k$ then –

(A) $a\in R$ (B) $|a|<2$
(C) $|a|>2$ (D) $1<|a|<2$

Q.13 Over $x\in(0,10)$ the function $[\sin x]+[\sin 2x]$, (where [.] denotes greatest function) is discontinuous at –

(A) 11 points (B) 10 points
(C) 9 points (D) 8 points

Q.14 The left hand derivative of $f(x)=[x]\tan\pi x$, where [.] denotes greatest integer function, at $x=k$, where k is integer is –

(A) $(-1)^k\pi(k-1)$ (B) $(-1)^{k+1}\pi(k-1)$
(C) $(-1)^{2k}\pi(k-1)$ (D) $-\pi(k-1)$

Q.15 If $f(x)=\begin{cases}2x-[x]+x.\sin(x-[x]), & x\ne 0\\ 0 & , x=0\end{cases}$ where [.] is G.I.F. then –

(A) f(x) is differentiable at $x=0$
(B) f(x) is differentiable at $x=2$
(C) f(x) is continuous but not differentiable at $x=0$
(D) Discontinuous at $x=0$

Q.16 Let $f(x)=\frac{1-\tan x}{4x-\pi}, x\ne\frac{\pi}{4}, x\in\left[0,\frac{\pi}{2}\right]$. If f(x) is continuous in $\left[0,\frac{\pi}{2}\right]$, then $f\left(\frac{\pi}{4}\right)$ is –

(A) –1 (B) 1/2
(C) –1/2 (D) 1

Q.17 For the function $f(x)=\begin{cases}|x-3| & , x\ge 1\\ \left(\frac{x^2}{4}\right)-\left(\frac{3x}{2}\right)+\left(\frac{13}{4}\right) & , x<1\end{cases}$

choose the incorrect option –

(A) continuous at $x=1$ (B) differentiable at $x=1$
(C) continuous at $x=3$ (D) differentiable at $x=3$

Q.18 f(x) is a function such that $f''(x)=-f(x)$ and $f'(x)=g(x)$ and h(x) is a function such that $h(x)=[f(x)]^2+[g(x)]^2$ and $h(5)=11$, then find the value of h(10).

(A) 11 (B) 12
(C) 21 (D) 22

Q.19 The set of the points where $f(x)=x|x|$ is twice differentiable, will be –

(A) R (B) R_0
(C) R^+ (D) R^-

Q.20 If $f(x)=\frac{1}{1-x}$, then the points of discontinuity of the function f[f{f(x)}] are

(A) {0, –1} (B) {0,1}
(C) {1, –1} (D) {–1, 2}

Q.21 If $f(x)=\begin{cases}\dfrac{[x]-1}{x-1}, & x\ne 1\\ 0 & , x=1\end{cases}$ then f(x) is

(A) continuous as well as differentiable at $x=1$
(B) differentiable but not continuous at $x=1$
(C) continuous but not differentiable at $x=1$
(D) neither continuous nor differentiable at $x=1$

Q.22 If $f(x)=\sqrt{1-\sqrt{1-x^2}}$, then at $x=0$,

(A) f(x) is differentiable as well as continuous
(B) f(x) is differentiable but not continuous
(C) f(x) is continuous but not differentiable
(D) f(x) is neither continuous nor differentiable

Q.23 Let $f(x)=|x|+|x-1|$, then –

(A) f(x) is continuous at $x=0$
(B) f(x) is continuous at $x=1$
(C) f(x) is not continuous at $x=0$ and $x=1$
(D) Both (A) and (B)

Q.24 Consider two function $y=f(x)$ and $y=g(x)$ defined as

$$f(x)=\begin{cases}ax^2+b & , 0\le x\le 1\\ 2bx+2b & , 1<x\le 3\\ (a-1)x+2a-3 & , 3<x\le 4\end{cases}$$ and

$$g(x)=\begin{cases}cx^2+d & , 0\le x\le 2\\ dx+3-c & , 2<x<3\\ x^2+b+1 & , 3\le x\le 4\end{cases}$$

g(x) is continuous at $x=2$ if –

(A) $c=1, d=2$ (B) $c=2, d=3$
(C) $c=1, d=-1$ (D) $c=1, d=4$

Q.25 If $f(x) = \min\{1, x^2, x^3\}$, then –

(A) $f(x)$ is discontinuous $\forall\ x \in R$

(B) $f(x) > 0\ \forall\ x \in R$

(C) $f(x)$ is not differentiable but continuous $\forall\ x \in R$

(D) $f(x)$ is not differentiable for two values of x

Q.26 The value of $\lim\limits_{x\to\infty} [\cos\sqrt{x+1} - \cos\sqrt{x}]$ is –

(A) 2 (B) 1

(C) 0 (D) None of these

Q.27 Evaluate $\lim\limits_{x\to\infty} \dfrac{3x^2+5x+1}{7x^2+3x-1}$

(A) 1/2 (B) 1/7

(C) 3/2 (D) 3/7

Q.28 Find the value of $\lim\limits_{x\to 0} \dfrac{\sqrt{(1+x^2)} - \sqrt{1-x^2}}{x^2}$

(A) –2 (B) 1/2

(C) 0 (D) 1

Q.29 Find the value of $\lim\limits_{n\to\infty} \dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + ... + \dfrac{1}{n(n+1)}$

(A) –2 (B) 1/2

(C) 1 (D) 0

Q.30 Evaluate $\lim\limits_{x\to 0} \left(\dfrac{a^x+b^x}{2}\right)^{1/x}$

(A) $\sqrt{ab}$ (B) $2\sqrt{ab}$

(C) $\sqrt{2ab}$ (D) $\sqrt{a/b}$

Q.31 Evaluate $\lim\limits_{x\to\infty} \left(\dfrac{x+5}{x-1}\right)^x$

(A) e^2 (B) e^4

(C) e^6 (D) e^5

Q.32 If $f(x) = \dfrac{2}{\pi}\cot^{-1}\left(\dfrac{3x^2+1}{(x-1)(x-2)}\right)$, then the quadratic equation with root's $\lim\limits_{x\to 1^-} f(x)$ and $\lim\limits_{x\to 2^-} f(x)$ will be

(A) $x^2 - 2x + 1 = 0$ (B) $x^2 - x = 0$

(C) $x^2 - 3x + 2 = 0$ (D) $x^2 - 2x = 0$

Q.33 $\lim\limits_{x\to -1^+} \dfrac{\sqrt{\pi} - \sqrt{\cos^{-1} x}}{\sqrt{x+1}}$ is equal to –

(A) $\dfrac{1}{\sqrt{2}}$ (B) $\dfrac{1}{\sqrt{2\pi}}$ (C) $\dfrac{1}{\sqrt{\pi}}$ (D) $\dfrac{-1}{\sqrt{2\pi}}$

Directions : Assertion-Reason type questions.

(A) Statement- 1 is True, Statement-2 is True, Statement2 is a correct explanation for Statement -1

(B) Statement -1 is True, Statement -2 is True; Statement2 is NOT a correct explanation for Statement - 1

(C) Statement - 1 is True, Statement- 2 is False

(D) Statement -1 is False, Statement -2 is True

Q.34 Let $f(x) = 2 + \cos x$ for all real x.

Statement 1 : For each real t, there exists a point c in $[t, t+\pi]$ such that $f'(c) = 0$.

Statement 2 : $f(t) = f(t + 2\pi)$ for each real t.

Q.35 **Statement -1 :** $f(x) = \sin x + [x]$ is discontinuous at $x = 0$, where [.] denotes the greatest integer function.

Statement -2 : If g (x) is continuous and h (x) is discontinuous at x = a, then g (x) + h (x) will necessarily be discontinuous at x = a.

Passsage (Q.36-Q.37) :

Consider the function $f(x) = \begin{cases} (x-3)\dfrac{10^{\frac{1}{x-3}}+1}{10^{\frac{1}{x-3}}-1}, & x \neq 3 \\ 0, & x = 3 \end{cases}$

Q.36 The function f (x) is –

(A) odd (B) even

(C) neither even nor odd (D) None of these

Q.37 The function f (x) is –

(A) continous for all x

(B) continous for all except x = 3

(C) continous for all except x = 0

(D) None of these

Passage (Q.38-Q.40) :

For $0 < x < \dfrac{\pi}{2}$, let

$f_1(x) = \sum\limits_{r=1}^{n} \sec\left(x + \dfrac{r\pi}{6}\right)\sec\left(x + (r-1)\dfrac{\pi}{6}\right)$ and

$f_1(x) - 2f_2(x) = 2\tan\left(x + \dfrac{n\pi}{6}\right)$.

Also, $f_2(x) + f_3(x) = 0$ and

$$f_4(x) = \begin{cases} (\exp.)\dfrac{e^{(e^{x+f_2(x)+\tan x}-1)}}{2(e^x-1)} & ; x < 0 \\ k_1 & ; x = 0 \\ (1+|f_2(x)|)^{\frac{k_2}{f_3(x)}} & ; x > 0 \end{cases}$$

Q.38 The value of k_1 and k_2, if $f_4(x)$ is continuous at x = 0 is –

(A) $\dfrac{1}{\sqrt{e}}, 2$ (B) e, 1 (C) $\dfrac{e}{2}, 2$ (D) $\sqrt{e}, \dfrac{1}{2}$

Q.39 $y = f_3(x)$ is –
(A) discontinuous and non-derivable at $x = \pi/4$ and $\pi/3$
(B) neither continuous nor derivable at $x = 2\pi/5$
(C) continuous and derivable in $(0, \pi/2)$
(D) continuous but not derivable at $x = 2\pi/5$

Q.40 For $n = 3$, the solution of equation $f_1(x) + 4 = 0$, in $(0, \pi/2)$ is–
(A) $\pi/6$ (B) $\pi/4$
(C) $\pi/3$ (D) non existent

NOTE : The answer to each question is a NUMERICAL VALUE.

Q.41 Find the number of points of discontinuity of $y = [\sin x]$, $x \in [0, 2\pi)$ where [.] represents greatest integer function.

Q.42 If $\lim\limits_{n\to\infty} \dfrac{n.3^n}{n\,(x-2)^n + n.3^{n+1} - 3^n} = \dfrac{1}{3}$ where $n \in N$, then find the number of integer(s) in the range of x.

Q.43 $\lim\limits_{n\to\infty} \dfrac{1}{n}\left[\tan\dfrac{\pi}{4n} + \tan\dfrac{2\pi}{4n} + ... + \tan\dfrac{n\pi}{4n}\right] = \dfrac{X}{\pi}\log 2$ then find the value of X

Q.44 For positive integers k = 1, 2, 3, ...n, let S_k denotes the area of ΔAOB_k (where O is origin) such that $\angle AOB_k = \dfrac{k\pi}{2n}$, OA = 1 and $OB_K = k$. The value of the $\lim\limits_{n\to\infty} \dfrac{1}{n^2}\sum\limits_{k=1}^{n} S_k$ is $\dfrac{X}{\pi^2}$. Find the value of X.

Q.45 If $f(x) = \dfrac{\sin 3x + A\sin 2x + B\sin x}{x^5}\,(x \ne 0)$ is continuous at $x = 0$, then find $f(0)$.

Q.46 Let $\lim\limits_{n\to\infty}\left(\dfrac{1}{2n+1} + \dfrac{1}{2n+3} + \dfrac{1}{2n+5} + ... \dfrac{1}{4n-1}\right) = \dfrac{A}{B}\ln C$, where $A, B, C \in N$.

Q.47 If $\lim\limits_{x\to\infty} \dfrac{a\,(2x^3 - x^2) + b\,(x^3 + 5x^2 - 1) - c\,(3x^3 + x^2)}{a\,(5x^4 - x) - bx^4 + c\,(4x^4 + 1) + 2x^2 + 5x} = 1$, then the value of (a + b + c) can be expressed in the lowest form as $\dfrac{p}{q}$. The value of (p + q) is.

Q,48 Find the value of $\lim\limits_{x\to 0} \dfrac{1 - \cos^5 x\cos^3 2x\cos^3 3x}{x^2}$

Q.49 The integer n for which $\lim\limits_{x\to 0} \dfrac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite nonzero number is –

Q.50 For $x > 0$ value of $\lim\limits_{x\to 0} ((\sin x)^{1/x} + (1/x)^{\sin x})$ is –

Q.51 Let p(x) be a polynomial of degree 4 having extremum at $x = 1, 2$ and $\lim\limits_{x\to 0}\left(1 + \dfrac{p(x)}{x^2}\right) = 2$. Then the value of p(2) is

Q.52 The largest value of the non-negative integer a for which
$$\lim_{x\to 1}\left\{\frac{-ax + \sin(x-1) + a}{x + \sin(x-1) - 1}\right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$$ is –

Q.53 Let $f : R \to R$ and $g : R \to R$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define $h : R \to R$ by
$$h(x) = \begin{cases} \max\{f(x), g(x)\} & \text{if } x \le 0, \\ \max\{f(x), g(x)\} & \text{if } x > 0. \end{cases}$$
The number of points at which h(x) is not differentiable is

Q.54 Let m and n be two positive integers greater than 1. If $\lim\limits_{\alpha\to 0}\left(\dfrac{e^{\cos(\alpha^n)} - e}{\alpha^m}\right) = -\left(\dfrac{e}{2}\right)$, then the value of $\dfrac{m}{n}$ is –

Q.55 Let $\alpha, \beta \in R$ be such that $\lim\limits_{x\to 0}\dfrac{x^2\sin(\beta x)}{\alpha x - \sin x} = 1$, Then $6(\alpha + \beta)$ equals

Q.56 Find the value of $\underset{x\to\infty}{\text{Limit}}\ \dfrac{\cot^{-1}\left(x^{-a}\log_a x\right)}{\sec^{-1}\left(a^x\log_x a\right)}$ $(a > 1)$

Q.57 Find the value of $\underset{x\to\infty}{\text{Limit}}\ \dfrac{\cot^{-1}\left(\sqrt{x+1} - \sqrt{x}\right)}{\sec^{-1}\left\{\left(\dfrac{2x+1}{x-1}\right)^x\right\}}$

EXERCISE - 3 [PREVIOUS YEARS JEE MAIN QUESTIONS]

Q.1 If $f(1)=1, f'(1)=2$, then $\lim_{x\to 1}\frac{\sqrt{f(x)}-1}{\sqrt{x}-1}=$ **[AIEEE 2002]**

(A) 2 (B) 1
(C) 3 (D) 4

Q.2 The value of $\lim_{x\to 0}\frac{(1-\cos 2x)\sin 5x}{x^2\sin 3x}$ is- **[AIEEE 2002]**

(A) 10/3 (B) 3/10
(C) 6/5 (D) 5/6

Q.3 $\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+3}\right)^x=$ **[AIEEE 2002]**

(A) e^4 (B) e^2
(C) e^3 (D) e

Q.4 $\lim_{x\to\infty}\frac{\log x^n-[x]}{[x]}$, $n\in N$, (where [x] denotes greatest integer less than or equal to x) **[AIEEE-2002]**

(A) Has value – 1 (B) Has value 0
(C) Has value 1 (D) Does not exist

Q.5 If $f(x)=\begin{cases}x & x\in Q\\ -x & x\notin Q\end{cases}$, then f is continuous at- **[AIEEE-2002]**

(A) Only at zero (B) only at 0, 1
(C) all real numbers (D) all rational numbers

Q.6 If $f(x)=\begin{cases}x\,e^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x\neq 0\\ 0, & x=0\end{cases}$ then f(x) is **[AIEEE 2003]**

(A) discontinuous everywhere
(B) continuous as well as differentiable for all x
(C) continuous for all x but not differentiable at x = 0
(D) neither differentiable nor continuous at x = 0

Q.7 If $\lim_{x\to 0}\frac{\log(3+x)-\log(3-x)}{x}=k$, the value of k is -

(A) – 2/3 (B) 0 **[AIEEE 2003]**
(C) – 1/3 (D) 2/3

Q.8 Let f(a) = g(a) = k and their n^{th} derivatives $f^n(a)$, $g^n(a)$ exist and are not equal for some n. Further if

$\lim_{x\to a}\frac{f(a)\,g(x)-f(a)-g(a)\,f(x)+g(a)}{g(x)-f(x)}=4$ then the value of k is- **[AIEEE 2003]**

(A) 0 (B) 4
(C) 2 (C) 1

Q.9 $\lim_{x\to\frac{\pi}{2}}\frac{\left[1-\tan\left(\frac{x}{2}\right)\right][1-\sin x]}{\left[1+\tan\left(\frac{x}{2}\right)\right][\pi-2x]^3}$ is- **[AIEEE 2003]**

(A) ∞ (C) 1/8
(B) 0 (D) 1/32

Q.10 If $\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x}=e^2$, then the values of a and b, are- **[AIEEE 2004]**

(A) $a\in R, b\in R$ (B) $a=1, b\in R$
(C) $a\in R, b=2$ (D) a = 1 and b = 2

Q.11 Let $f(x)=\frac{1-\tan x}{4x-\pi}$, $x\neq\frac{\pi}{4}$, $x\in\left[0,\frac{\pi}{2}\right]$. If f(x) is continuous in $\left[0,\frac{\pi}{2}\right]$, then $f\left(\frac{\pi}{4}\right)$ is- **[AIEEE 2004]**

(A) 1 (B) 1/2
(C) – 1/2 (D) –1

Q.12 If f is a real-valued differentiable function satisfying $|f(x)-f(y)|\le(x-y)^2$, $x, y\in R$ and f(0) = 0, then f (1) equals **[AIEEE-2005]**

(A) –1 (B) 0
(C) 2 (D) 1

Q.13 Suppose f(x) is differentiable at x = 1 and $\lim_{h\to 0}\frac{1}{h}f(1+h)=5$, then f'(1) equals - **[AIEEE-2005]**

(A) 3 (B) 4
(C) 5 (D) 6

Q.14 Let α and β be the distinct roots of $ax^2+bx+c=0$, then $\lim_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$ is equal to - **[AIEEE-2005]**

(A) $\frac{a^2}{2}(\alpha-\beta)^2$ (B) 0

(C) $\frac{-a^2}{2}(\alpha-\beta)^2$ (D) $\frac{1}{2}(\alpha-\beta)^2$

Q.15 The set of points where $f(x)=\frac{x}{1+|x|}$ is differentiable is

(A) $(-\infty,-1)\cup(-1,\infty)$ (B) $(-\infty,\infty)$ **[AIEEE 2006]**
(C) $(0,\infty)$ (D) $(-\infty,0)\cup(0,\infty)$

Q.16 The function $f: R\backslash\{0\}\to R$ given by $f(x)=\frac{1}{x}-\frac{2}{e^{2x}-1}$ can be made continuous at x = 0 by defining f (0) as

(A) 2 (B) –1 **[AIEEE 2007]**
(C) 0 (D) 1

Q.17 Let $f: R\to R$ be a function defined by $f(x)=\text{Min}\{x+1, |x|+1\}$. Then which of the following is true ? **[AIEEE 2007]**

(A) $f(x)\ge 1$ for all $x\in R$
(B) f (x) is not differentiable at x = 1
(C) f (x) is differentiable everywhere
(D) f (x) is not differentiable at x = 0

Q.18 Let $f(x)=\begin{cases}(x-1)\sin\frac{1}{x-1} & \text{if } x\neq 1\\ 0 & \text{if } x=1\end{cases}$ Then which one of the following is true ? **[AIEEE 2008]**

(A) f is differentiable at x = 0 and at x = 1
(B) f is differentiable at x = 0 but not at x = 1
(C) f is differentiable at x = 1 but not at x = 0
(D) f is neither differentiable at x = 0 nor at x = 1

Q.19 Let $f(x)=x|x|$ and $g(x)=\sin x$. **[AIEEE 2009]**
Statement – 1 : gof is differentiable at x = 0 and its derivative is continuous at that point.
Statement – 2 : gof is twice differentiable at x = 0.
(A) Statement -1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement -1
(B) Statement -1 is true, Statement-2 is true; Statement -2 is **not** a correct explanation for Statement -1.
(C) Statement -1 is true, Statement -2 is false.
(D) Statement -1 is false, Statement -2 is true.

