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DIFFERENTIATION

INTRODUCTION

* The rate of change of a quantity 'y' with respect to another quantity 'x' is called the derivative or differential coefficient of y with respect to x.
* This is equivalent to finding the slope of the tangent line to the function at a point.

DIFFERENTIAL COEFFICIENT

Let y = f(x) be a continuous function of a variable quantity x, where x is independent and y is dependent variable quantity. Let δx be an arbitrary small change in the value of x and δy be the corresponding change in y then $\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ if it exists, is called the derivative or differential coefficient of y with respect to x and it is denoted by $\frac{dy}{dx}$, y', y_1 or Dy.

So, $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$

$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$

This definition of derivative is also called the **first principle of derivative.**

If we again differentiate (dy/dx) with respect to x then the new derivative so obtained is called second derivative of y with respect to x and it is denoted by $\left(\frac{d^2y}{dx^2}\right)$ or y" or y_2 or D^2y. Similarly, we can find successive derivatives of y which may be denoted by

$\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots\ldots\ldots\ldots \frac{d^ny}{dx^n}, \ldots\ldots\ldots\ldots$

Note:

(i) $\frac{\delta y}{\delta x}$ is a ratio of two quantities δy and δx where as $\frac{dy}{dx}$ is not a ratio, it is single quantity i.e. $\frac{dy}{dx} \neq dy \div dx$.

(ii) $\frac{dy}{dx}$ is $\frac{d}{dx}$ (y) in which d/dx is simply a symbol of operation and not 'd' divided by dx.

Example 1 :

Find the derivative of the function $y = \frac{x}{x^2+1}$ with respect to x using first principle.

Sol. $y = \frac{x}{x^2+1} = f(x)$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{x+h}{(x+h)^2+1} - \frac{x}{x^2+1}}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)(x^2+1) - x[(x^2+1) + 2hx + h^2]}{h(x^2+1)[(x+h)^2+1]}$$

$$= \lim_{h \to 0} \frac{(x^2+1) - x(2x+h)}{(x^2+1)[(x+h)^2+1]}$$

$$= \lim_{h \to 0} \frac{(x^2+1) - 2x^2}{(x^2+1)^2} = \lim_{h \to 0} \frac{1-x^2}{(x^2+1)^2}$$

STANDARD DERIVATIVES

1. $\frac{d}{dx}(x^n) = nx^{n-1}$

2. $\frac{d}{dx}(e^x) = e^x$

3. $\frac{d}{dx}(a^x) = a^x \log_e a$

4. $\frac{d}{dx}(\log_e x) = 1/x$

5. $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$

6. $\frac{d}{dx}(\sin x) = \cos x$

7. $\frac{d}{dx}(\cos x) = -\sin x$

8. $\frac{d}{dx}(\tan x) = \sec^2 x$

9. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

10. $\frac{d}{dx}(\sec x) = \sec x \tan x$

11. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

THEOREMS ON DIFFERENTIATION

Theorem I :

Differentiation of a constant function is zero i.e. $\frac{d}{dx}(c) = 0$

Note : Geometrically, graph of a constant function is a straight line parallel to x-axis. So tangent at every point is parallel to x-axis. Consequently slope of the tangent is zero,

i.e., $\frac{dy}{dx} = 0$

Theorem II :

Let f(x) be a differentiable function and let c be a constant. Then c. f (x) is also differentiable such that

$$\frac{d}{dx}\{c.\, f(x)\} = c.\, \frac{d}{dx}(f(x))$$

i.e. The derivative of constant times a function is the constant times the derivative of the function.

Example 2 :

Find the derivative of $e^{\log x}$.

Sol. $y = e^{\log x} = x \Rightarrow \frac{dy}{dx} = 1$

Theorem III :

If f (x) and g(x) are differentiable functions, then f(x) ± g(x) are also differentiable such that

$$\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}\{f(x)\} \pm \frac{d}{dx}\{g(x)\}$$

i.e. The derivative of the sum or difference of two functions is the sum or difference of their derivatives

Note : The above result can be extended to a finite number of differentiable functions. Thus we have

$$\frac{d}{dx}\{f_1(x) \pm f_2(x) \pm ..... \pm f_n(x)\}$$

$$= \frac{d}{dx}\{f_1(x)\} \pm \frac{d}{dx}\{f_2(x)\} \pm .... \pm \frac{d}{dx}\{f_n(x)\}$$

Example 3 :

Find $\frac{d}{dx}(\cos^{-1} x + \sin^{-1} x)$

Sol. $\frac{d}{dx}(\cos^{-1} x + \sin^{-1} x) = \frac{d}{dx}\left(\frac{\pi}{2}\right) = 0$

Example 4 :

If $y = (1 + x^{1/4})(1 + x^{1/2})(1 - x^{1/4})$,

then find the value of $\frac{dy}{dx}$.