Q.20 Let f: R → R be a continuous function defined by
$f(x)=\frac{1}{e^x+2e^{-x}}$ **[AIEEE 2010]**
Statement-1: $f(c)=1/3$, for some $c\in R$.
Statement-2: $0<f(x)\le\frac{1}{2\sqrt{2}}$, for all $x\in R$
(A) Statement-1 is true, Statement-2 is true; Statement-2 is not the correct explanation for Statement-1.
(B) Statement-1 is true, Statement-2 is false.
(C) Statement-1 is false, Statement-2 is true.
(D) Statement-1 is true, Statement-2 is true; Statement-2 is the correct explanation for Statement-1

Q.21 Let f : R → R be a positive increasing function with $\lim_{x\to\infty}\frac{f(3x)}{f(x)}=1$. Then $\lim_{x\to\infty}\frac{f(2x)}{f(x)}=$ **[AIEEE 2010]**
(A) 2/3 (B) 3/2
(C) 3 (D) 1

Q.22 The value of p and q for which the function

$$f(x)=\begin{cases}\frac{\sin(p+1)x+\sin x}{x}, & x<0\\ q, & x=0\\ \frac{\sqrt{x+x^2}-\sqrt{x}}{x^{3/2}}, & x>0\end{cases}$$

is continuous for all x in R, are – **[AIEEE 2011]**

(A) $p=\frac{1}{2},q=-\frac{3}{2}$ (B) $p=\frac{5}{2},q=\frac{1}{2}$
(C) $p=-\frac{3}{2},q=\frac{1}{2}$ (D) $p=\frac{1}{2},q=\frac{3}{2}$

Q.23 $\lim_{x\to 2}\left(\frac{\sqrt{1-\cos\{2(x-2)\}}}{x-2}\right)$ **[AIEEE 2011]**
(A) does not exist (B) equals $\sqrt{2}$
(C) equals $-\sqrt{2}$ (D) equals $\frac{1}{\sqrt{2}}$

Q.24 If f : R → R is a function defined by $f(x)=[x]\cos\left(\frac{2x-1}{2}\right)\pi$, where [x] denotes the greatest integer function, then f is :
(A) continuous for every real x. **[AIEEE 2012]**
(B) discontinuous only at x = 0.
(C) discontinuous only at non-zero integral values of x.
(D) continuous only at x = 0.

Q.25 Consider the function, $f(x)=|x-2|+|x-5|, x\in R$.
Statement-1 : $f'(4)=0$ **[AIEEE 2012]**
Statement-2 : f is continuous in [2, 5], differentiable in (2, 5) and f (2) = f (5).
(A) Statement-1 is false, Statement-2 is true.
(B) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1.
(C) Statement-1 is true, statement-2 is true; statement-2 is not a correct explanation for Statement-1.
(D) Statement-1 is true, statement-2 is false.

Q.26 $\lim_{x\to 0}\frac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to
(A) –1/4 (B) 1/2 **[JEE MAIN 2013]**
(C) 1 (D) 2

Q.27 $\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}$ is equal to – **[JEE MAIN 2014]**
(A) π/2 (B) 1
(C) –π (D) π

Q.28 If the function, $g(x)=\begin{cases}k\sqrt{x+1}, & 0\le x\le 3\\ mx+2, & 3<x\le 5\end{cases}$
is differentiable, the value of k + m is **[JEE MAIN 2015]**
(A) 16/5 (B) 10/3
(C) 4 (D) 2

Q.29 $\lim_{x\to 0}\frac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}=$ **[JEE MAIN 2015]**
(A) 3 (B) 2
(C) 1/2 (D) 4

Q.30 For $x\in R$, $f(x)=|\log 2-\sin x|$ and $g(x)=f(f(x))$, then :
(A) $g'(0)=\cos(\log 2)$ **[JEE MAIN 2016]**
(B) $g'(0)=-\cos(\log 2)$
(C) g is differentiable at x = 0 and $g'(0)=-\sin(\log 2)$
(D) g is not differentiable at x = 0

Q.31 Let $p=\lim_{x\to 0^+}(1+\tan^2\sqrt{x})^{1/2x}$ then log p is equal to –
(A) 1 (B) 1/2 **[JEE MAIN 2016]**
(C) 1/4 (D) 2

Q.32 For $x\in R$, $f(x)=|\log 2-\sin x|$ and $g(x)=f(f(x))$, then :
(A) $g'(0)=\cos(\log 2)$ **[JEE MAIN 2017]**
(B) $g'(0)=-\cos(\log 2)$
(C) g is differentiable at x = 0 and $g'(0)=-\sin(\log 2)$
(D) g is not differentiable at x = 0

Q.33 $\lim_{x\to\pi/2} \frac{\cot x - \cos x}{(\pi - 2x)^3}$ equals – **[JEE MAIN 2017]**

(A) 1/8 (B) 1/4
(C) 1/24 (D) 1/16

Q.34 For each $t \in R$, let [t] be the greatest integer less than or equal to t. Then $\lim_{x\to 0^+} x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+.....+\left[\frac{15}{x}\right]\right)$ **[JEE MAIN 2018]**

(A) is equal to 120 (B) does not exist (in R)
(C) is equal to 0 (D) is equal to 15

Q.35 Let $S = \{t \in R : f(x) = |x - \pi| \cdot (e^{|x|} - 1)\sin|x|$ is not differentiable at t}. Then the set S is equal to **[JEE MAIN 2018]**

(A) $\{\pi\}$ (B) $\{0, \pi\}$
(C) ϕ (an empty set) (D) $\{0\}$

Q.36 $\lim_{y\to 0} \frac{\sqrt{1+\sqrt{1+y^4}} - \sqrt{2}}{y^4}$ **[JEE MAIN 2019 (Jan)]**

(A) exists and equals $1/4\sqrt{2}$ (B) does not exist
(C)exists & equals $\frac{1}{2\sqrt{2}}$ (D) exists & equals $\frac{1}{2\sqrt{2}\,(\sqrt{2}+1)}$

Q.37 Let $f : R \to R$ be a function defined as :

$$f(x) = \begin{cases} 5, & \text{if } x \le 1 \\ a + bx, & \text{if } 1 < x < 3 \\ b + 5x, & \text{if } 3 \le x < 5 \\ 30, & \text{if } x \ge 5 \end{cases}$$ **[JEE MAIN 2019 (Jan)]**

Then, f is :
(A) continuous if a = 5 and b = 5
(B) continuous if a = –5 and b = 10
(C) continuous if a = 0 and b = 5
(D) not continuous for any values of a and b

Q.38 $\lim_{x\to 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1+\cos x}}$ equals : **[JEE MAIN 2019 (April)]**

(A) $2\sqrt{2}$ (B) $4\sqrt{2}$
(C) $\sqrt{2}$ (D) 4

Q.39 Let $f : R \to R$ be a differentiable function satisfying $f'(3) + f'(2) = 0$. Then

$\lim_{x\to 0}\left(\frac{1+f(3+x)-f(3)}{1+f(2-x)-f(2)}\right)^{1/x}$ is equal to **[JEE MAIN 2019 (April)]**

(A) e^2 (B) e
(C) e^{-1} (D) 1

Q.40 Let $f : [-1,3] \to R$ be defined as

$$f(x) = \begin{cases} |x| + [x], & -1 \le x < 1 \\ x + |x|, & 1 \le x < 2 \\ x + [x], & 2 \le x \le 3 \end{cases}$$

where [t] denotes the greatest integer less than or equal to t. Then, f is discontinuous at: **[JEE MAIN 2019 (April)]**

(A) four or more points (B) only one point
(C) only two points (D) only three points

Q.41 Let $f(x) = 15 - |x - 10|$; $x \in R$. Then the set of all values of x, at which the function, $g(x) = f(f(x))$ is not differentiable, is : **[JEE MAIN 2019 (April)]**

(A) {5, 10, 15, 20} (B) {10, 15}
(C) {5, 10, 15} (D) {10}

Q.42 If the function f defined on $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ by

$$f(x) = \begin{cases} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}, & x \ne \frac{\pi}{4} \\ k, & x = \frac{\pi}{4} \end{cases}$$ is continuous, then k is equal to **[JEE MAIN 2019 (April)]**

(A) 1/2 (B) 1
(C) $1/\sqrt{2}$ (D) 2

Q.43 Evaluate $\lim_{x\to 2} \frac{3^x + 3^{x-1} - 12}{3^{-x/2} - 3^{1-x}}$ **[JEE MAIN 2020 (Jan)]**

Q.44 If f (x) is defined in $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

$$f(x) = \begin{cases} \left(\frac{1}{x}\right)\log_e\left(\frac{1+3x}{1-2x}\right), & x \ne 0 \\ k, & x = 0 \end{cases}$$

Find k such that f (x) is continuous. **[JEE MAIN 2020 (Jan)]**

Q.45 $\lim_{x\to 0}\left(\frac{3x^2+2}{7x^2+2}\right)^{1/x^2}$ is equal to – **[JEE MAIN 2020 (JAN)]**

(A) e^{-2} (B) e^2
(C) $e^{2/7}$ (D) $e^{3/7}$

Q.46 $\lim_{x\to 0} \frac{\int_0^x t\sin(10t)\,dt}{x}$ is equal to – **[JEE MAIN 2020 (JAN)]**

(A) 1 (B) 10
(C) 5 (D) 0

Q.47 Let f be any function continuous on [a, b] and twice differentiable on (a, b). If for all $x \in (a, b)$, $f'(x) > 0$ and $f''(x) < 0$, then for any $c \in (a, b)$, $\frac{f(c) - f(a)}{f(b) - f(c)}$ is greater than : **[JEE MAIN 2020 (JAN)]**

(A) $\frac{b+a}{b-a}$ (B) $\frac{b-a}{c-a}$
(C) $\frac{c-a}{b-c}$ (D) 1

Q.48 If $f(x) = \begin{cases} \dfrac{\sin(a+2)x + \sin x}{x} & ; x < 0 \\ b & ; x = 0 \\ \dfrac{(x+3x^2)^{1/3} - x^{-1/3}}{x^{4/3}} & ; x > 0 \end{cases}$

is continuous at x = 0, then a + 2b is equal to :

[JEE MAIN 2020 (JAN)]

(A) –1
(B) 1
(C) –2
(D) 0

Q.49 Let [t] denote the greatest integer $\le$ t and $\lim\limits_{x \to 0} x\left[\dfrac{4}{x}\right] = A$.

Then the function, $f(x) = [x^2] \sin(\pi x)$ is discontinuous, when x is equal to :

[JEE MAIN 2020 (JAN)]

(A) $\sqrt{A+5}$
(B) $\sqrt{A+1}$
(C) $\sqrt{A}$
(D) $\sqrt{A+21}$

ANSWER KEY

EXERCISE - 1

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	C	A	C	C	B	A	A	B	A	C	B	D	D	D	B	D	C	D	C	D	D	C	D	A	C
Q	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
A	B	B	A	D	B	C	D	B	A	D	A	A	A	C	C	A	B	B	D	C	D	A	A	C	D
Q	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72			
A	C	C	D	D	A	D	D	B	B	B	D	B	B	B	B	B	D	D	B	C	C	A			

EXERCISE - 2

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	D	D	B	D	B	D	B	C	D	C	B	C	C	C	D	C	D	A	B	B	D	C	D	A	C
Q	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
A	C	D	D	C	A	C	D	B	D	A	C	A	D	C	B	2	5	2	2	1	5	167	22	3	1
Q	51	52	53	54	55	56	57																		
A	0	2	3	2	7	1	1																		

EXERCISE - 3

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	A	A	A	A	A	C	D	B	D	B	C	B	C	A	B	D	C	B	C	D	D	C	A	A	C
Q	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	
A	D	D	D	B	A	B	A	D	A	C	A	D	B	D	D	C	A	72	5	A	D	C	D	B	

CHAPTER- 4: LIMIT, CONTINUITY AND DIFFERENTIABILITY OF FUNCTION

SOLUTIONS TO TRY IT YOURSELF

TRY IT YOURSELF-1

(1) (i) $f(x)=\frac{|x|}{x}$ has no limit at $x=0$: $f(0^+)=1$, $f(0^-)=-1$

(ii) $\lim\limits_{x\to 0}\frac{1}{\ln|x|}$ exists at $x=0$ $f(0^-)=f(0^+)=0$

even if f (0) is not defined.

(iii) $\lim\limits_{x\to 0}[x]+\sqrt{\{x\}}$ exists at $x=0$ as

$\lim\limits_{x\to 0^+} f(x)=0+0=0$, $\lim\limits_{x\to 0^-} f(x)=-1+\sqrt{1}=0$

(iv) $\lim\limits_{x\to 0}\sin^{-1}[\sec x]$, where [ ] denotes greatest integer function, exists and is equal to $\pi/2$.

(2) L.H.L. of f(x) at x = 4

$$\lim_{x\to 4^-} f(x)=\lim_{h\to 0} f(4-h)=\lim_{h\to 0}\frac{|4-h-4|}{4-h-4}$$

$$=\lim_{h\to 0}\frac{|-h|}{-h}=\lim_{h\to 0}\frac{h}{-h}=\lim_{h\to 0}-1=-1$$

R.H.L. of f(x) at x = 4

$$\lim_{x\to 4^+} f(x)=\lim_{h\to 0} f(4+h)=\lim_{h\to 0}\frac{|4+h-4|}{4+h-4}$$

$$=\lim_{h\to 0}\frac{|h|}{h}=\lim_{h\to 0}\frac{h}{h}=\lim_{h\to 0}1=1$$

(3) $$\lim_{x\to 1}\frac{x^2+x\log_e x-\log_e x-1}{(x^2-1)}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 1}\frac{(x-1)(\log_e x+x+1)}{(x+1)(x-1)}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 1}\frac{\log_e x+x+1}{(x+1)}=\frac{\log_e 1+1+1}{1+1}=\frac{0+2}{2}=1$$

(4) When x = 0, the expression $\frac{\sqrt{2+x}-\sqrt{2}}{x}$ takes the form $\frac{0}{0}$, Rationalizing the numerator, we have

$$\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}$$

$$=\lim_{x\to 0}\frac{(\sqrt{2+x}-\sqrt{2})(\sqrt{2+x}+\sqrt{2})}{x(\sqrt{2+x}+\sqrt{2})}$$

$$=\lim_{x\to 0}\frac{2+x-2}{(\sqrt{2+x}+\sqrt{2})x}=\lim_{x\to 0}\frac{1}{\sqrt{2+x}+\sqrt{2}}=\frac{1}{2\sqrt{2}}$$

(5) $$\lim_{x\to 2}\frac{x^{10}-1024}{x^5-32}=\lim_{x\to 2}\frac{x^{10}-2^{10}}{x^5-2^5}=\lim_{x\to 2}\frac{\frac{x^{10}-2^{10}}{x-2}}{\frac{x^5-2^5}{x-2}}$$

$$=\frac{\lim\limits_{x\to 2}\frac{x^{10}-2^{10}}{x-2}}{\lim\limits_{x\to 2}\frac{x^5-2^5}{x-2}}=\frac{10\times 2^{10-1}}{5\times 2^{5-1}}=64$$

(6) $$\lim_{x\to\infty}\frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3}=\lim_{x\to\infty}\frac{|x|\sqrt{3-\frac{1}{x^2}}-|x|\sqrt{2-\frac{1}{x^2}}}{4x+3}$$

$$=\lim_{x\to\infty}\frac{\sqrt{3-\frac{1}{x^2}}-\sqrt{2-\frac{1}{x^2}}}{4+\frac{3}{x}}=\frac{\sqrt{3}-\sqrt{2}}{4}$$

(7) We have, $\lim\limits_{x\to\infty}(\sqrt{25x^2-3x}+5x)$ $(\infty-\infty$ form)

$$=\lim_{y\to\infty}(\sqrt{25y^2+3y}-5y)\text{, where } y=-x$$

$$=\lim_{y\to\infty}\frac{25y^2+3y-25y^2}{\sqrt{25y^2+3y}+5y}=\lim_{y\to\infty}\frac{3y}{\sqrt{25y^2+3y}+5y}$$

$$=\lim_{y\to\infty}\frac{3}{\sqrt{25+\frac{3}{y}}+5}=\frac{3}{5+5}=\frac{3}{10}$$

(8) **(A).** Let y = cos x

$$\lim_{x\to 0}\frac{(\cos x)^{1/3}-(\cos x)^{1/2}}{1-\cos^2 x}=\lim_{y\to 1}\frac{y^{1/3}-y^{1/2}}{1-y^2}$$

$$=\lim_{y\to 1}\frac{\left(\frac{y^{1/3}-1}{y-1}\right)-\left(\frac{y^{1/2}-1}{y-1}\right)}{\left(\frac{1-y^2}{1-y}\right)}=\frac{\frac{1}{3}(1)^{2/3}-\frac{1}{2}(1)^{-1/2}}{-2}$$

$$=\frac{\frac{1}{3}-\frac{1}{2}}{-2}=\frac{1}{12}$$

(9) **(B).** $\lim_{x\to 0}\frac{\ln(\sin 2x)}{\ln(\sin x)}=\lim_{x\to 0}\frac{\ln 2+\ln\sin x+\ln\cos x}{\ln\sin x}$

$$=\lim_{x\to 0}\left(1+\frac{\ln 2}{\ln\sin x}+\frac{\ln\cos x}{\ln\sin x}\right)=1+0+0=1$$

(10) $\lim_{x\to 0}\left(\begin{matrix}1^{\operatorname{cosec}^2 x}+2^{\operatorname{cosec}^2 x}+3^{\operatorname{cosec}^2 x}\\ +.......+100^{\operatorname{cosec}^2 x}\end{matrix}\right)^{\sin^2 x}$ $[\infty^0]$

$$100\lim_{x\to 0}\left(\begin{matrix}\left(\frac{1}{100}\right)^{\operatorname{cosec}^2 x}+\left(\frac{2}{100}\right)^{\operatorname{cosec}^2 x}\\ +...+\left(\frac{99}{100}\right)^{\operatorname{cosec}^2 x}+1\end{matrix}\right)^{\sin^2 x}$$

$$=100\,(0+0+........0+1)^0=100$$

(11) We know that $\sin^{-1}\left(\frac{2x}{1+x^2}\right)=2\tan^{-1}x$, for $-1\le x\le 1$

$$\Rightarrow\ \lim_{x\to 0}\frac{1}{x}\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\lim_{x\to 0}\frac{2\tan^{-1}x}{x}=2$$

(12) $\lim_{x\to\infty}2^{x-1}\tan\left(\frac{a}{2^x}\right)=\lim_{x\to\infty}\frac{a}{2}\frac{\tan\left(\frac{a}{2^x}\right)}{\left(\frac{a}{2^x}\right)}$ $\left(\frac{0}{0}\text{ form}\right)$

$$=\frac{a}{2}\lim_{y\to 0}\frac{\tan y}{y},\text{ where } y=\frac{a}{2^x}=\frac{a}{2}$$

(13) We have, $\lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}$ $\left(\frac{0}{0}\text{ form}\right)$

$$=\lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}\frac{(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)}$$

$$=\lim_{x\to 0}\frac{2^x-1}{x}\lim_{x\to 0}(\sqrt{1+x}+1)$$

$$=(\log 2)\,(\sqrt{1+0}+1)=2\log 2$$

(14) Let $x-a=h$, then if $x\to a$, $h\to 0$

$$\Rightarrow\ \lim_{x\to a}\frac{\log x-\log a}{x-a}\ \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{h\to 0}\frac{\log(a+h)-\log a}{h}=\lim_{h\to 0}\frac{\log\left(1+\frac{h}{a}\right)}{\frac{h}{a}a}=\frac{1}{a}$$

(15) $L=\lim_{x\to 0}\frac{\log(\tan^2 2x)}{\log(\tan^2 x)}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

Using L'hospital rule

We have $L=\lim_{x\to 0}\dfrac{\left(\frac{1}{\tan^2 2x}2\tan 2x\sec^2 2x\right)\times 2}{\frac{1}{\tan^2 2x}2\tan x\sec^2 x}$

$$=\lim_{x\to 0}\frac{2\left(\frac{1}{\sin 2x\cos 2x}\right)}{\frac{1}{\sin x\cos x}}=\lim_{x\to 0}\frac{\left(\frac{1}{\sin 2x\cos 2x}\right)}{\left(\frac{1}{\sin 2x}\right)}$$

$$=\lim_{x\to 0}\frac{1}{\cos 2x}=1$$

(16) $\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3}$

$$=\lim_{x\to 0}\frac{(1-x^2)-\sqrt{1-x^2}}{3x^2\sqrt{1-x^2}\,(1+x^2)}\quad\text{(Using L'hospital's rule)}$$

$$=\lim_{x\to 0}\frac{(1+x^2)-\sqrt{1-x^2}}{3x^2\sqrt{1-x^2}\,(1+x^2)}$$

$$=\lim_{x\to 0}\frac{(1+x^2)^2-(1-x^2)}{3x^2\sqrt{1-x^2}\,(1+x^2)}\times\frac{1}{(1+x^2)+\sqrt{1-x^2}}$$

(Rationalizing)

$$=\lim_{x\to 0}\frac{x^4+3x^2}{3x^2\sqrt{1-x^2}\,(1+x^2)}\times\frac{1}{(1+x^2)+\sqrt{1-x^2}}$$

$$=\lim_{x\to 0}\frac{x^2+3}{3\sqrt{1-x^2}\,(1+x^2)}\times\frac{1}{(1+x^2)+\sqrt{1-x^2}}=\frac{1}{2}$$

(17) $\lim_{x\to\tan^{-1}3}\frac{[\tan^2 x]-2[\tan x]-3}{[\tan^2 x]-4[\tan x]+3}$

$$=\lim_{x\to\tan^{-1}3}\frac{8-4-3}{8-8+3}=\frac{1}{3}$$

(18) $\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}=\lim_{x\to 0}\frac{e^{\frac{1}{x}\ln(1+x)}-e}{x}$

$$=\lim_{x\to 0}\frac{e\left(e^{\frac{\ln(1+x)-x}{x}}-1\right)}{\left(\frac{\ln(1+x)-x}{x}\right)}\cdot\left(\frac{\ln(1+x)-x}{x^2}\right)$$

$$= e \lim_{x\to 0} \frac{e^{\frac{\ln(1+x)}{x}-1}}{\left(\frac{\ln(1+x)}{x}-1\right)} \cdot \lim_{x\to 0}\left(\frac{\ln(1+x)-x}{x^2}\right)$$

$$= e\,(1)\cdot\left(\frac{-1}{2}\right) = \frac{-e}{2}$$

TRY IT YOURSELF-2

(1) $f(x) = \frac{1}{1-e^{\frac{x+1}{x-2}}}$

f (x) is discontinuous when $x-2=0$ also

when $1-e^{\frac{x+1}{x-2}} = 0 \Rightarrow x = 2$ and $e^{\frac{x+1}{x-2}} = 1$

$x = 2$ and $\frac{x-1}{x-2} = 0 \Rightarrow x = 2$ and $x = 1$

(2) $\lim_{x\to 0} f(x) = \lim_{x\to 0} (\cos x)^{\cot^2 x}$

$$e^{\lim_{x\to 0}(\cos x - 1)^{\cot^2 x}} = e^{\lim_{x\to 0} \frac{-(1-\cos x)}{x^2}\cdot\frac{x^2}{\tan^2 x}} = e^{-1/2} = f(0)$$

$\Rightarrow$ f (x) is continuous at $x = 0$.

(3) $f(0^-) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1-\cos 4x}{x^2} = 8$

$$f(0^+) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x}\cdot(\sqrt{16+\sqrt{x}}+4)}{16+\sqrt{x}-16}$$

$$= \lim_{x\to 0^+} (\sqrt{16+\sqrt{x}}+4) = 8$$

$f(0^-) = f(0^+) = 8 = f(0) \Rightarrow a = 8$

(4) **(B).** $k = \lim_{x\to 0} \frac{(e^{2x}+1)-(x+1)(e^x+e^{-x})}{x(e^x-1)}$

$$= \lim_{x\to 0} \frac{(e^{2x}+1)-(x+1)(e^x+e^{-x})}{x^2}$$

$$= \lim_{x\to 0} \frac{2e^{2x}-(x+1)(e^x-e^{-x})-(e^x+e^{-x})}{2x}$$

By L'Hospital rule

$$= \lim_{x\to 0} \frac{4e^{2x}-(x+1)(e^x+e^{-x})-(e^x-e^{-x})-(e^x-e^{-x})}{2}$$

$$= \frac{4-2-0-0}{2} = 1$$

(5) **(B).** $x^2 - k \neq 0 \; \forall\, x \in R$

$\Rightarrow k < 0$ .... (1)

$x^2 + x + 1 \geq 0 \; \forall\, x \in R$

$\Rightarrow k^2 - 4 \leq 0 \Rightarrow -2 \leq k \leq 2$ .... (2)

From eq. (1) and eq. (2), $k \in [-2, 0)$

(6) $\sin(x-2)$ and $\cos(x-2)$ are continuous for all x.

Since [x] is not continuous at integral point.

So, f (x) is continuous in [4, 6] if

$$\left[\frac{(x-2)^3}{a}\right] = 0 \;\; \forall\, x \in [4, 6]$$

Now, $(x-2)^3 \in [8, 64]$ for $x \in [4, 6]$.

$\Rightarrow a > 64$ for $\left[\frac{(x-2)^3}{a}\right] = 0$

(7) $f(x) = \text{sgn}(2\sin x + a)$ is continuous for all x.

then $2\sin x + a \neq 0$ for any real x.

$\Rightarrow \sin x \neq -\frac{a}{2} \Rightarrow \left|\frac{a}{2}\right| > 1 \Rightarrow a < -2$ or $a > 2$

(8) We have, $f(x) = \begin{cases} kx+1 & \text{if } x \leq \pi \\ \cos x & \text{if } x > \pi \end{cases}$

L.H.L. $= \lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^-} (kx+1)$

Put $x = \pi - h$ as $x \to \pi^-$, $h \to 0$

$\therefore \lim_{h\to 0} k(\pi - h) + 1 = \lim_{h\to 0} k\pi - kh + 1 = k\pi + 1$

R.H.L. $= \lim_{x\to\pi^+} f(x) = \lim_{x\to\pi^+} \cos x$

Put $x = \pi + h$ as $x \to \pi^+$, $h \to 0$

$\therefore \lim_{h\to 0} \cos(\pi + h) = \lim_{h\to 0} -\cos h = -1$

$f(\pi) = k\pi + 1$ $\quad [\because f(x) = kx + 1]$

Thus f (x) is continuous at $x = \pi$.

$\therefore$ L.H.L. = R.H.L. $= f(\pi) \Rightarrow k\pi + 1 = -1 \Rightarrow k = -2/11$

(9) We have $f(x) = \begin{cases} 5 & \text{if } x \leq 2 \\ ax+b & \text{if } 2 < x < 10 \\ 21 & \text{if } x \geq 10 \end{cases}$

At $x = 2$, L.H.L. $= \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (5) = 5$

R.H.L. $= \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (ax+b)$

Put $x = 2 + h$ as $x \to 2^+$, $h \to 0$

$\therefore \lim_{h\to 0} [a(2+h)+b] = \lim_{h\to 0} (2a + ah + b) = 2a + b$

$f(2) = 5$,

Since f (x) is continuous at $x = 2$

$\therefore$ L.H.L. = R.H.L. $= f(2) \Rightarrow 2a + b = 5$ .... (1)

At $x = 10$, L.H.L. $= \lim_{x\to 10^-} f(x) = \lim_{x\to 10^-} (ax+b)$

Put $x = 10 - h$ as $x \to 10^-$, $h \to 0$

$\therefore \lim_{h\to 0}[a(10-h)+b]=\lim_{h\to 0}[10a-ah+b]=10a+b$

R.H.L. $=\lim_{x\to 10^+} f(x)=\lim_{x\to 10^+}(21)=21$

$f(10)=21$

Since $f(x)$ is continuous at $x=10$

$\therefore$ L.H.L. = R.H.L. $=f(10)\Rightarrow 10a+b=21$ .... (2)

Solving eq. (1) and (2), $a=1, b=1$

(10) Let $g(x)=|x|$ and $h(x)=|x+1|$

Now g (x) = | x | is the absolute value function, so it is continuous function.

h (x) = | x + 1 | is the absolute value function, so it is continuous function.

Since g (x) and h (x) are both continuous functions, so difference of two continuous function is a continuous function.

Thus $f(x)=\sin|x|-|x+1|$ is a continuous function at all points

There is no point at which f (x) is discontinuous.

TRY IT YOURSELF-3

(1) $f(x)=\begin{cases} x^m\cdot\sin\left(\frac{1}{x}\right) & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$

For continuity at $x=0$

$\lim_{h\to 0} f(0+h)=f(0)\Rightarrow \lim_{h\to 0} h^m\sin\left(\frac{1}{n}\right)=0\Rightarrow m>0$

For function to be not differentiable at $x=0$

$\lim_{x\to 0}\frac{f(0+h)-f(0)}{h}=$ DNE

$\lim_{h\to 0}\frac{h^m\sin(1/h)-0}{h}=\lim_{h\to 0} h^m\sin\left(\frac{1}{h}\right)=$ DNE

$\Rightarrow\ m-1\le 0\Rightarrow m\le 1$

$\therefore\ m\in[0,1]$ for functiion $f(x)$ to be continuous but not differentiable at $x=0$.

(2) $f(x)=\begin{cases}\sin x & \text{if } x<\pi\\ mx+n & \text{if } x\ge\pi\end{cases}$

Since $f(x)$ is continuous at $x=\pi$

$0=m\pi+n$ ...... (1)

Now, $f(x)$ is differentiable at $x=\pi$

$\cos x|_{x=\pi}=m\Rightarrow m=-1\Rightarrow n=\pi$

(3) Limit $=\lim_{h\to 0}\frac{f(a+2h^2)-f(a-2h^2)}{h^2}$ put $t=h^2$

$=\lim_{h\to 0}\frac{f(a+2t)-f(a-2t)}{t}$ differentiating numerator and denominator.

$=\lim_{h\to 0}\frac{2f'(a+2t)-2f'(a-2t)}{1}$

$=2f'(a)+2f'(a)=4f'(a)=1$

(4) We have $f(x)=|x-1|$

$\begin{cases}x-1 & \text{if } x\ge 1\\ 1-x & \text{if } x<1\end{cases}$

L.H.D. at $x=1$

$\lim_{h\to 0}\frac{f(1-h)-f(1)}{(1-h-1)}=\lim_{h\to 0}\frac{1-(1-h)-(1-1)}{-h}$

$=\lim_{h\to 0}\frac{h}{-h}=-1$

R.H.D. at $x=1$

$\lim_{h\to 0}\frac{f(1+h)-f(1)}{(1+h-1)}=\lim_{h\to 0}\frac{(1+h)-1-(1-1)}{h}=\lim_{h\to 0}\frac{h}{h}=1$

$\therefore$ L.H.D. $\neq$ R.H.D.

Thus $f(x)$ is not differentiable at $x=1$

(5) We have $f(x)=[x]$

R.H.D. at $x=1$

$\lim_{h\to 0}\frac{f(1+h)-f(1)}{(1+h-1)}=\lim_{h\to 0}\frac{1-1}{h}=0$

L.H.D. at $x=1$

$\lim_{h\to 0}\frac{f(1-h)-f(1)}{(1-h-1)}=\lim_{h\to 0}\frac{0-1}{-h}=\lim_{h\to 0}\frac{1}{h}$ = not defined. $\therefore$ L.H.D. $\neq$ R.H.D.

Thus $f(x)$ is not differentiable at $x=1$

R.H.D. at $x=2$, $\lim_{h\to 0}\frac{f(2+h)-f(2)}{(2+h-2)}=\lim_{h\to 0}\frac{2-2}{h}=0$

L.H.D. at $x=2$,

$\lim_{h\to 0}\frac{f(2-h)-f(2)}{(2-h-2)}=\lim_{h\to 0}\frac{1-2}{-h}=\lim_{h\to 0}\frac{-1}{-h}=\lim_{h\to 0}\frac{1}{h}$

= not defined.

$\therefore$ L.H.D. $\neq$ R.H.D.

Thus $f(x)$ is not differentiable at $x=2$.

(6) $f(x)=|x|+|x-1|$.

f (x) is continuous everywhere at | x | and | x – | are continuous for all x.

Also $|x|$ and $|x-1|$ are non-differentiable at $x=0$ and $x=1$, respectively.

Hence, $f(x)$ is non-differentiable at $x=0$ and $x=1$.

TRY IT YOURSELF-4

(1) We have, $f(x)=x^3-6x^2+11x-6$ (Polynomial)

We know that a polynomial function is everywhere continuous and differentiable. Therefore,

(i) It is continuous on [1, 3]

(ii) It is differentiable on (1, 3)

(iii) Also, $f(1)=1^3-6\times 1^2+11\times 1-6=0$

and $f(3)=3^3-6\times 3^2+11\times 3-6=0\Rightarrow f(1)=f(3)$

Thus, all the conditions of Rolle's theorem are satisfied.

So, there must exist some $c\in(1,3)$, such that $f'(c)=0$

Now, $f'(c)=3c^2-12c+11=0$

$\Rightarrow c = \frac{12 \pm \sqrt{144-132}}{6} \quad \Rightarrow c = \left(2 \pm \frac{1}{\sqrt{3}}\right)$

Clearly, both the values of c lie in the interval (1, 3). Hence, Rolle's theorem is verified.

(2) Rolle's theorem is applicable to a function f (x), if the function possesses a differential coefficient for every value of x in the domain (–1, 1). But, f (x) = | x | is not differentiable at x = 0. Hence, Rolle's theorem is not applicable to the function $f(x) = |x|$ in the interval [–1, 1].

(3) We have, $f(x) = \log x$ on [1, 2].

(i) Since, logarithmic function is continuous everywhere, therefore f (x) is continuous on the closed interval [1,2].

(ii) Since, $f'(x) = 1/x$, which exists for all $x \neq 0$. So f (x) is derivable on open interval (1, 2).

Thus, both the conditions of Lagrange's mean value theorem are satisfied.

Hence, there exists at least one $c \in (1, 2)$ such that,

$$\frac{f(b) - f(a)}{b - a} = f'(c) \qquad \text{............ (1)},$$

where a = 1 and b = 2.

Now, $f'(x) = 1/x \Rightarrow f'(c) = 1/c$

$f(b) = f(2) = \log 2$ and $f(a) = f(1) = \log 1$

From (1), we get, $\frac{\log 2 - \log 1}{2-1} = \frac{1}{c} \Rightarrow \log 2 - \log 1 = \frac{1}{c}$

$\Rightarrow \log 2 = \frac{1}{c} \Rightarrow c = \frac{1}{\log 2} = \log_2 e \quad [\because \log 1 = 0]$

Nearly, $\log_2 e$ lies in the interval (1, 2).

Hence, Lagrange Mean value theorem is varified.

(4) Let $f(x) = x - \log_e(1+x) \Rightarrow f'(x) = 1 - \frac{1}{1+x}$

$f'(x) > 0$ for $x > 0$

f (x) is increasing for $x > 0$; $x > \log_e(1+x)$ .... (1)

Let $F(x) = \log_e(1+x) - x + \frac{x^2}{2}$;

$F'(x) = \frac{1}{1+x} - 1 + x$; $\quad F'(x) > 0$ for $x > 0$

F '(x) is increasing for $x > 0$ i.e., $\log(1+x) > x - \frac{x^2}{2}$ ...... (2)

From eq. (1) and (2), we have, $x > \log_e(1+x) > x - \frac{x^2}{2}$

(5) Let $f(x) = x - \sin x$, defined on the interval [0, x], where $x > 0$

Clearly, f (x) is everywhere continuous and differentiable. So, (i) f (x) is continuous on the closed interval [0, x]

(ii) f (x) is differentiable on the open interval (0, x)

Thus, both the conditions of Lagrange's Mean value theorem are satisfied and therefore there exists $c \in (0, x)$.

$f'(c) = \frac{f(x) - f(0)}{x - 0}$ [By Lagrange's Mean value theorem]

$\Rightarrow 1 - \cos c = \frac{x - \sin x}{x} \Rightarrow \frac{x - \sin x}{x} > 0 \quad [\because 1 - \cos c > 0]$

$\Rightarrow x - \sin x > 0 \quad [\because x > 0]$

$\Rightarrow x > \sin x \Rightarrow \sin x < x$ for all $x > 0$

CHAPTER-4 : LIMITS, CONTINUITY AND DIFFERENTIABILITY
EXERCISE-1

(1) **(C).** $\lim_{x\to 3} x+3 = 3+3=6$

(2) **(A).**
$$\lim_{x\to 0}\frac{\sin ax}{\sin bx} = \lim_{x\to 0}\left[\frac{\sin ax}{ax}\times ax\times\frac{1}{\frac{\sin bx}{bx}\times bx}\right]$$
$$=\lim_{x\to 0}\left[\frac{\sin ax}{ax}\times\frac{1}{\frac{bx}{bx}}\times\frac{ax}{bx}\right]=\frac{a}{b}\lim_{x\to 0}\left[\frac{\sin ax}{ax}\times\frac{1}{\frac{\sin bx}{bx}}\right]$$
$$=\frac{a}{b}\times 1\times 1=\frac{a}{b}$$

(3) **(C).** We have, $\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$
$$=\frac{1}{\pi}\lim_{x\to\pi}\frac{\sin(\pi-x)}{(\pi-x)}=\frac{1}{\pi}\lim_{x\to\pi}\frac{\sinh}{h}$$
$$[\because x\to\pi\Rightarrow \pi-x\to h]$$
$$=\frac{1}{\pi}.1=\frac{1}{\pi}\qquad\left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

(4) **(C).** $\lim_{x\to 0}\frac{\cos x}{\pi-x}=\frac{\cos 0}{\pi-0}=\frac{1}{\pi}$

(5) **(B).** $\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}=\lim_{x\to 0}\frac{1-\cos 2x}{1-\cos x}$
$$=\lim_{x\to 0}\frac{2\sin^2 x}{2\sin^2 x/2}=\lim_{x\to 0}\frac{(2\sin x/2\cos x/2)^2}{\sin^2 x/2}$$
$$=\lim_{x\to 0}\frac{4\sin^2(x/2)\cos^2(x/2)}{\sin^2 x/2}=\lim_{x\to 0}4\cos^2(x/2)$$
$$=4\times 1=4$$

(6) **(A).** $\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}=\lim_{x\to 0}\left(\frac{ax}{b\sin x}+\frac{x\cos x}{b\sin x}\right)$
$$=\frac{a}{b}\lim_{x\to 0}\frac{x}{\sin x}+\frac{1}{b}\lim_{x\to 0}\frac{x\cos x}{\sin x}$$
$$=\frac{a}{b}(1)+\frac{1}{b}\lim_{x\to 0}\frac{x}{\tan x}\qquad\left[\because \lim_{x\to 0}\frac{x}{\sin x}=1\right]$$
$$=\frac{a}{b}+\frac{1}{b}(1)=\frac{a+1}{b}\qquad\left[\because \lim_{x\to 0}\frac{x}{\tan x}=1\right]$$

(7) **(A).** Here, $\lim_{x\to 0}x\sec x=\lim_{x\to 0}x\times\frac{1}{\cos x}=\lim_{x\to 0}\frac{x}{\cos x}=\frac{0}{1}=0$

(8) **(B).** We have, $\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}=\lim_{x\to 0}\frac{\frac{\sin ax}{x}+\frac{bx}{x}}{\frac{ax}{x}+\frac{\sin bx}{x}}$
$$=\frac{\lim_{x\to 0}a\times\frac{\sin x}{x}+\lim_{x\to 0}b}{\lim_{x\to 0}a+\lim_{x\to 0}b\times\frac{\sin x}{x}}=\frac{a\lim_{x\to 0}\frac{\sin x}{x}+\lim_{x\to 0}b}{\lim_{x\to 0}a+b\lim_{x\to 0}\frac{\sin x}{x}}$$
$$=\frac{a(1)+b}{a+b(1)}=\frac{a+b}{a+b}=1\qquad\left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

(9) **(A).** Here, $\lim_{x\to 0}(\operatorname{cosec}x-\cot x)=\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right)$
$$=\lim_{x\to 0}\frac{1-\cos x}{\sin x}=\lim_{x\to 0}\frac{2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)}$$
$$=\lim_{x\to 0}\tan(x/2)=0$$

(10) **(C).**
$$\lim_{x\to\pi/2}\frac{\tan 2x}{x-\frac{\pi}{2}}\left[\frac{0}{0}\text{ form}\right]\quad\text{Put } x=\frac{\pi}{2}+y \text{ as } x\to\frac{\pi}{2},\ y\to 0$$
$$\therefore\ \lim_{y\to 0}\frac{\tan 2\left(\frac{\pi}{2}+y\right)}{\frac{\pi}{2}+y-\frac{\pi}{2}}=\lim_{y\to 0}\frac{\tan(\pi+2y)}{y}$$
$$=\lim_{y\to 0}\frac{\tan 2y}{y}=\lim_{y\to 0}\frac{\tan 2y}{2y}\times 2=1\times 2=2$$

(11) **(B).** $f(x)=\begin{cases}2x+3, & x\le 0\\ 3(x+1), & x>0\end{cases}$

(i) We have to find $\lim_{x\to 0}f(x)$

Left hand limit
$$=\lim_{x\to 0^-}f(x)=\lim_{x\to 0-h}2(x)+3=\lim_{h\to 0}[2(0-h)+3]=3$$
Right hand limit
$$=\lim_{x\to 0^+}f(x)=\lim_{x\to 0+h}3(x+1)=\lim_{h\to 0}[3(0+h)+1]=3$$
Here, $\lim_{x\to 0^-}f(x)=\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}f(x)=3$

(ii) $\lim_{x\to 1}f(x)=\lim_{x\to 1}3(x+1)=3(1+1)=6$

(12) **(D).** Here, $f(x)=\begin{cases}x^2-1, & x\le 1\\ -x^2-1, & x>1\end{cases}$

L.H.L. $= \lim_{x\to1^-} f(x) = \lim_{x\to1^-} (x^2-1)$

Put $x = 1 - h$ as $x \to 1, h \to 0$

$\therefore \lim_{h\to0} [(1+h)^2 - 1] = \lim_{h\to0} [1+h^2-2h-1]$

$= (0)^2 - 2\times0 = 0$

R.H.L. $= \lim_{x\to1^+} f(x) = \lim_{x\to1^+} (-x^2-1)$

Put $x = 1 + h$ as $x \to 1, h \to 0$

$\therefore \lim_{h\to0} [-(1+h)^2 - 1] = \lim_{h\to0} [-1-h^2-2h-1]$

$= -(0)^2 - 2\times0 - 2 = -2$

$\therefore$ LHL $\neq$ RHL

This limit does not exist at $x = 1$.

(13) **(D).**

We have to find $\lim_{x\to0} f(x)$, where $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

LHL $= \lim_{x\to0^-} f(x) = \lim_{x\to0-h} \frac{|x|}{x} = \lim_{x\to h} \frac{|-h|}{h} = -1$

Right hand limit $= \lim_{x\to0^+} f(x) = \lim_{x\to0+h} \frac{|x|}{x} = \lim_{x\to h} \frac{|h|}{h} = 1$

$\lim_{x\to0^-} f(x) \neq \lim_{x\to0^+} f(x)$. Limit does not exist at $x = 0$

(14) **(D).** LHL $= \lim_{h\to0} \frac{-h+|h|}{-h} = \lim_{h\to0} (0) = 0$

RHL $= \lim_{h\to0} \frac{h+|h|}{h} = 2$

LHL $\neq$ RHL $\Rightarrow$ does not exist.

(15) **(B).** $\lim_{x\to\infty} \frac{\sin x}{x}$

$= \lim_{x\to\infty}$ (a finite number between -1 and 1)/∞

$= 0$

(16) **(D).** LHL $= \text{Lim}_{h\to0} \frac{1+e^{1/h}}{1-e^{1/h}}$

$= \text{Lim}_{h\to0} \frac{e^{-1/h}+1}{e^{-1/h}-1} = -1$

RHL $= \text{Lim}_{h\to0} \frac{1+e^{-1/h}}{1-e^{-1/h}} = \frac{1+0}{1-0} = 1$

LHL $\neq$ RHL, so given limit does not exist.

(17) **(C).** $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \sqrt{\frac{\{1-(\sin x/x)\}}{\{1+(\cos^2 x/x)\}}}$

$= \sqrt{\frac{1-0}{1+0}} = 1.$

(18) **(D).** Here $G(1) = -\sqrt{25-x^2} = -\sqrt{24}$

$\therefore$ Given limit $= \lim_{x\to1} \frac{-\sqrt{25-x^2}+\sqrt{24}}{x-1} \left(\frac{0}{0}\text{form}\right)$

$= \lim_{x\to1} \frac{x}{\sqrt{25-x^2}} = \frac{1}{\sqrt{24}}$ (By L Hospital rule)

(19) **(C).** Given limit is in 0/0 form, using Hospital rule, we get

Limit $= \lim_{x\to9} \frac{\frac{1}{2\sqrt{f(x)}}.f(x)}{\frac{1}{2\sqrt{x}}} = \frac{f(9).\sqrt{9}}{\sqrt{f(9)}} = \frac{4.3}{3} = 4$

(20) **(D).** Limit $= \lim_{x\to0} \frac{\sin(\pi/180)x}{x}$ $\left(\frac{0}{0}\text{form}\right)$

$= \lim_{x\to0} \frac{(\pi/180)\cos(\pi/180)x}{1} = \frac{\pi}{180}$

(21) **(D).** Here $\text{Lim}_{x\to0^+} f(x) = \text{Lim}_{x\to0^+} x^2 = 0$

and $\text{Lim}_{x\to0^-} f(x) = \text{Lim}_{x\to0^-} (x-1) = -1$

$\because \text{Lim}_{x\to0^+} f(x) \neq \text{Lim}_{x\to0^-} f(x) \therefore \text{Lim}_{x\to0} f(x)$ does not exist.

(22) **(C).** $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x = \lim_{x\to0} (1+ax)^{1/x} = e^a$

(23) **(D).** $\lim_{x\to0} \frac{2x}{x(\sqrt{3+x}+\sqrt{3-x})} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}.$

(24) **(A).** $\lim_{x\to0^-} f(x) = \lim_{h\to0} f(0-h) = \lim_{h\to0} f(0-h) = 0$

and $\lim_{x\to0^+} f(x) = \lim_{h\to0} f(0+h) = \lim_{h\to0} -(0+h) = 0$

$\therefore \lim_{x\to0} f(x) = 0, \left(\because \lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x)\right).$

(25) **(C).** $\lim_{x\to2}\left(\frac{3^{x/2}-3}{3^x-9}\right) = \lim_{x\to2}\left(\frac{3^{x/2}-3}{(3^{x/2})^2-3^2}\right)$

$= \lim_{x\to2} \frac{1}{3^{x/2}+3} = \frac{1}{6}.$

(26) **(B).** Let $x = a \in Q$

$f(a) = a$; $f(a+) = 1 - a$ or a ; $f(a-) = 1 - a$ or a

continuous at where $1 - a = a \Rightarrow a = 1/2$

$\Rightarrow$ continuous at one point

(27) $$f(g(x)) = \begin{cases} 1, & x < -1 \\ 0, & x = -1 \\ -1, & -1 < x < 0 \\ 0, & x = 0 \\ 1, & 0 < x < 1 \\ 0, & x = 1 \\ -1, & x > 1 \end{cases}$$

$\therefore$ Points of discontinuity are $x = -1, 0, 1$

(28) **(A).** $f(3) = 6$ (given)

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)} = 6$$

$\because \lim_{x\to 3} f(x) = f(3)$ $\therefore$ $f(x)$ is continuous at $x = 3$

(29) **(D).** Here $f(2) = 0$

$$\lim_{x\to 2-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} |2-h-2| = 0$$

$$\lim_{x\to 2-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} |2-h-2| = 0$$

Hence it is continuous at $x = 2$.

(30) **(B).** $f(a) = 0$

$$\lim_{x\to a-} f(x) = \lim_{x\to a-}\left(\frac{x^2}{a} - a\right) = \lim_{h\to 0}\left\{\frac{(a-h)^2}{a} - a\right\} = 0$$

and $$\lim_{x\to a+} f(x) = \lim_{h\to 0}\left\{a - \frac{(a+h)^2}{a}\right\} = 0$$

Hence it is continuous at $x = a$.

(31) **(C).** $\lim_{x\to 0^+} f(x) = x^2 \sin\frac{1}{x}$, but $-1 \le \sin\frac{1}{x} \le 1$ and $x \to 0$

Therefore, $\lim_{x\to 0^+} f(x) = 0 = \lim_{x\to 0^-} f(x) = f(0)$

Hence $f(x)$ is continuous at $x = 0$

(32) **(D).** $\lim_{x\to 0-} f(x) = 0$; $f(0) = 0$, $\lim_{x\to 0+} f(x) = -4$

$f(x)$ discontinuous at $x = 0$

and $\lim_{x\to 1-} f(x) = 1$ and $\lim_{x\to 1+} f(x) = 1, f(1) = 1$

Hence $f(x)$ is continuous at $x = 1$.

Also $\lim_{x\to 2-} f(x) = 4(2)^2 - 3.2 = 10$

$f(2) = 10$ and $\lim_{x\to 2+} f(x) = 3(2) + 4 = 10$

Hence $f(x)$ is continuous at $x = 2$

(33) **(B).** $\lim_{x\to 2-} f(x) = \frac{1}{2}$ and $\lim_{x\to 2+} f(x) = \frac{1}{2}$ and $f(2) = 1$

(34) **(A).** $$\lim_{x\to 0-} f(x) = \lim_{x\to 0-}\left(\frac{2\sin^2 2x}{(2x)^2}\right)4 = 8$$

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \sqrt{16+\sqrt{x}+4} = 8$$. Hence $a = 8$.

(35) **(D).** $$\lim_{x\to 4-} f(x) = \lim_{h\to 0} f(4-h) = \lim_{h\to 0} \frac{4-h-4}{|4-h-4|} + a$$

$$= \lim_{h\to 0} -\frac{h}{h} + a = a - 1.$$

$$= \lim_{x\to 4+} f(x) = \lim_{h\to 0} f(4+h) = \lim_{h\to 0} \frac{4+h-4}{|4+h-4|} + b = b + 1$$

and $f(4) = a + b$

Since $f(x)$ is continuous at $x = 4$

Therefore $\lim_{x\to 4-} f(x) = f(4) = \lim_{x\to 4+} f(x)$

$\Rightarrow a - 1 = a + b = b + 1 \Rightarrow b = -1$ and $a = 1$

(36) **(A).** $$\lim_{x\to 0} (\cos x)^{1/x} = k \Rightarrow \lim_{x\to 0} \frac{1}{x}\log(\cos x) = \log k$$

$$\Rightarrow \lim_{x\to 0} \frac{1}{x} \lim_{x\to 0} \log\cos x = \log k$$

$$\Rightarrow \lim_{x\to 0} \frac{1}{x} \times 0 = \log_e k \Rightarrow k = 1.$$

(37) **(A).** If f is continuous at $x = 0$, then

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) \Rightarrow f(0) = \lim_{x\to 0^-} f(x)$$

$$k = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{\cos\frac{\pi}{2}[0-h]}{[0-h]}$$

$$k = \lim_{h\to 0} \frac{\cos\frac{\pi}{2}[-h]}{[-h]} = \lim_{h\to 0} \frac{\cos\frac{\pi}{2}[-h-1]}{[-h-1]}$$

$$k = \lim_{h\to 0} \frac{\cos\left(-\frac{\pi}{2}\right)}{-1}; k = 0.$$

(38) **(A).** $$f(5) = \lim_{x\to 5} f(x) = \lim_{x\to 5} \frac{x^2 - 10x + 25}{x^2 - 7x + 10}$$

$$= \lim_{x\to 5} \frac{(x-5)^2}{(x-2)(x-5)} = \frac{5-5}{5-2} = 0.$$

(39) **(C).** $f(x)$ is continuous at $x = \frac{\pi}{2}$, then

$$\lim_{x\to \pi/2} f(x) = f(0) \text{ or } \lambda = \lim_{x\to \pi/2} \frac{1-\sin x}{\pi - 2x}, \left(\frac{0}{0} \text{ form}\right)$$

Applying L-Hospital's rule,

$$\lambda = \lim_{x\to \pi/2} \frac{-\cos x}{-2} \Rightarrow \lambda = \lim_{x\to \pi/2} \frac{\cos x}{2} = 0.$$

(40) (C). $f(x) = \dfrac{2\cos^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}} = \dfrac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}}$

$= \tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$ at $x = \pi$, $f(\pi) = -\tan\frac{\pi}{4} = -1$.

(41) (A). The function f (x) = [x] in discontinuous at every integral value of x. If n is any integer, then

$\lim_{x\to n^-}[x] = n-1$ and $\lim_{x\to n^+}[x] = n$

$\therefore$ LHL $\neq$ RHL. $f(x) = [x]$ is not continuous for rational or real values of x as both contain integers.

(42) (B). $k = \lim_{x\to 1}\frac{\log x}{x-1} = \lim_{x\to 1}\frac{1/x}{1}$ [Using L'Hospital Rule]

(43) (B). $\lim_{x\to 0}\left(\frac{3\sin \pi x}{5x}\right) = 2k$

$\pi\frac{3}{5}\lim_{x\to 0}\frac{\sin\pi x}{x\pi} = 2k$; $\frac{3\pi}{5} = 2k \Rightarrow k = \frac{3\pi}{10}$

(44) (D). LHL = RHL = f(5) $\Rightarrow 2k = 15 - 8$ $\therefore$ k = 7/2

(45) (C).

(46) (D). Statement (D) is true, because differentiable function is always continuous.

(47) (A). Since this function is continuous at x = 0

Now for differentiability

$f(x) = |x| = |0| = 0$ and $f(0+h) = f(h) = |h|$

$\therefore \lim_{h\to 0-}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0-}\frac{|h|}{h} = -1$

and $\lim_{h\to 0+}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0+}\frac{|h|}{h} = 1$.

Therefore it is continuous and non-differentiable.

(48) (A). $f'(k-0) = \lim_{h\to 0}\frac{[k-h]\sin\pi(k-h) - [k]\sin\pi k}{-h}$

$= \lim_{h\to 0}\frac{(-1)^{k-1}(k-1)\sin\pi h - k\times 0}{-h}$

$= \lim_{h\to 0}\frac{(-1)^{k-1}(k-1)\sin\pi h}{-h} = (-1)^k.(k-1)\pi$.

(49) (C). $f(x) = 1 + \sin(3x)g(x)$

$f'(x) = 3\cos 3x\, g(x) + \sin 3x\, g'(x) = f(x)\cos 3x$.

(50) (D). $f(x) = x^2 - 2x + 4$; $f'(x) = 2x - 2$

At $x = c$, $f'(c) = 2c - 2$

$f(5) = 5^2 - 2(5) + 4 = 19$; $f(1) = 1^2 - 2(1) + 4 = 3$

$\frac{f(5)-f(1)}{5-1} = f(c) \Rightarrow \frac{19-3}{5-1} = 2c - 2 \Rightarrow \frac{16}{4} = 2c - 2$

$\Rightarrow 4c = 2c - 2 \Rightarrow 2c = 6$ or $c = 3$.

(51) (C). $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2x-1, & x > 1 \end{cases}$

$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0}(1-h) = 1$

$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} 2(1+h) - 1 = 1$

$\because \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 1$

$\therefore$ Function is continuous at x = 1.

$Lf'(1) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{(1-h)-1}{-h} = 1$

$Rf'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{-h} = \lim_{h\to 0}\frac{2+2h-1-1}{h} = 2$

$\therefore Lf'(1) \neq Rf'(1)$

$\therefore$ Function is not differentiation at x = 1

(52) (C). f (x) possesses derivative at x = 0, so it is both continuous and differentiable at x = 0.

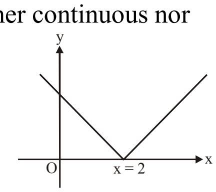
Now $f(0+0) = 0, f(0-0) = b, f(0) = b, \therefore b = 0$

Also $Rf'(0) = 0, Lf'(0) = 0, \forall a \in R$

$\therefore f'(0) = 0$ if $b = 0$.

(53) (D). Since the function is defined for $x \ge 0$ i.e. not defined for $x < 0$. Hence the function neither continuous nor differentiable at $x = 0$.

(54) (D). $f(x) = |x-2| + x$ is continuous at x = 2 and x = 0



(55) (A). $f(x) = e^{-2x}\sin 2x \Rightarrow f'(x) = 2e^{-2x}(\cos 2x - \sin 2x)$

Now, $f'(c) = 0$

$\Rightarrow \cos 2c - \sin 2c = 0 \Rightarrow \tan 2c = 1 \Rightarrow c = \frac{\pi}{8}$.

(56) (D). $f(x) = x^3 - 6x^2 + ax + b$

$\Rightarrow f'(x) = 3x^2 - 12x + a$

$\Rightarrow f'(c) = 0 \Rightarrow f'\left(2 + \frac{1}{\sqrt{3}}\right) = 0$

$\Rightarrow 3\left(2 + \frac{1}{\sqrt{3}}\right)^2 - 12\left(2 + \frac{1}{\sqrt{3}}\right) + a = 0$

$\Rightarrow 3\left(4 + \frac{1}{3} + \frac{4}{\sqrt{3}}\right) - 12\left(2 + \frac{1}{\sqrt{3}}\right) + a = 0$

$12 + 1 + 4\sqrt{3} - 24 - 4\sqrt{3} + a = 0 \Rightarrow a = 11$.

(57) (D). $f(x) = \sqrt{x}$

$\therefore f(a) = \sqrt{4} = 2,\ f(b) = \sqrt{9} = 3$; $f'(x) = \frac{1}{2\sqrt{x}}$

Also, $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{3-2}{9-4} = \frac{1}{5}$

$\therefore \frac{1}{2\sqrt{c}} = \frac{1}{5} \Rightarrow c = \frac{25}{4} = 6.25$.

(58) **(B).** $f(b) = f(2) = 8 - 24a + 10 = 18 - 24a$

$f(a) = f(1) = 1 - 6a + 5 = 6 - 6a$

$f'(x) = 3x^2 - 12ax + 5$

From Lagrange's mean value theorem,

$f'(x) = \frac{f(b) - f(a)}{b - a} = \frac{18 - 24a - 6 + 6a}{2 - 1} \quad \therefore f'(x) = 12 - 18a$

At $x = \frac{7}{4}, 3 \times \frac{49}{16} - 12a \times \frac{7}{4} + 5 = 12 - 18a$

$\Rightarrow 3a = \frac{147}{16} - 7 \Rightarrow 3a = \frac{35}{16} \Rightarrow a = \frac{35}{48}$.

(59) **(B).** Let $x \in [0, 2]$. Since f (x) satisfies all the conditions of L.M.V. Theorem on [0, 2].

$\therefore$ it also satisfied on $[0, x] \subseteq [0, 2]$

$\therefore \frac{f(x) - f(0)}{x - 0} = f'(x_1)$

where $0 < x_1 < x < 2$ i.e. $0 < x_1 < 2$

$\Rightarrow f(x) = x f'(x_1) \quad [\because f(0) = 0]$

$\Rightarrow |f(x)| = |x f'(x_1)|$

$= |x||f'(x_1)| \le 2 . \frac{1}{2} = 1 \quad \{\because |x| \le 2 \text{ and } |f'(x_1)| \le \frac{1}{2}]$

$\Rightarrow |f(x)| \le 1$

(60) **(B).** Here $\frac{f(b) - f(a)}{b - a} = f'(c)$

$\Rightarrow \frac{1 - 0}{1 - 0} = 2 - 2c \quad \left\{\begin{matrix} \because b = 1, a = 0 \\ \Rightarrow f(1) = 1, f(0) = 0 \end{matrix}\right\}$

$\Rightarrow -2c = -1 \Rightarrow c = \frac{1}{2} \quad \left\{\begin{matrix} \because f'(x) = 2 - 2x \\ f'(c) = 2 - 2c \end{matrix}\right\}$

(61) **(D).** f (x) and g (x) are both continuous is [–2, 2] and differentiable is (–2, 2).

$\therefore$ f (x) and g (x) satisfy Mean Value Theorem

Now $f(-2) = -8, f(2) = 8 \therefore f(-2) \ne f(2) | g(1) = g(-2)$

$\therefore$ f (x) doesn't satisfy Rolle's theorem.

(62) **(B).** $f \lim_{x \to 0} \frac{e^{2x} - (1 + 4x)^{1/2}}{\ln(1 - x^2)}$; $\lim_{x \to 0} \frac{\frac{e^{2x} - (1 + 4x)^{1/2}}{\ln(1 - x^2)}}{-x^2}(-x^2)$

$\lim_{x \to 0} \frac{(1 + 4x)^{1/2} - e^{2x}}{x^2}$

$$\lim_{x \to 0} \frac{\left(1 + \frac{1}{2}4x + \frac{1}{2}\left(\frac{1}{2} - 1\right)\frac{1}{2!}16x^2 + ...\right) - \left(1 + \frac{2x}{1!} + \frac{4x^2}{2!} + ...\right)}{x^2} = -2 - 2 = -4$$

(63) **(B).**

$f'(3^+) = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \to 0} \frac{(2 - e^h) - 1}{h} = -\lim_{h \to 0} \left(\frac{e^h - 1}{h}\right) = -1$

$f'(3^-) = \lim_{h \to 0} \frac{f(3 - h) - f(3)}{-h} = \lim_{h \to 0} \frac{\sqrt{10 - (3 - h)^2} - 1}{-h} = -\lim_{h \to 0} \frac{\sqrt{1 + (6h - h^2)} - 1}{-h}$

$= \lim_{h \to 0} \frac{6h - h^2}{-h(\sqrt{1 + 6h - h^2} + 1)} = \lim_{h \to 0} \frac{h(h - 6)}{h(\sqrt{1 + 6h - h^2} + 1} = \frac{-6}{2} = -3$

Hence, $f'(3^+) \ne f(3^-)$

(64) **(B). Case (i) : $x^2 > 1$**

$$f(x) = \lim_{n \to \infty} \frac{\frac{1}{x^{2n}}\log(2 + x) - \sin x}{\left(\frac{1}{x^{2n}} + 1\right)} = -\sin x$$

Case (ii) : $x^2 < 1$ i.e., $-1 < x < 1$

$$f(x) = \lim_{n \to \infty} \frac{\log(2 + x) - x^{2n}\sin x}{1 + x^{2n}} = \log(2 + x)$$

$(\because \text{if } |x| < 1, \lim x^{2n} = 0)$

Case (iii) : x = 1

$$f(x) = f(1) = \lim_{n \to \infty} \frac{\log 3 - \sin 1}{2} = \frac{\log 3 - \sin 1}{2}$$

(65) **(B). Differentiability at x = 0**

$R[f'(0)] = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$

$= \lim_{h \to 0} \frac{(0 + h)^2 - 0}{h} = \lim_{h \to 0} h = 0$

$L[f'(0)] = \lim_{h \to 0} \frac{f(0 - h) - f(0)}{-h} = \lim_{h \to 0} \frac{-(0 - h) - 0}{-h} = -1$

$\therefore R[f'(0)] \ne L[f'(0)] \therefore$ f (x) is not differentiable at x = 0

Differentiability at x = 1

$R[f'(1)] = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h}$

$= \lim_{h \to 0} \frac{(1 + h)^3 - (1 + h) + 1 - 1}{h} = \lim_{h \to 0} \frac{2h + 3h^2 + h^3}{h}$

$L[f'(1)] = \lim_{h \to 0} \frac{f(1 - h) - f(1)}{-h} = \lim_{h \to 0} \frac{f(1 - h) - 1}{-h}$

$= \lim_{h \to 0} \frac{-2h + h^2}{-h} = 2$. Thus $R[f'(1)] = L f'(1)]$

$\because$ Function f(x) is differentiable at x = 1

(66) **(B).** $\because$ f(x) is continuous at x = a

$$\therefore f(a) = \lim_{x\to a} f(x) = \lim_{x\to a} \frac{x-a}{\sqrt{x}-\sqrt{a}}$$

$$= \lim_{x\to a} \frac{\left(\sqrt{x}-\sqrt{a}\right)\left(\sqrt{x}+\sqrt{a}\right)}{\sqrt{x}-\sqrt{a}} = \lim_{x\to a}\left(\sqrt{x}+\sqrt{a}\right) = 2\sqrt{a}$$

(67) **(D).** $f(x) = \lim_{n\to\infty} \sin^{2n} x = \lim_{n\to\infty} (\sin^2 x)^n$

$$= \begin{cases} 1, & x = (2n+1)\dfrac{\pi}{2}, n \in I \\ 0, & x \neq (2n+1)\dfrac{\pi}{2}, n \in I \end{cases}$$

Clearly, f(x) is discontinuous at $x = (2n+1)\dfrac{\pi}{2}, n \in I$

(68) **(D).** $\because$ f(x) is continuous at x = 0, so $f(0) = \lim_{x\to 0} f(x)$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} (x+1)^{\cot x}$$

$$\Rightarrow \log A = \lim_{x\to 0} \cot x . \log(1+x)$$

$$= \lim_{x\to 0} \frac{\log(1+x)}{\tan x} \left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to 0} \frac{x - \frac{x^2}{2} + ...}{x + \frac{x^3}{3} + ...} = 1$$

$\therefore A = e^1 = e \Rightarrow f(0) = e.$

(69) **(B).** $\lim_{h\to 0} f(0-h) = f(0) = \lim_{h\to 0} f(0+h)$

f(0) = a

$$\lim_{h\to 0} f(0-h) = \lim_{h\to 0} (\cos h + \sin h)^{-\operatorname{cosec} h}$$

$$= e^{\lim_{h\to 0}(-\operatorname{cosec} h)(\cos h + \sin h - 1)} = e^{\lim_{h\to 0}\frac{\sin\frac{h}{2} - \cos\frac{h}{2}}{\cos\frac{h}{2}}} = e^{-1}$$

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{e^{\frac{1}{h}} + e^{\frac{2}{h}} + e^{\frac{3}{h}}}{a.e^{\frac{2}{h}} + be^{\frac{3}{h}}}$$

divided by $e^{3/h}$

We get $\lim_{h\to 0} \frac{1}{b} = \frac{1}{b}$; $a = \frac{1}{e} = \frac{1}{b} \Rightarrow (a, b) \equiv \left(\frac{1}{e}, e\right)$

(70) **(C).** In the definition of the function, b ≠ 0, for then f (x) will be undefined in x > 0

$\because$ f (x) is continuous at x = 0

$\therefore$ LHL = RHL = f(0)

$$\Rightarrow \lim_{\substack{x\to 0\\x<0}} \frac{\sin(a+1)x + \sin x}{x} = \lim_{\substack{x\to 0\\x>0}} \frac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}} = c$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{\sin(a+1)x}{x} + \frac{\sin x}{x}\right) = \lim_{x\to 0}\frac{\sqrt{1+bx}-1}{bx} = c$$

$$\Rightarrow (a+1)+1 = \lim_{x\to 0}\frac{(1+bx)-1}{bx\,(\sqrt{1+bx}+1)} = c$$

$$\Rightarrow a+2 = \lim_{x\to 0}\frac{1}{\sqrt{1+bx}} = c \Rightarrow a+2 = \frac{1}{2} = c$$

$$\therefore a = -\frac{3}{2}, c = \frac{1}{2}, b \neq 0$$

(71) **(C).** Since f(x) is continuous in (– ∞ , 6). so it is continuous at x = 1 and x = 3.

Now, $f(1) = 2 = f(1-0)$

$f(1+0) = a+b$

But f (x) is continuous at x = 1

$\Rightarrow f(1-0) = f(1+0) = f(1) \Rightarrow a+b = 2$ .... (1)

Also $f(3) = 6 = f(3+0)$

$f(3-0) = 3a+b$

$\therefore$ f (x) is continuous at x = 3

$\Rightarrow f(3-0) = f(3+0) = f(3) \Rightarrow 3a+b = 6$ .... (2)

from (1) & (2) $\Rightarrow a = 2, b = 0$

(72) **(A).** $f(x) = [x](\sin kx)^p$

$(\sin kx)^p$ is continuous and differentiable function

$\forall\ x \in R, k \in R$ and $p > 0$.

[X] is discontinuous at x ∈ I

For $k = n\pi,\ n \in I$

$f(x) = [x](\sin(n\pi x))^p$

$\lim_{x\to a} f(x) = 0, a \in I$ and $f(a) = 0$

So, f (x) becomes continuous for all x ∈ R.

EXERCISE-2

(1) **(D).**

(A) $\lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$ exist finitely

$$\therefore \lim_{h\to 0^+} f(a+h) - f(a) = \lim_{h\to 0^+}\left(\frac{f(a+h)-f(a)}{h}\right) h = 0$$

$$\Rightarrow \lim_{h\to 0^+} f(a+h) = f(a)$$

Similarly, $\lim_{h\to 0^-} f(a+h) = f(a)$

$\therefore$ f is continuous at x = a

(B) Function is not differentiable at 5x = (2n + 1) π/2 only, which are not in domain

(C) Let $f(x)=\frac{1}{x^2}$ and $g(x)=-\frac{1}{x^2}$,

$\lim_{x\to 0} f(x)+g(x)$ exists whatever $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ does not exist.

(2) **(D).** $\lim_{x\to 0^+}\frac{e^{e/x}-e^{-e/x}}{e^{1/x}+e^{-1/x}}=\lim_{x\to 0^+}\frac{e^{\frac{e-1}{x}}(1-e^{-2e/x})}{(1+e^{-2/x})}=+\infty$

$\lim_{x\to 0^-}\frac{e^{e/x}-e^{-e/x}}{e^{1/x}+e^{-1/x}}=\lim_{x\to 0^-}\frac{e^{-e/x}(e^{2e/x}-1)}{e^{-e/x}(e^{+2/x}+1)}$

$=\lim_{x\to 0^+} e^{-\left(\frac{e-1}{x}\right)}\left(\frac{e^{2e/x}-1}{e^{2/x}+1}\right)=-\infty$

Limit doesn't exist, so $f(x)$ is discontinuous.

(3) **(B).** $f(x)=\lim_{x\to 0}\left(\frac{2x-\sin^{-1}x}{2x+\tan^{-1}x}\right)=f(0),\left(\frac{0}{0}\right)$

Applying L-Hospital's rule,

$$f(0)=\lim_{x\to 0}\frac{\left(2-\frac{1}{\sqrt{1-x^2}}\right)}{\left(2+\frac{1}{1+x^2}\right)}=\frac{2-1}{2+1}=\frac{1}{3}$$

(4) **(D).** $f(x)=\begin{cases}\frac{1-|x|}{1+x}, & x\neq -1\\ 1, & x=-1\end{cases}$ and $f(x)=\begin{cases}1, & x<0\\ \frac{1-x}{1+x}, & x\geq 0\end{cases}$

$$f(2x)=\begin{cases}1, & x<0\\ \frac{1-[2x]}{1+[2x]}, & x>0\end{cases}\Rightarrow f(2x)=\begin{cases}1, & x<0\\ 1, & 0\leq x<\frac{1}{2}\\ 0, & \frac{1}{2}\leq x\leq 1\\ -\frac{1}{3}, & 1\leq x<\frac{3}{2}\end{cases}$$

$\Rightarrow$ f (x), for all values of x where $x<\frac{1}{2}$ a continous function and for $x=\frac{1}{2}$ and $x=1$, f (x)be a discontinous function.

(5) **(B).** $f(x)=\begin{cases}\frac{1-\cos 4x}{8x^2}, & x\neq 0\\ k, & x=0\end{cases}$

If f (x) is continous function at point $x=0$ then

$\lim_{x\to 0+}[f(x)=\lim_{x\to 0-}[f(x)]$

$\lim_{x\to 0}[f(x)]=\lim_{h\to 0-}[f(0+h)]$

$=\lim_{h\to 0}[f(h)]=\lim_{h\to 0}\frac{1-\cos 4h}{8h^2}=\lim_{h\to 0}\frac{2\sin^2 2h}{8h^2}=\lim_{h\to 0}\frac{\sin^2 2h}{4h^2}$

$=\lim_{h\to 0}\left(\frac{\sin 2h}{2h}\right)^2=(1)^2=1$

$\lim_{x\to 0^-} f(x)=\lim_{h\to 0}[f(0-h)]$

$=\lim_{h\to 0}[f(-h)]=\lim_{h\to 0}\frac{1-\cos 4(-h)}{8(-h)^2}=\lim_{h\to 0}\frac{1-\cos 4h}{8h^2}=1$

$f(0)=1\Rightarrow k=1$.

(6) **(D).** Since $|x-3|=x-3$, if $x\geq 3$ $=-x+3$, if $x<3$

$\therefore$ The given function can be defined as

$$f(x)=\begin{cases}\frac{1}{4}x^2-\frac{3}{2}x+\frac{13}{4}, & x<1\\ 3-x, & 1\leq x<3\\ x-3, & x\geq 3\end{cases}$$

Now proceed to check the continuity and differentiability at $x=1$

(7) **(B).** Given $f(x)$ is differentiable at $x=0$.
Hence, f (x) will be continuous at $x=0$.

$\therefore \lim_{x\to 0^-}(e^x+ax)=\lim_{x\to 0^+} b(x-1)^2$

$\Rightarrow e^0+a\times 0=b(0-1)^2 \quad\Rightarrow b=1$ …..(i)

But f (x) is differentiable at $x=0$, then

$Lf'(x)=Rf'(x) \quad\Rightarrow \frac{d}{dx}(e^x+ax)=\frac{d}{dx}b(x-1)^2$

$\Rightarrow e^x+a=2b(x-1)$

At $x=0$, $e^0+a=-2b\Rightarrow a+1=-2b\Rightarrow a=-3$

$\Rightarrow (a, b)=(-3, 1)$.

(8) **(C).** f is continuous at

$x=0$, $\therefore f(0^-)=f(0^+)=f(0)=-1$

Also $Lf'(0)=Rf'(0)$

$\Rightarrow \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

$\Rightarrow \lim_{h\to 0}\left(\frac{e^{-2h}-1+1}{-h}\right)=\lim_{h\to 0}\left(\frac{ah+\frac{bh^2}{2}-1+1}{h}\right)$

$$\Rightarrow \lim_{h\to 0}\left(\frac{-2e^{-2h}}{-1}\right) = \lim_{h\to 0}\left(a+\frac{bh}{2}\right)$$

$\Rightarrow 2 = a + 0 \Rightarrow a = 2$, b any number.

(9) **(D).** $\lim_{x\to 0} f(x) = x^2 \sin\left(\frac{1}{x}\right)$, but

$-1 \le \sin\left(\frac{1}{x}\right) \le 1$ and $x \to 0$

$\therefore \lim_{x\to 0^+} f(x) = 0 = \lim_{x\to 0^-} f(x) = f(0)$

Therefore f (x) is continuous at x = 0. Also, the function

$f(x) = x^2 \sin\frac{1}{x}$ is differentiable because

$Rf'(x) = \lim_{h\to 0}\frac{h^2\sin\frac{1}{h} - 0}{h} = 0$,

$Lf'(x) = \lim_{h\to 0}\frac{h^2\sin(1/-h)}{-h} = 0$.

(10) **(C).** $(gof)(x) = g[f(x)]$

$= g[1-\cos x] = e^{1-\cos x}$, for $x \le 0$

$(gof)'(x) = e^{1-\cos x}.\sin x$, for $x \le 0$

$(gof)'(0) = 0$

(11) **(B).** $y' = \frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}}\cdot\frac{2(1+x^2)-4x^2}{(1+x^2)^2}$

$$= \frac{2(1-x^2)}{\sqrt{(1-x^2)^2.(1+x^2)}} \Rightarrow y' = \begin{cases} \frac{2}{1+x^2} & \text{for } |x|<1 \\ \frac{-2}{1+x^2} & \text{for } |x|>1 \end{cases}$$

Hence for $|x| = 1$, the derivative does not exist.

(12) **(C).** $\lim_{x\to k^-} f(x) = 3 + n\, f(k) = 3$

$f(k^-) > f(k)$ and $f(k^+) > f(k)$; $a^2 - 2 + 1 > 3$

$|a| > 2$

(13) **(C).** [sin x] will be discontinuous at those points, where sin x becomes 0 and 1 and is continuous when sin x = – 1 or elsewhere.

Now sin x = 0 and 1.

If $x = n\pi$ or $x = (4n+1)\pi/2$

i.e., $x = \frac{\pi}{2}, \pi, 2\pi, \frac{5\pi}{2}, 3\pi$ and $\sin 2x = 0$ or 1

If $2x = n\pi$ or $2x = (4n+1)\frac{\pi}{2}$

$$2x = \frac{\pi}{2}, \pi, 2\pi, \frac{5\pi}{2}, 3\pi, 4\pi, \frac{9\pi}{2}, 5\pi, 6\pi$$

$$\Rightarrow x = \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, 2\pi, \frac{9\pi}{4}, \frac{5\pi}{2}, 3\pi$$

$\Rightarrow$ [sin x] + [sin 2x] is discontinuous at 9 points.

(14) **(C).** $f(x) = (k-1)\tan \pi x$; $x<k$

$f'(x) = (k-1)\pi \sec^2 \pi x$; $x<k$

$f(k^-) = (k-1)(-1)^{2k}\pi$; $x<k$

(15) **(D).** Continuity $f(2^+) = 2 + 2\sin(0) = 2$

$f(2^-) = 3 + 2\sin(0) = 3$ discontinuous at x = 2

At x = 0, $f(0^+) = 2(0) - 0 - 0 \times \sin(0[0]) = 0$

$f(0^-) = 2(0) - (-1) + 0 \times \sin(0-(-1))$

Discontinuous at x = 0

(16) **(C).** $f\left(\frac{\pi}{4}\right) = \lim_{x\to\pi/4}\frac{1-\tan x}{4x-\pi} = \frac{0}{0}$ form

$$= \lim_{x\to\pi/4}\frac{-\sec^2 x}{4} = \frac{-\sec^2 \pi/4}{4} = -\frac{2}{4} = -\frac{1}{2}$$

(17) **(D).** $f(x) = \begin{cases} x-3 & \text{if } x \ge 3 \\ 3-x & \text{if } 1 \le x < 3 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4} & \text{if } x < 1 \end{cases}$

$f'(1^+) = \text{Limit}_{h\to 0}\frac{f(1+h)-f(1)}{h}$

$= \text{Limit}_{h\to 0}\frac{3-(1+h)-2}{h} = -1$

$f'(1^-) = \text{Limit}_{h\to 0}\frac{\frac{(1-h)^2}{4} - \frac{3}{2}(1-h) + \frac{13}{4} - 2}{-h}$

$= \text{Limit}_{h\to 0}\frac{(1-h)^2 - 6(1-h) + 5}{-4h}$

$= \text{Limit}_{h\to 0}\frac{h^2 - 2h + 6h}{-4h} = -1$

$\Rightarrow$ f is continuous at x = 1

(18) **(A).** $h'(x) = 2f(x)f'(x) + 2g(x)g'(x)$

$= 2f(x)g(x) + 2g(x)f''(x)$

$= 2f(x)g(x) - 2f(x)g(x) = 0$ $[\because f''(x) = -f(x)]$

$\Rightarrow h(x) = c \Rightarrow h(10) = h(5) = 11$

(19) **(B).** $\because f(x) = \begin{cases} x^2, & x \ge 0 \\ -x^2, & x < 0 \end{cases}$

$\Rightarrow f'(x) = 2x$, when $x > 0$ and $f'(x) = -2x$, when $x < 0$

Also $f'(0+0) = 0$, $f'(0-0) = 0 \Rightarrow f'(0) = 0$

$\therefore f'(x) = \begin{cases} 2x & , x>0 \\ 0 & , x=0 \\ -2x & , x<0 \end{cases} \Rightarrow f''(x) = \begin{cases} 2 & , x>0 \\ -2 & , x<0 \end{cases}$

Also $f''(0+0)=2, f''(0-0)=-2 \Rightarrow f''(0)$ does not exist.
Hence f(x) is twice differentiable in R_0

(20) **(B).** We have, $f(x) = \dfrac{1}{1-x}$.
As at x = 1, f(x) is not defined, x = 1 is a point of discontinuity of f(x).

If $x \neq 1, f[f(x)] = f\left(\dfrac{1}{1-x}\right) = \dfrac{1}{1-1/(1-x)} = \dfrac{x-1}{x}$

$\therefore x = 0, 1$ are points of discontinuity of $f[f(x)]$.
If $x \neq 0, x \neq 1$

$f[f\{f(x)\}] = f\left(\dfrac{x-1}{x}\right) = \dfrac{1}{1-\dfrac{(x-1)}{x}} = x$.

(21) **(D).** We have $f(x) = \begin{cases} \dfrac{-1}{x-1}, 0<x<1 \\ \dfrac{1-1}{x-1}, 1<x<2 \\ 0 \quad , x=1 \end{cases}$

$\lim\limits_{h\to 0} f(1-h) = \lim\limits_{h\to 0} \dfrac{-1}{(1-h)-1} = \lim\limits_{h\to 0} \dfrac{1}{h} = \infty$

$\therefore$ f (x) is not continuous and hence not differentiable at x = 1.

(22) **(C).** $L f'(0) = \lim\limits_{h\to 0} \dfrac{f(0-h)-f(0)}{-h} = \lim\limits_{h\to 0} \dfrac{\sqrt{1-\sqrt{1-h^2}}}{-h}$

$= \lim\limits_{h\to 0} \dfrac{\sqrt{1-\sqrt{1-h^2}}}{-h} \times \dfrac{1}{\sqrt{1+\sqrt{1-h^2}}}$

$= \lim\limits_{h\to 0} \dfrac{-1}{\sqrt{1+\sqrt{1-h^2}}} = \dfrac{-1}{\sqrt{2}}$

$R f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{1-\sqrt{1-h^2}}}{h}$

$= \lim\limits_{h\to 0} \dfrac{1}{\sqrt{1+\sqrt{1-h^2}}} = \dfrac{1}{\sqrt{2}}$

Therefore, f(x) is not differentiable at x = 0.
Since L f '(0) and R f '(0) are finite therefore, f(x) is continuous at x = 0.
Hence f (x) is continuous but not differentiable at x = 0.

(23) **(D).** For x = 0, f(0) = 1

For x = 1, f(0) = 1 ; $\lim\limits_{x\to 0^+} f(x) = x+1-x = 1$

$\lim\limits_{x\to 0^-} f(x) = -x+1-x = 1-2x = 1$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} x+x-1 = 1$

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} x+1-x = 1$

$\Rightarrow$ f (x) is continous at x = 0 and x = 1 also.

(24) **(A).** $\lim\limits_{x\to 2^-} g(x) = \lim\limits_{x\to 2^-} cx^2+d = 4c+d$

$\lim\limits_{x\to 2^+} g(x) = \lim\limits_{x\to 2^+} (dx+3-c) = 2d+3-c$

$g(2) = 4c+d \quad \therefore \ 4c+d = 2d+3-c \quad \therefore \ d = 5c-3$

(25) **(C).** Obviously, $f(x) = \begin{cases} x^3, & x \le 1 \\ 1 \ , & x>1 \end{cases}$

$\because$ f (x) = 1 = f (1), so f (x) is continuous at x = 1 and as such f (x) is continuous $\forall\ x \in R$
Further, we note that f (1 – 0) = 3 and f' (1 + 0) = 0
$\Rightarrow$ f (x) is not differentiable at x = 1
Also f' (x) exists $\forall\ x \in R$. x ≠ 1. Hence (C) are correct.

(26) **(C).** $-\lim\limits_{x\to\infty} 2\sin\dfrac{\sqrt{x+1}+\sqrt{x}}{2}\sin\dfrac{\sqrt{x+1}-\sqrt{x}}{2}$

$= \lim\limits_{x\to\infty} 2\sin\dfrac{\sqrt{x+1}+\sqrt{x}}{2} \sin\dfrac{\dfrac{\sqrt{x+1}-\sqrt{x}}{2}}{\dfrac{\sqrt{x+1}-\sqrt{x}}{2}}\left(\dfrac{\sqrt{x+1}+\sqrt{x}}{2}\right)$

$= (\text{finite}) \times 1 \times \text{zero} = 0 \qquad \left(\because \lim\limits_{x\to\infty} \dfrac{\sqrt{x+1}-\sqrt{x}}{2} = 0\right)$

(27) **(D).** Put $x = \dfrac{1}{y}$; limit = $\lim\limits_{y\to 0} \dfrac{y^2+5y+3}{-y^2+3y+7} = \dfrac{3}{7}$

Shortcut :

$\lim\limits_{x\to\infty} \dfrac{3x^2+5x+1}{7x^2+3x-1}\left(\dfrac{\infty}{\infty}\right) = \dfrac{\text{coeff. of highest power}}{\text{coeff. of highest power}} = \dfrac{3}{7}$

(28) **(D).** $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x^2} \cdot \dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$

$\Rightarrow \lim\limits_{x\to 0} \dfrac{1+x^2-1+x^2}{x^2\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}$

$\Rightarrow \lim\limits_{x\to 0} \dfrac{2x^2}{x^2\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)} = \dfrac{2}{\sqrt{1}+\sqrt{1}} = \dfrac{2}{2} = 1$

(29) **(C).** $\lim_{n\to\infty} \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+....+\frac{1}{n(n+1)}$

$\Rightarrow$

$$\lim_{n\to\infty}\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+....+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$\Rightarrow \lim_{n\to\infty} 1-\frac{1}{(n+1)} \Rightarrow 1-\frac{1}{\infty}=1$$

(30) **(A).** Let $A = \lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{1/x}$

$$\Rightarrow \log A = \lim_{x\to 0}\frac{\log(a^x+b^x)-\log 2}{x} \quad \left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to 0}\frac{a^x\log a + b^x \log b}{a^x+b^x} = \frac{1}{2}\log ab \therefore A = \sqrt{ab}$$

(31) **(C).** This is in 1^∞ form.So

$$\text{Limit} = \lim_{x\to\infty}\left[1+\left(\frac{x+5}{x-1}-1\right)\right]^x$$

$$= \lim_{x\to\infty}\left[1+\frac{6}{x-1}\right]^x = e^{\lim_{x\to\infty}\left[\frac{6}{x-1}\right]x} = e^{\lim_{x\to\infty}\left[\frac{6}{1-1/x}\right]} = e^6$$

(32) **(D).** $\lim_{x\to 1^-} f(x) = \lim_{h\to 0}\frac{2}{\pi}\cot^{-1}\left(\frac{3(1-h)^2+1}{(-h)(-h-1)}\right) = 0$

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 0}\frac{2}{\pi}\cot^{-1}\left(\frac{3(2-h)^2+1}{(1-h)(-h)}\right) = \frac{2}{\pi}.\pi = 2$$

$\therefore$ Quadratic equation whose roots are 0 and 2 is $x^2-2x=0$

(33) **(B).** Put $\cos^{-1}x = y \Rightarrow x = \cos y$ and as $x\to -1$; $y\to\pi$

$$\therefore \text{limit} = \lim_{y\to\pi}\frac{\sqrt{\pi}-\sqrt{y}}{\sqrt{1+\cos y}} ; \lim_{y\to\pi}\frac{\sqrt{\pi}-\sqrt{y}}{\sqrt{2}\cos\frac{y}{2}} = \frac{1}{\sqrt{2\pi}}$$

(34) **(D).** S-1 : f (x) = 2 + cos x $\to$ always continuous and differentiable $f(t) = 2+\cos t$

$f(t+\pi) = 2+\cos(\pi+t) = 2-\cos t$

$\because f(t) \ne f(\pi+t) \Rightarrow$ Rolle's theorem is not applicable

$\Rightarrow$ Statement 1 is false.

S-2 : $f(t) = 2+\cos t$

$f(2\pi+t) = 2+\cos(2\pi+t) = 2+\cos t$

$\Rightarrow f(t) = f(2\pi+t)$

(35) **(A).** $\lim_{x\to 0^+}(\sin x+[x]) = 0$, $\lim_{x\to 0^-}(\sin x+[x]) = -1$

Thus, limit does not exist, hence f (x) is discontinuous at x = 0. S 2 is fundamental property and is a correct explanation of statement 1.

(36) (C), (37) (A).

(i) $f(-x) = (-x-3)\frac{10^{\frac{1}{x-3}}+1}{10^{\frac{1}{x-3}}-1} = -(x+3)\frac{10^{\frac{1}{-(x+3)}}+1}{10^{\frac{1}{-(x-3)}}-1}$

which is neither equal to f (x) nor equal to – f (x)

$\Rightarrow$ f (x) is neither even nor odd.

(ii) It is evident that f (x) is continuous for all x except possible at x = 3.

$$\text{At } x=3,\ \text{RHL} = \lim_{x\to 0}(3+h-3)\frac{10^{\frac{1}{3+h-3}}+1}{10^{\frac{1}{3+h-3}}-1}$$

$$= \lim_{x\to 0}\frac{10^{\frac{1}{h}}+1}{10^{\frac{1}{h}}-1} = \lim_{x\to 0}\frac{1+10^{\frac{1}{h}}}{1-10^{\frac{1}{h}}} = 0$$

$$\text{LHL} = \lim_{x\to 0}(3-h-3)\frac{10^{\frac{1}{3-h-3}}+1}{10^{\frac{1}{3-h-3}}-1} = \lim_{x\to 0}(-h)\frac{10^{-\frac{1}{h}}+1}{10^{-\frac{1}{h}}-1} = 0$$

Since f (3) = 0 (given). f (x) is continuous at x = 3

(38) (D), (39) (C), (40) (B).

$$f_1(x) = 2\sum_{r=1}^{n}\frac{\sin\left(\left(x+\frac{r\pi}{6}\right)-\left(x+(r-1)\frac{\pi}{6}\right)\right)}{\cos\left(x+(r-1)\frac{\pi}{6}\right).\cos\left(x+\frac{r\pi}{6}\right)}$$

$$= 2\left[\left(\tan\left(x+\frac{\pi}{6}\right)-\tan x\right)+\left(\tan\left(x+\frac{2\pi}{6}\right)-\tan\left(x+\frac{\pi}{6}\right)\right) + \left(\tan\left(x+\frac{3\pi}{6}\right)-\tan\left(x+\frac{2\pi}{6}\right)\right) + ... + \left(\tan\left(x+\frac{n\pi}{6}\right)-\tan\left(x+(n-1)\frac{\pi}{6}\right)\right)\right]$$

$$\Rightarrow f_1(x) = 2\left(\tan\left(x+\frac{n\pi}{6}\right)-\tan x\right)$$

$$\text{for } n=3,\ f_1(x) = 2\left(\tan\left(\frac{\pi}{2}+x\right)-\tan x\right)$$

$$= 2(-\cot x-\tan x) = -2\frac{1}{\sin x\cos x}$$

$$2f_2(x) = f_1(x) - 2\tan\left(x+\frac{n\pi}{6}\right)$$

$$= 2\tan\left(x+\frac{n\pi}{6}\right) - 2\tan x - 2\tan\left(x+\frac{n\pi}{6}\right)$$

$\therefore\ f_2(x) = -\tan x$

$f_3(x) = -f_2(x)$, so

$f_3(x) = \tan x$

Now, $f_4(x) = \begin{cases} \dfrac{e^{(e^x-1)}-1}{e^{2(e^x-1)}} & ; x < 0 \\ k_1 & ; x = 0 \\ (1+|\tan x|)^{\frac{k_2}{\tan x}} & ; x > 0 \end{cases}$

Clearly, $f(0^-) = e^{\frac{1}{2}\ln e} = \sqrt{e}$

$f(0^-) = e^{k_2}\ \&\ f(0) = k_1$

(i) As $f_4(x)$ is continuous at x = 0, so be definition of $f(0^-) = f(0^+) = f(0)$

$\sqrt{e} = e^{k_2} = k_1 \quad \therefore\ k_1 = \sqrt{e}\ \&\ k_2 = \dfrac{1}{2}$

(ii) As $y = f_3(x) = \tan x$

Clearly $f_3(x)$ is continuous as well as derivable everywhere in $(0, \pi/2)$

(iii) $n = 3,\ f_1(x) = -4$

$\Rightarrow\ -2(\tan x + \cot x) = -4$

$\Rightarrow\ \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x} = 2$

$\Rightarrow\ 1 = 2\sin x \cdot \cos x \qquad \therefore\ x = \dfrac{\pi}{4} \in \left(0, \dfrac{\pi}{2}\right)$

(41) **2.** $y = [\sin x] = \begin{cases} 0, & 0 \le x < \dfrac{\pi}{2} \\ 1, & x = \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} < x \le \pi \\ -1, & \pi < x < 2\pi \\ 0, & x = 2\pi \end{cases}$

$\therefore$ position of discontinuity are $\pi/2, \pi$

(42) **5.** We have, $\displaystyle\lim_{n\to\infty} \frac{n.3^n}{n(x-2)^n + n.3^{n+1} - 3^n} = \frac{1}{3}$

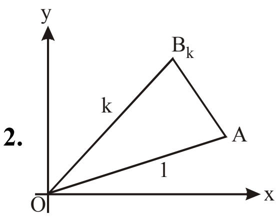
So, $\displaystyle\lim_{n\to\infty} \frac{1}{\left(\frac{x-2}{3}\right)^n + 3 - \frac{1}{n}} = \frac{1}{3}$

Clearly, $-1 < \dfrac{x-2}{3} < 1 \Rightarrow -1 < x < 5$

(43) **2.** $\displaystyle I = \lim_{n\to\infty} \Sigma \frac{1}{n}\tan\frac{\pi}{4}\frac{r}{n} = \int_0^1 \tan\frac{\pi}{4}x\,dx$

$\displaystyle = \frac{4}{\pi}\int_0^{\pi/4} \tan t\,dt = \frac{4}{\pi}[\log\sec t]_0^{\pi/4} = \frac{4}{\pi}\log\sqrt{2} = \frac{2}{\pi}\log 2$

(44) **2.**



$OB_k = k$

$\angle AOB_k = \dfrac{k\pi}{2n}$; $S_k = \dfrac{1}{2}k\sin\dfrac{k\pi}{2n}$ (Using $\Delta = \dfrac{1}{2}ab\sin\theta$)

$\displaystyle\therefore\ L = \frac{k}{2n^2}\sum_{n=1}^{\infty}\sin\frac{k\pi}{2n} = \frac{1}{2n}\sum_{n=1}^{\infty}\sin\frac{k\pi}{2n} = \frac{1}{2}\int_0^1 x.\sin\frac{\pi x}{2}dx$

$\displaystyle = \frac{1}{2}\left[\underbrace{-\frac{2}{\pi}x\cos\frac{\pi x}{2}\Big|_0^1}_{\text{zero}} + \frac{2}{\pi}\int_0^1\cos\frac{\pi x}{2}dx\right]$

$\displaystyle = \frac{1}{2}\left[0 + \frac{2}{\pi}.\frac{2}{\pi}\left(\sin\frac{\pi x}{2}\right)_0^1\right] = \frac{2}{\pi^2}$

(45) **1.** $\displaystyle f(0) = \lim_{x\to 0}\frac{\left[3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \ldots\right] + A\left[2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \ldots\right] + B\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right]}{x^5}$ .... (1)

Now, $f(x)$ is continuous at $x = 0$, so we must have

$2A + 3 + B = 0$ .... (2)

and $\dfrac{27}{6} + \dfrac{8A}{6} + \dfrac{B}{6} = 0 \Rightarrow 8A + B = -27$ .... (3)

$\therefore$ On solving (2) and (3), we get $A = -4, B = 5$

Hence, $f(0) = 1$

Alternatively : We have $f(0)$

$\displaystyle = \lim_{x\to 0}\frac{3\sin x - 4\sin^3 x + 2A\sin x\cos x + B\sin x}{x^5}\left(\frac{0}{0}\right)$ form ..... (1)

$[N^r = 3 + 2A + B = 0]$

$\displaystyle = \lim_{x\to 0}\frac{3\sin x - 4\sin^3 x + 2A\sin x\cos x - (3+2A)\sin x}{x^5}\left(\frac{0}{0}\right)$ form

$$= \lim_{x\to 0} \frac{\sin x}{x}\left[\frac{3-4\sin^2 x+2A\cos x-3-2A}{x^4}\right]\left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x\to 0} \frac{-2A\,(1-\cos x)-4\sin^2 x}{x^4}\left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x\to 0} \frac{-4A\sin^2\frac{x}{2}-4\sin^2 x}{x^4}$$

$$= \lim_{x\to 0} \frac{-4A\sin^2\frac{x}{2}-16\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{x^4}$$

$$= -\lim_{x\to 0} \frac{\sin^2\frac{x}{2}}{x^4/4}\left[\frac{A+4\cos^2\frac{x}{2}}{x^2}\right]$$

$\therefore\ A+4=0 \Rightarrow A=-4 \Rightarrow B=5$

Also, $f(0)=1$

(46) **5.** Let given limit = L, then

$$L = \lim_{n\to\infty}\left(\frac{1}{2n+1}+\frac{1}{2n+2}+\frac{1}{2n+3}+.....+\frac{1}{4n}\right)$$

$$-\lim_{n\to\infty}\left(\frac{1}{2n+2}+\frac{1}{2n+4}+\frac{1}{2n+6}+.....+\frac{1}{4n}\right)$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=1}^{2n}\frac{n}{2n+r}-\frac{1}{n}\sum_{r=1}^{2n}\frac{n}{2n+2r}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=1}^{2n}\frac{1}{2+\frac{r}{n}}-\frac{1}{n}\sum_{r=1}^{n}\frac{1}{2+2\,(r/n)}\right]$$

$$= \int_0^2\frac{1}{2+x}dx-\int_0^1\frac{1}{2+2x}dx$$

$$= [\ln(2+x)]_0^2-\frac{1}{2}[\ln(1+x)]_0^1 = \ln 4-\ln 2-\frac{1}{2}\ln 2$$

$$= \left(2-\frac{3}{2}\right)\ln 2 = \frac{1}{2}\ln 2 = \frac{A}{B}\ln C$$

Hence least value of $A+B+C = 1+2+2 = 5$

(47) **167.** For limit to exist and equal to 1

Coefficient of x^4 in denominator = 0

(as degree of x in $D^r > N^r$)

Now degree of x in D^r is 2 and degree of x in N^r is 3

$\therefore$ coefficient of x^3 in $N^r = 0$ otherwise $L \neq 1$ and will be

infinity and $\dfrac{\text{coefficient of } x^2 \text{ in } N^r}{\text{coefficient of } x^2 \text{ in } D^r} = 1$

now coefficient of x^4 in $D^r = 5a-b+4c=0$ ....(1)

coefficient of x^3 in $N^r = 2a+b-3c=0$ ....(2)

$$\frac{\text{coefficiently of } x^2 \text{ in } N^r}{\text{coefficiently of } x^2 \text{ in } D^r} = 1 \Rightarrow \frac{-a+5b-c}{2} = 1$$

$\therefore\ a-5b+c+2=0$ ....(3)

Solving (1), (2) and (3) we get

$$a = \frac{-2}{109},\ b = \frac{46}{109} \text{ and } c = \frac{14}{109}$$

$$\Rightarrow a+b+c = \frac{58}{109} = \frac{p}{q} \Rightarrow p+q = 167$$

(48) **22.** Using L'Hospital's rule

$$\lim_{x\to 0}\frac{\begin{array}{l}5\cos^4 x(\sin x)\cos^3 2x\cos^3 3x\\ +3\cos^2 2x\,(2\sin 2x)\cos^5 x\cos^3 3x\\ +3\cos^2 3x\,(3\sin 3x)\cos^5 x\cos^3 2x\end{array}}{2x}$$

$$= \frac{5}{2}+6+\frac{9}{2}\times 3 = 22$$

Alternatively : Using expansion :

$\cos^5 x\cos^3 3x\cos^3 3x$

$$= \left(1-\frac{x^2}{2!}+....\right)^5\cdot\left(1-\frac{(2x)^2}{2!}+....\right)^3\cdot\left(1-\frac{9x^2}{2!}+....\right)^3$$

$$= \left(1-\frac{5x^2}{2}+....\right)\left(1-6x^2+....\right)\left(1-\frac{27x^2}{2}+....\right)$$

$$= \left(1-\left(6+\frac{5}{2}\right)x^2+....\right)\left(1-\frac{27x^2}{2}+....\right)$$

$$= 1-\frac{27x^2}{2}-6x^2-\frac{5x^2}{2}+...... = 1-22x^2+........$$

$$\therefore \lim_{x\to 0}\frac{1-\cos^5 x.\cos^3 2x.\cos^3 3x}{x^2}$$

$$= \lim_{x\to 0}\frac{1-(1-22x^2+......)}{x^2} = 22$$

(49) **3.** Given that, $\displaystyle\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$

= finite non-zero number

$$= \lim_{x\to 0}\frac{(\cos x-1)(1+\cos x)(e^x-\cos x)}{x^n\,(1+\cos x)}$$

$$= \lim_{x\to 0}\left(\frac{\sin^2 x}{x}\right)\cdot\left(\frac{e^x-\cos x}{x^{n-2}}\right)\cdot\left(\frac{1}{1+\cos x}\right)$$

$$= 1^2 \cdot \frac{1}{2} \lim_{x\to 0} \frac{\left[1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+...\infty\right]\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+...\infty\right]}{x^{n-2}}$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{x\left(1+x+\frac{x^2}{3!}+\frac{2x^3}{4!}+.....\infty\right)}{x^{n-2}}$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{\left(1+x+\frac{x^2}{3!}+\frac{2x^3}{4!}+.....\infty\right)}{x^{n-3}}$$

For this limit to be finite $n - 3 = 0 \Rightarrow n = 3$

(50) **1.** $\lim_{x\to 0} [(\sin x)^{1/x} + (1/x)^{\sin x}]$

$$= \lim_{x\to 0} (\sin x)^{1/x} + \lim_{x\to 0} (1/x)^{\sin x}$$

$$= 0 + e^{\lim_{x\to 0} \sin x \log(1/x)} \quad [\because |\sin x| < 1 \text{ when } x \to 0]$$

$$= e^{\lim_{x\to 0} \frac{-\log x}{\operatorname{cosec} x}} = e^{\lim_{x\to 0} \frac{-1/x}{-\operatorname{cosec} x \cot x}}$$

[Using L'Hospital rule]

$$= e^{\lim_{x\to 0} \frac{\sin x}{x} \cdot \tan x} = e^0 = 1$$

(51) **0.** $p(x) = ax^4 + bx^3 + cx^2 + dx + e$

$p'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$p'(1) = 4a + 3b + 2c + d = 0$ .....(i)

$p'(2) = 32a + 12b + 4c + d = 0$ .....(ii)

$$\lim_{x\to 0}\left(1+\frac{p(x)}{x^2}\right) = 2$$

$$\lim_{x\to 0} \frac{ax^4 + bx^3 + (c+1)x^2 + dx + e}{x^2} = 2$$

$c + 1 = 2, d = 0, e = 0$

$c = 1$

Now equation (i) and (ii) are

$4a + 3b = -2$ and $32a + 12b = -4$

$\Rightarrow a = 1/4$ and $b = -1$

(52) **2.** $\lim_{x\to 1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}}$

$$\lim_{x\to 1}\left\{\frac{a(1-x)+\sin(x-1)}{(x-1)+\sin(x-1)}\right\}^{1+\sqrt{x}}$$

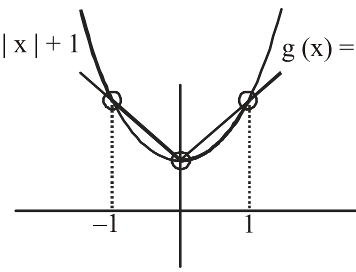
$$\lim_{x\to 1}\left\{\frac{-a+\frac{\sin(x-1)}{x-1}}{1+\frac{\sin(x-1)}{x-1}}\right\}^{1+\sqrt{x}}$$

$$\left\{\frac{-a+1}{2}\right\}^2 = \frac{1}{4}\ ;\ \frac{-a+1}{2} = \pm\frac{1}{2}$$

$-a + 1 = \pm 1 \Rightarrow a = 0, a = 2$

$\therefore\ a = 2$

(53) **3.**



(54) **2.** $m \ge 2$ and $n \ge 2$

$$= \lim_{a\to 0} \frac{e\,(e^{\cos(a^n)-1}-1)}{(\cos(a^n)-1)} \times \frac{\cos(a^n)-1)}{(a^n)^2} \frac{a^{2n}}{a^m}$$

$$= e \times \lim_{a\to 0}\left(\frac{e^{\cos(a^n)-1}-1)}{(\cos(a^n)-1)}\right) \times \lim_{a\to 0}\left(\frac{\cos(a^n)-1)}{a^{2n}}\right) \times \lim_{a\to 0} a^{2n-m}$$

$$= e \times 1 \times -\frac{1}{2} \times \lim_{a\to 0} a^{2n-m}$$

Now, $\lim_{a\to 0} a^{2n-m}$ must be equal to 1.

i.e., $2n - m = 0$

$$\frac{m}{n} = 2$$

(55) **7.** $\lim_{x\to 0} \frac{x^2 \sin(\beta x)}{\alpha x - \sin x} \equiv 1$; $\lim_{x\to 0} \frac{\beta x^3}{\alpha x - \sin x} \equiv 1$

$\alpha \equiv 1$; $6\beta \equiv 1 \Rightarrow \beta = 1/6$

$$6(\alpha+\beta) = 6\left(1+\frac{1}{6}\right) \equiv 7$$

(56) **1.** $\text{Limit}_{x\to\infty} \frac{\cot^{-1}\left(\frac{\log_a x}{x^a}\right)}{\sec^{-1}\left(\frac{a^x}{\log_a x}\right)}$

as $_{x\to\infty} \left(\frac{\log_a x}{x^a}\right) \to 0$ & $\left(\frac{a^x}{\log_a x}\right) \to \infty$ (using L'opital rule)

$$\therefore\ l = \frac{\pi/2}{\pi/2} = 1$$

(57) **1.** $\text{Limit}_{x\to\infty} \sqrt{x+1}-\sqrt{x} = 0 \Rightarrow \cot^{-1}(0)=\pi/2$

$\text{Limit}_{x\to\infty}\left(\frac{2x+1}{x-1}\right)^x = \infty \Rightarrow \sec^{-1}(\infty)=\pi/2 \quad \therefore\ l=1$

EXERCISE-3

(1) **(A).** $f(1)=1, f'(1)=2$

$$\lim_{x\to1}\frac{\sqrt{f(x)}-1}{\sqrt{x}-1}=\lim_{x\to1}\frac{\sqrt{f(x)}-1}{\sqrt{x}-1}\times\frac{\sqrt{x}+1}{\sqrt{x}+1}\times\frac{\sqrt{f(x)}+1}{\sqrt{f(x)}+1}$$

$$=\lim_{x\to1}\frac{f(x)-1}{x-1}\times\frac{\sqrt{x}+1}{\sqrt{f(x)}+1}$$

$$=\lim_{x\to1}\frac{f(x)-f(1)}{x-1}\times\frac{\sqrt{x}+1}{\sqrt{f(x)}+1}\quad\{\because\ f(1)=1\}$$

$$=\left(\lim_{x\to1}\frac{f(x)-f(1)}{x-1}\right)\times\lim_{x\to1}\left(\frac{\sqrt{x}+1}{\sqrt{f(x)}+1}\right)$$

$$=f'(1)\times\left(\frac{1+1}{1+1}\right)=2\times1=2$$

(2) **(A).** $\lim_{x\to0}\frac{(1-\cos2x)\sin5x}{x^2\sin3x}=\lim_{x\to0}\frac{2\sin^2x\sin5x}{x^2\sin3x}$

$$=\lim_{x\to0}2\left(\frac{\sin x}{x}\right)^2\frac{\sin5x}{5x}\times\frac{3x}{\sin3x}\times\frac{5}{3}$$

Applying limit $=2\times(1)^2\times1\times1\times\frac{5}{3}=\frac{10}{3}$

(3) **(A).** $\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+3}\right)^x\ (1^\infty\text{ form})$

$$=e^{\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+3}-1\right)\times x}=e^{\lim_{x\to\infty}\left(\frac{x^2+5x+3-x^2-x-3}{x^2+x+3}\right)\times x}$$

$$=e^{\lim_{x\to\infty}\left(\frac{4x}{x^2+x+3}\right)x}=e^{\lim_{x\to\infty}\frac{x^2}{x^2}\left(\frac{4}{1^2+1/x+3/x^2}\right)}$$

$$=e^{\lim_{x\to\infty}\frac{4}{1+\frac{1}{x}+\frac{3}{x^2}}}=e^4$$

(4) **(A).** $\lim_{x\to\infty}\frac{\log x^n-[x]}{[x]}, n\in N$

$\because\ x\to\infty;\ [x]=x$

$$\therefore\ \lim_{x\to\infty}\frac{n\log x-x}{x}=\lim_{x\to\infty}\frac{n\log x}{x}-1$$

Applying D.L. Hospital rule

$$\Rightarrow\ \lim_{x\to\infty}\frac{n.\frac{1}{x}}{1}-1\Rightarrow0-1=-1$$

(5) **(A).** $f(x)=\begin{cases}x, & x\in Q\\ -x, & x\notin Q\end{cases}$

at $x=0$

Let neighbourbodd of 0 is irrational

$\lim_{h\to0}f(0+h)=\lim_{h\to0}(-(0+h))=0$

$\lim_{h\to0}f(0-h)=\lim_{h\to0}-(0-h)=0$ and $f(0)=0$

$\therefore$ L.H.L. = R.H.L. = $f(0)=0$

& if neighbourhood of 0 is rational

$\therefore\ \lim_{h\to0}f(0+h)=\lim_{h\to0}(0+h)=0$

$\lim_{h\to0}f(0-h)=\lim_{h\to0}(0-h)=0\ \&\ f(0)=0$

L.H.L. = R.H.L. = $f(0)=0$

$\therefore$ function is continuous at $x=0$

(6) **(C).** $f(x)=\begin{cases}xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & ;\ x\neq0\\ 0 & ;\ x=0\end{cases}$

$$=\begin{cases}xe^{-\left(\frac{1}{x}+\frac{1}{x}\right)} & ;\ x<0\\ xe^{-\left(\frac{1}{x}+\frac{1}{x}\right)} & ;\ x>0\\ 0 & ;\ x=0\end{cases}=\begin{cases}x & ;\ x<0\\ xe^{-2/x} & ;\ x>0\\ 0 & ;\ x=0\end{cases}$$

$f(x)$ is a differentiable as well as continuous everywhwere except possibly at $x=0$

$$\therefore\ Lf'(0)=\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^-}\frac{x}{x}=1$$

$$Rf'(0)=\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0}\frac{x.e^{-2/x}}{x}=e^{-\infty}=0$$

$\therefore\ Lf'(0)\neq Rf'(0)$

$\therefore\ f(x)$ is not differentiable at $x=0$

Again $\lim_{x\to0^-}f(x)=\lim_{x\to0^-}x=0$

$\therefore\ \lim_{x\to0^+}f(x)=\lim_{x\to0^+}x.e^{-2/x}=0$

$\therefore\ \lim_{x\to0^-}f(x)=0=f(0)$

$\therefore\ f(x)$ is continuous at $x=0$

(7) **(D).** $\lim_{x\to 0}\frac{\log(3+x)-\log(3-x)}{x}=k$

$\Rightarrow \lim_{x\to 0}\frac{\log 3(1+x/3)-\log 3(1-x/3)}{x}=k$

$\Rightarrow \lim_{x\to 0}\frac{\log 3+\log(1+x/3)-\log 3-\log(1-x/3)}{x}=k$

$\Rightarrow \lim_{x\to 0}\frac{\log(1+x/3)-\log(1-x/3)}{x}=k$

$\Rightarrow \lim_{x\to 0}\frac{\log(1+x/3)}{x}-\lim_{x\to 0}\frac{\log(1-x/3)}{x}=k$

$\Rightarrow \lim_{x\to 0}\frac{\log(1+x/3)}{3.\frac{x}{3}}-\lim_{x\to 0}\frac{\log(1-x/3)}{-3\left(\frac{x}{-3}\right)}=k$

$\Rightarrow \frac{1}{3}-\left(-\frac{1}{3}\right)=k \quad \left\{\because \lim_{x\to 0}\frac{\log(1+x)}{x}=1\right\}$

$\Rightarrow \frac{2}{3}=k$

(8) **(B).** $\lim_{x\to a}\frac{f(a)\,g(x)-f(a)-g(a)f(x)+g(a)}{g(x)-f(x)}=4$

$\because f(a)=g(a)=k$ .... (1)

$\therefore \lim_{x\to a}\frac{k\,g(x)-k-k\,f(x)+k}{g(x)-f(x)}=4$ {from (1)}

$\Rightarrow \lim_{x\to a}\frac{k\{g(x)-f(x)\}}{g(x)-f(x)}=4 \Rightarrow k=4$

(9) **(D).** $\lim_{x\to\pi/2}\frac{(1-\tan x/2)(1-\sin x)}{(1+\tan x/2)(\pi-2x)^3}$

Let $x=\frac{\pi}{2}-h$ if $x\to\pi/2$ then $h\to 0$

$\Rightarrow \lim_{h\to 0}\frac{1-\tan(\pi/4-h/2)}{1-\tan(\pi/4-h/2)}\,\frac{1-\sin(\pi/2-h)}{[\pi-2(\pi/2-h)]^3}$

$\Rightarrow \lim_{h\to 0}\tan\left(\frac{\pi}{4}-\frac{\pi}{4}+\frac{h}{2}\right)\frac{1-\cos h}{(2h)^3}$

$\Rightarrow \lim_{h\to 0}\tan\frac{h}{2}\,\frac{2\sin^2 h/2}{8h^3}$

$\Rightarrow \frac{1}{4}\lim_{h\to 0}\frac{\tan h/2}{2h/2}\left(\frac{\sin h/2}{h/2}\right)\times\frac{1}{4}=\frac{1}{4}\times\frac{1}{2}\times\frac{1}{4}=\frac{1}{32}$

(10) **(B).** $\lim_{x\to\infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x}=e^2$

$\Rightarrow e^{\lim_{x\to\infty}\left(\frac{a}{x}+\frac{b}{x^2}\right)\times 2x}=e^2$

$\Rightarrow e^{\lim_{x\to\infty}\left(a+\frac{b}{x}\right)\times 2}=e^2 = e^{2a}=e^2$

$\Rightarrow 2a=2 \Rightarrow a=1$

and b can take any real value

$\therefore$ a = 1 and $b\in R$

(11) **(C).** Since f(x) is continous in $[0,\pi/2]$

$\therefore$ it is continuous at $x=\pi/4$

$\therefore f(\pi/4)=\lim_{x\to\pi/4}f(x)=\lim_{x\to\pi/4}\frac{1-\tan x}{4x-\pi}\left(\frac{0}{0}\text{form}\right)$

Applying D.L. hospital rule

$=\lim_{x\to\pi/4}\frac{-\sec^2 x}{4}=\frac{-2}{4}=-\frac{1}{2}$

(12) **(B).** $|f(x)-f(y)|\le(x-y)^2$

$\Rightarrow \frac{|f(x)-f(y)|}{|x-y|}\le|x-y|$

$\Rightarrow \lim_{x\to y}\frac{|f(x)-f(y)|}{|x-y|}\le\lim_{x\to y}|x-y|$

$\Rightarrow |f'(y)|\le 0$ $\{\because |f'(y)|$ can't be –ve}

$\Rightarrow f'(y)=0$

$\Rightarrow$ f ' (y) is constant function but f (0) = 0 (given)

$\therefore f(y)=0$

$\Rightarrow f(1)=0$

(13) **(C).** $f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$

$=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}-\lim_{h\to 0}\frac{f(1)}{h} \quad \because \lim_{h\to 0}\frac{f(1+h)}{h}=5$

$\therefore \lim_{h\to 0}\frac{f(1)}{h}$ must be finite as f ' (1) exists and

$\lim_{h\to 0}\frac{f(1)}{h}=0 \qquad \therefore f'(1)=\lim_{h\to 0}\frac{f(1+h)}{h}=5$

(14) **(A).** $\because$ α and β are roots of equation $ax^2+bx+c=0$

$\therefore ax^2+bx+c=0$ .... (1)

$ax^2+bx+c=a(x-\alpha)(x-\beta)$

Now, $\lim_{x\to\infty}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$

$=\lim_{x\to\infty}\frac{2\sin^2\left(\frac{ax^2+bx+c}{2}\right)}{(x-\alpha)^2}$

$$= \lim_{x\to\infty} \frac{2\sin^2\left(\frac{a}{2}(x-\alpha)(x-\beta)\right)}{(x-\alpha)^2}$$

$$= 2\lim_{x\to\infty} \frac{\left[\sin\left(\frac{a}{2}(x-\alpha)(x-\beta)\right)\right]^2}{\frac{a^2}{4}(x-\alpha)^2(x-\beta)^2} \times \frac{a^2}{4}(x-\beta)^2$$

$$= 2.1.\frac{a^2}{4}(\alpha-\beta)^2 = \frac{a^2}{2}(\alpha-\beta)^2$$

(15) **(B).** Here $f(x) = \dfrac{x}{1+|x|}$

$\Rightarrow D_f = R$

and $$f'(x) = \frac{(1+|x|)\times 1 - x\left(0+\frac{x}{|x|}\right)}{(1+|x|)^2}$$

$$= \frac{1+|x| - \frac{x^2}{|x|}}{(1+|x|)^2} \quad \begin{cases} \because \text{ if } f(x) = |x| \text{ then } f'(x) = -1\,;\, x<0 \\ \text{and } \frac{x}{|x|} = 1;\ x>0 \\ = -1;\ x<0 \\ \because x^2 = |x|^2 \end{cases}$$

$$= \frac{1+|x| - \frac{|x|^2}{|x|}}{(1+|x|)^2} = \frac{1}{(1+|x|)^2}, \ \ldots\ldots\ldots\ldots \in R$$

(16) **(D).** $f(x) = \dfrac{1}{x} - \dfrac{2}{e^{2x}-1}$

$\because$ for function to be continuous at any point a

L.H.L. = R.H.L. = f(a)

or $\lim_{x\to a} f(x) = f(a)$

If function is continuous at x = 0

$$\therefore \lim_{x\to 0} \frac{1}{x} - \frac{2}{e^{2x}-1} = f(0)$$

$$f(0) = \lim_{x\to 0} \frac{e^{2x}-1-2x}{x(e^{2x}-1)} \left(\frac{0}{0}\text{ form}\right)$$

Applying D.L. hospital rule

$$= \lim_{x\to 0} \frac{2e^{2x}-2}{2xe^{2x}+(e^{2x}-1).1} \quad \left(\frac{0}{0}\text{ form}\right)$$

Again applying

$$= \lim_{x\to 0} \frac{4.e^{2x}}{2x.e^{2x}+2e^{2x}.1+2e^{2x}} = \frac{4.1}{2+2} = 1 \Rightarrow f(0) = 1$$

(17) **(C).** $f(x) = \text{Min}\{(x+1), |x|+1\}$

for $x \geq 0$, $x+1$ and $|x|+1$ are same

for $x < 0$

$x < |x|$

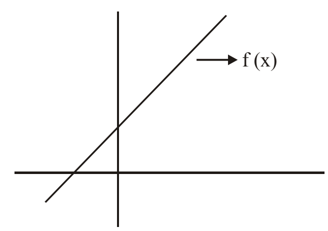
$\Rightarrow x+1 < |x|+1$

$$f(x) = \begin{cases} x+1, & x<0 \\ |x|+1, & x\geq 0 \end{cases} \qquad \{\text{if } x\geq 0\ |x| = x\}$$

$$f(x) = \begin{cases} x+1, & x<0 \\ x+1, & x\geq 0 \end{cases}$$

$f(x) = x+1,\ x \in R$

If we plot graph of f (x) that is a straight line



$\therefore$ it is differentiable every where

$\therefore$ slope of this line is 1

or $f(x) = x+1$

$\therefore f'(x) = 1$

$\therefore$ it is differentiable everywhere.

(18) **(B).** $$f(x) = \begin{cases} (x-1)\sin\frac{1}{x-1}, & x\neq 1 \\ 0, & x=1 \end{cases}$$

we will check differntiability at x = 0 and x = 1 at x = 0

$$\text{L.H.D.} = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{(-h-1)\sin\frac{1}{-h-1} - \sin 1}{-h} \quad \left(\frac{0}{0}\text{ form}\right)$$

Applying D.L. Hospital rule

$$\lim_{h\to 0} \frac{(-h-1)\cos\left(\frac{1}{-h-1}\right)\times\frac{1}{(h+1)^2} + \sin\left(\frac{1}{-h-1}\right)(-1) - 0}{-1}$$

$$= \frac{-\cos 1 + \sin 1}{-1} = \cos 1 - \sin 1$$

$$\text{R.H.D.} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{(h-1)\sin\frac{1}{h-1} - \sin 1}{h} \quad \left(\frac{0}{0}\text{ form}\right)$$

Applying D.L. Hospital rule

$$= \lim_{h\to 0} \frac{(h-1)\cos\left(\frac{1}{h-1}\right)\times\frac{-1}{(h-1)^2} + \sin\left(\frac{1}{h-1}\right)\times(1) - 0}{1}$$

= cos (–1) + sin (–1) = cos 1 – sin 1

$\because$ L.H.D. = R.H.D.

$\Rightarrow f(x)$ is differentiable at $x = 0$

At $x = 1$,

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{(1+h-1)\sin\dfrac{1}{1+h-1}-0}{h}$$

$$= \lim_{h\to 0}\frac{h\sin\dfrac{1}{h}}{h} = \lim_{h\to 0}\sin\frac{1}{h} = \text{does not exist}$$

$$\text{L.H.D.} = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{(1-h-1)\sin\dfrac{1}{1-h-1}-0}{h} = \lim_{h\to 0}\frac{-h\sin\dfrac{1}{h}}{-h}$$

$$= \lim_{h\to 0} -\sin\frac{1}{h}$$

$\Rightarrow$ L.H.D. $\neq$ R.H.D.

$f(x)$ is not differentiable at $x = 1$

(19) **(C).** $h(x) = g(f(x)) = \sin^2 x, x \geq 0 = -\sin x^2, x < 0$

$h'(x) = 2x\cos x^2, x \geq 0$

$= -2x\cos x^2, x < 0$

$\Rightarrow h'(0^+) = h'(0^-) = 0$

$h''(x) = -4x^2\sin x^2 + 2\cos x^2, x \geq 0$

$= -[-4x^2\sin x^2 + 2\cos x^2], x < 0$

$\Rightarrow h''(0^+) = 2$

$h''(0^-) = -2$

(20) **(D).** $f(x) = \dfrac{1}{e^x + 2e^{-x}} = \dfrac{e^x}{e^{2x}+2}$

$$f'(x) = \frac{(e^{2x}+2)e^x - 2e^{2x}.e^x}{(e^{2x}+2)^2}$$

$f'(x) = 0 \Rightarrow e^{2x} + 2 = 2e^{2x}$

$e^{2x} = 2 \Rightarrow e^x = \sqrt{2}$

$$\text{maximum } f(x) = \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}}$$

$$0 < f(x) \leq \frac{1}{2\sqrt{2}} \quad \forall x \in R$$

Since $0 < \dfrac{1}{3} < \dfrac{1}{2\sqrt{2}} \Rightarrow$ for same $c \in R$

$f(c) = 1/3$

(21) **(D).** f(x) is a positive increasing function

$\Rightarrow 0 < f(x) < f(2x) < f(3x)$

$$\Rightarrow 0 < 1 < \frac{f(2x)}{f(x)} < \frac{f(3x)}{f(x)}$$

$$\Rightarrow \lim_{x\to\infty} 1 \leq \lim_{x\to\infty}\frac{f(2x)}{f(x)} \leq \lim_{x\to\infty}\frac{f(3x)}{f(x)}$$

By sandwich theorem.

$$\Rightarrow \lim_{x\to\infty}\frac{f(2x)}{f(x)} = 1$$

(22) **(C).** $f(0) = q$

$$f(0^+) = \lim_{x\to 0^+}\frac{(1+x)^{1/2}-1}{x} = \lim_{x\to 0^+}\frac{1+\frac{1}{2}x+......-1}{x} = \frac{1}{2}$$

$$f(0^-) = \lim_{x\to 0^-}\frac{\sin(p+1)x+\sin x}{x}$$

$$f(0^-) = \lim_{x\to 0^-}\frac{(\cos(p+1)x)(p+1)+\cos x}{1}$$

$= (p+1) + 1 = p + 2$

$$p + 2 = q = \frac{1}{2} \Rightarrow p = -\frac{3}{2}, q = \frac{1}{2}$$

(23) **(A).** $\displaystyle\lim_{x\to 2}\sqrt{2}\frac{|\sin x(x-2)|}{(x-2)}$

$\therefore$ does not exist

(24) **(A).** Doubtful points are $x = n, n \in I$

L.H.L

$$= \lim_{x\to n^-}[x]\cos\left(\frac{2x-1}{2}\right)\pi = (n-1)\cos\left(\frac{2n-1}{2}\right)\pi = 0$$

$$\text{R.H.L.} = \lim_{x\to n^+}[x]\cos\left(\frac{2n-1}{2}\right)\pi = n\cos\left(\frac{2n-1}{2}\right)\pi = 0$$

$f(n) = 0$

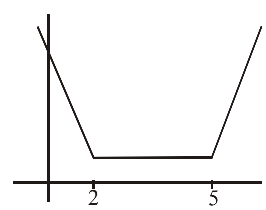
Hence continuous.

(25) **(C).**

$f(x) = 3 \quad 2 \leq x \leq 5$

$f'(x) = 0 \quad 2 < x < 5$

$f'(4) = 0$



(26) **(D).** $I = \displaystyle\lim_{x\to 0}\frac{(1-\cos 2x)}{x^2}\frac{(3+\cos x)}{1}.\frac{x}{\tan 4x}$

$$= \lim_{x\to 0}\frac{2\sin^2 x}{x^2}\frac{3+\cos x}{1}.\frac{x}{\tan 4x} = 2\times 4\times\frac{1}{4} = 2$$

(27) **(D).** $\displaystyle\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2} = \lim_{x\to 0}\frac{\sin(\pi - \pi\sin^2 x)}{x^2}$

$$= \lim_{x\to 0}\frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x}\times\frac{\pi\sin^2 x}{x^2} = \pi$$

(28) **(D).** $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \leq x \leq 3 \\ mx+2, & 3 < x \leq 5 \end{cases}$; g is constant at $x = 3$

$k\sqrt{4} = 3m + 2$; $2k = 3m + 2$ ... (1)

Also, $\left(\dfrac{k}{2\sqrt{x+1}}\right)_{x=3} = m$; $\dfrac{k}{4} = m$; $k = 4m$ ... (2)

$8m = 3m + 2$; $m = 2/5, k = 8/5$; $m + k = \dfrac{2}{5} + \dfrac{8}{5} = 2$

(29) **(B).** $\lim\limits_{x\to 0} \dfrac{2\sin^2 x}{x^2} \dfrac{(3+\cos x)}{\dfrac{\sin 4x}{4x}} \dfrac{\cos 4x}{4} = 2$

(30) **(A).** $g'(0) = f'(f(0))\, f'(0)$

For $x \to 0$, $\log 2 > \sin x$ $\quad \therefore f(x) = \log 2 - \sin x$

$\therefore$ $f'(x) = -\cos x \Rightarrow f'(0) = -1$

Also, $x \to \log 2$, $\log 2 > \sin x$

$\therefore$ $f(x) = \log 2 - \sin x$

$\therefore$ $f'(x) = -\cos x \Rightarrow f'(\log 2) = -\cos(\log 2)$

$\therefore$ $g'(0) = (-\cos(\log 2))(-1) = \cos(\log 2)$

(31) **(B).** $p = \lim\limits_{x\to 0^+} (1 + \tan^2\sqrt{x})^{1/2x} = \lim\limits_{x\to 0^+} e^{\frac{1}{2x}(\tan^2\sqrt{x})} = e^{1/2}$

$\log_e p = 1/2$

(32) **(A).** $g'(0) = f'(f(0))\, f'(0)$. For $x \to 0$, $\log 2 > \sin x$

$\therefore$ $f(x) = \log 2 - \sin x$ $\quad\therefore\quad f'(x) = -\cos x \Rightarrow f'(0) = -1$

Also, $x \to \log 2$, $\log 2 > \sin x$ $\therefore$ $f(x) = \log 2 - \sin x$

$\therefore$ $f'(x) = -\cos x \Rightarrow f'(\log 2) = -\cos(\log 2)$

$\therefore$ $g'(0) = (-\cos(\log 2))(-1) = \cos(\log 2)$

(33) **(D).** $\lim\limits_{x\to\pi/2} \dfrac{\cot x\,(1-\sin x)}{-8\left(x - \dfrac{\pi}{2}\right)^3}$

$$= \lim_{x\to\pi/2} \frac{\tan\left(\frac{\pi}{2} - x\right)}{8\left(\frac{\pi}{2} - x\right)} \frac{\left(1 - \cos\left(\frac{\pi}{2} - x\right)\right)}{\left(\frac{\pi}{2} - x\right)^2} = \frac{1}{8} \times 1 \times \frac{1}{2} = \frac{1}{16}$$

(34) **(A).** Let $x\left(\left[\dfrac{1}{x}\right] + \left[\dfrac{2}{x}\right] + ... + \left[\dfrac{15}{x}\right]\right)$

$$= \lim_{x\to 0^+} x\left(\frac{1}{x} - \left\{\frac{1}{x}\right\} + \frac{2}{x} - \left\{\frac{2}{x}\right\} + ... + \frac{15}{x} - \left\{\frac{15}{x}\right\}\right)$$

$$= \lim_{x\to 0^+} (1 + 2 + 3 + ...15)$$

$$+ \lim_{x\to 0^+} x\left(\left\{\frac{1}{x}\right\} + \left\{\frac{2}{x}\right\} + ... + \left\{\frac{15}{x}\right\}\right)$$

Now, $0 \le \{x\} < 1 \ \ \forall\ \ x \in R = 120$

(35) **(C).** Doubtful points for differentiability are 0 & π

At $x = 0$, $\quad f'(0^+) = \lim\limits_{h\to 0^+} \dfrac{[h-\pi] \times (e^{|h|} - 1) \times \sin|h| - 0}{h}$

$$= \lim_{h\to 0^+} \frac{(\pi - h) \times (e^h - 1) \times \sin h}{h}$$

$\because$ $\lim\limits_{h\to 0^+} \dfrac{\sin h}{h} = 1$ and $\lim\limits_{h\to 0^+} e^h - 1 = 0$

$\therefore$ $f'(0^+) = \pi \times 0 \times 1 = 0$

$$f'(0^-) = \lim_{h\to 0^-} \frac{[-h-\pi] \times (e^{|-h|} - 1) \times \sin|-h| - 0}{-h}$$

$$= \lim_{h\to 0^-} \frac{(\pi + h) \times (e^h - 1) \times \sin h}{-h}$$

$= \lim\limits_{h\to 0^-} \dfrac{\sinh}{h} = 1$ and $\lim\limits_{h\to 0^-} e^{h-1} = 0$

$f'(0^-) = (-\pi) \times 0 \times 1 = 0$; $f'(0^+) = f'(0^-) = 0$

Similarly $f'(\pi^+) = f'(\pi^-) = 0$

Hence $f(x)$ is differentiable $\forall\ x \in R$

(36) **(A).**

$$\lim_{y\to 0} \frac{\sqrt{1+\sqrt{1+y^4}} - \sqrt{2}}{y^4} = \lim_{y\to 0} \frac{1 + \sqrt{1+y^4} - 2}{y^4\left(\sqrt{1+\sqrt{1+y^4}} + \sqrt{2}\right)}$$

$$= \lim_{y\to 0} \frac{(\sqrt{1+y^4} - 1)(\sqrt{1+y^4} + 1)}{y^4\left(\sqrt{1+\sqrt{1+y^4}} + \sqrt{2}\right)(\sqrt{1+y^4} + 1)}$$

$$= \lim_{y\to 0} \frac{1 + y^4 - 1}{y^4\left(\sqrt{1+\sqrt{1+y^4}} + \sqrt{2}\right)(\sqrt{1+y^4} + 1)}$$

$$= \lim_{y\to 0} \frac{1}{\left(\sqrt{1+\sqrt{1+y^4}} + \sqrt{2}\right)(\sqrt{1+y^4} + 1)} = \frac{1}{4\sqrt{2}}$$

(37) **(D).** $f(x) = \begin{cases} 5, & \text{if } x \le 1 \\ a + bx, & \text{if } 1 < x < 3 \\ b + 5x, & \text{if } 3 \le x < 5 \\ 30, & \text{if } x \ge 5 \end{cases}$

$f(1) = 5, f(1^-) = 5, f(1^+) = a + b$

$f(3^-) = a + 3b, f(3) = b + 15, \ f(3^+) = b + 15$

$f(5^-) = b + 25$; $f(5) = 30, \ f(5^+) = 30$

From above we concluded that f is not.

(38) **(B).** $\lim\limits_{x\to 0} \dfrac{\left(\dfrac{\sin^2 x}{x^2}\right)(\sqrt{2} + \sqrt{1+\cos x})}{\left(\dfrac{1-\cos x}{x^2}\right)} = \dfrac{(1)^2 \cdot (2\sqrt{2})}{1/2} = 4\sqrt{2}$

(39) **(D).** $\lim\limits_{x\to 0} \left(\dfrac{1 + f(3+x) - f(3)}{1 + f(2-x) - f(2)}\right)^{1/x}$ $(1^\infty$ form)

$\Rightarrow$ $e^{\lim\limits_{x\to 0} \frac{f(3+x) - f(2-x) - f(3) + f(2)}{x(1 + f(2-x) - f(2))}}$

Using L'Hopital

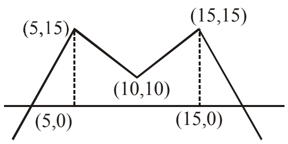
$\Rightarrow$ $e^{\lim\limits_{x\to 0} \frac{f'(3+x) + f'(2-x)}{f'(2-x) + (1 + f(2-x) - f(2))}}$

$\Rightarrow$ $e^{\frac{f'(3) + f'(2)}{1}} = 1$

(40) **(D).** $f(x)=\begin{cases}-(x+1), & -1\le x<0\\ x, & 0\le x<1\\ 2x, & 1\le x<2\\ x+2, & 2\le x<3\\ x+3, & x=3\end{cases}$

Function discontinuous at x = 0, 1, 3

(41) **(C).**



$f(x)=15-|x-10|, x\in R$

$f(f(x))=15-|f(x)-10|$

$=15-|15-|x-10|-10|=15-|5-|x-10||$

x = 5, 10, 15 are points of non differentiability

Aliter : At x = 10 f(x) is non differentiable also, when $15-|x-10|=10$

$\Rightarrow$ x = 5, 15

$\therefore$ Non-differentiability points are {5, 10, 15}

(42) **(A).** Function should be continuous at x = π/4

$$\lim_{x\to\pi/4} f(x)=f\left(\frac{\pi}{4}\right) \;;\; \lim_{x\to\pi/4}\frac{\sqrt{2}\cos x-1}{\cot x-1}=k$$

$$\lim_{x\to\pi/4}\frac{-\sqrt{2}\sin x}{\operatorname{cosec}^2 x}=k \text{ (Using L'Hopital rule)}$$

$$\lim_{x\to\pi/4}\sqrt{2}\sin^3 x=k \;;\; k=\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)^3=\frac{1}{2}$$

(43) **72.** Put $3^{x/2}=t$

$$\lim_{t\to3}\frac{\frac{4t^2}{3}-12}{-\frac{3}{t^2}+\frac{1}{t}}=\lim_{t\to3}\frac{4(t^2-9)t^2}{3(-3+t)}$$

$$=\lim_{t\to3}\frac{4t^2(3+t)}{3}=\frac{4\times9\times6}{3}=72$$

(44) **5.** $\lim_{x\to0} f(x)\;\lim_{x\to0}\left(\frac{1}{x}\ln\left(\frac{1+3x}{1-2x}\right)\right)$

$$=\lim_{x\to0}\left(\frac{\ln(1+3x)}{x}-\frac{\ln(1-2x)}{x}\right)$$

$$=\lim_{x\to0}\left(\frac{3\ln(1+3x)}{3x}-\frac{2\ln(1-2x)}{-2x}\right)$$

$=3+2=5$

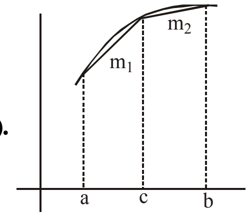
f(x) will be continuous if $f(0)=\lim_{x\to0}f(x)$

(45) **(A).** Let $L=\lim_{x\to0}\left(\frac{3x^2+2}{7x^2+2}\right)^{1/x^2}=e^{\lim_{x\to0}\frac{1}{x^2}\left\{\frac{3x^2+2}{7x^2+2}-1\right\}}$

$$=e^{\lim_{x\to0}\frac{1}{x^2}\left\{\frac{-4x^2}{7x^2+2}\right\}}=e^{-4/2}=e^{-2}$$

(46) **(D).** Using L'Hospital, $\lim_{x\to0}\frac{x\sin(10x)}{1}=0$

(47) **(C).**



It is clear from graph that $m_1>m_2$

$$\frac{f(c)-f(a)}{c-a}>\frac{f(b)-f(c)}{b-c} \;;\; \frac{f(c)-f(a)}{f(b)-f(c)}>\frac{c-a}{b-c}$$

(48) **(D).** $\lim_{x\to0^-}f(x)=\lim_{x\to0}\left(\frac{\sin(a+2)x}{x}+\frac{\sin x}{x}\right)=a+3$

$$\lim_{x\to0^-}f(x)=\lim_{x\to0}\frac{(x+3x^2)^{1/3}-x^{1/3}}{x^{4/3}}$$

$$=\lim_{x\to0}\frac{(1+3x)^{1/3}-1}{x}=1$$

$f(0)=b$

For continuity at x = 0

$$\lim_{x\to0^-}f(x)=f(0)=\lim_{x\to0^-}f(x)$$

$\Rightarrow a+3=b=1 \;\therefore a=-2, b=1 \;\therefore\; a+2b=0$

(49) **(B).** $A=\lim_{x\to0}x\left[\frac{4}{x}\right]=\lim_{x\to0}x\left(\frac{4}{x}\right)-x\left\{\frac{4}{x}\right\}=4$

$f(x)=[x^2]\sin(\pi x)$ will be discontinuous at non-integers

$\therefore\; x=\sqrt{A+1}$ i.e. $\sqrt{5}$