Sol. $y = (1 + x^{1/2})(1 - x^{1/2}) = 1 - x$

$\therefore \frac{dy}{dx} = -1$

Example 5 :

If $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\infty$, then find $\frac{dy}{dx}$.

Sol. $y = e^x \Rightarrow \frac{dy}{dx} = e^x = y$

Theorem IV

Product rule for differentiation : If f(x) and g(x) are two differentiable function, then f(x). g(x) is also differentiable

such that $\frac{d}{dx}[f(x).\, g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$

i.e. Derivative of the product of two functions
= [(First function) × (derivative of 2nd function)
+ (second function) × (derivative of first function)]

Note: The above result may also be expressed as

$(fg)' = f'g + fg'$

$\Rightarrow (fg)' = (fg)\left(\frac{f'}{f} + \frac{g'}{g}\right)$ [Dividing both sides by fg]

It can be generalized for the derivative of the product of more than two functions as given below

$$(fgh)' = (fgh)\left(\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right)$$

Generalisation of the product rule :

Let f (x), g(x), h (x) be three differentiable functions. Then,

$$\frac{d}{dx}\{f(x).g(x).h(x)\} = \left\{\frac{d}{dx}(f(x))\right\}g(x).h(x)$$

$$+ f(x)\left\{\frac{d}{dx}(g(x))\right\}h(x) + f(x)g(x).\left\{\frac{d}{dx}(h(x))\right\}$$

Example 6 :

If $y = x \tan\frac{x}{2}$, then find $(1 + \cos x)\frac{dy}{dx} - \sin x$.

Sol. $y = x\tan\frac{x}{2} \Rightarrow \frac{dy}{dx} = 1.\tan\frac{x}{2} + x.\sec^2\frac{x}{2}.\frac{1}{2}$

$$= \tan\frac{x}{2} + \frac{x}{2}\sec^2\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} + \frac{x}{2\cos^2\frac{x}{2}}$$

$$= \frac{2\sin\frac{x}{2}\cos\frac{x}{2} + x}{2\cos^2\frac{x}{2}} = \frac{\sin x + x}{1 + \cos x}$$

$$\Rightarrow (1 + \cos x)\frac{dy}{dx} - \sin x = x$$

Theorem V :

Quotient rule for differentiation :

If f(x) and g(x) are two differentiable functions and $g(x) \neq 0$,

then $\frac{f(x)}{g(x)}$ is also differentiable such that

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

or $$\frac{d}{dx}\left[\frac{N^r}{D^r}\right] = \frac{D^r\frac{d}{dx}[N^r] - N^r\frac{d}{dx}[D^r]}{[D^r]^2}$$

where N^r : Numerator ; D^r Denominator

Example 7 :

If $f(t) = \frac{1-t}{1+t}$, then find f' (1/t).

Sol. $$f'(t) = \frac{d}{dt}\left[\frac{1-t}{1+t}\right] = \frac{(1+t)(-1)-(1-t)\times(1)}{(1+t)^2}$$

$$= \frac{-1-t-1+t}{(1+t)^2} = \frac{-2}{(1+t)^2}$$

$$f'[1/t] = \frac{-2}{\left(1+\frac{1}{t}\right)^2} = \frac{-2t^2}{(t+1)^2}$$

Example 8 :

If $f(x) = \frac{x-4}{2\sqrt{x}}$ then find the value of f' (0).

Sol. $$f'(x) = \frac{\sqrt{x}.1-(x-4)\frac{1}{2\sqrt{x}}}{2x} = \frac{2x-x+4}{4x\sqrt{x}} = \frac{x+4}{4x\sqrt{x}}$$

which is not defined at x = 0 i.e. f' (0) does not exist.

Example 9 :

Find the differential coefficient of $\frac{x\sin x}{1+\cos x}$.

Sol. $$\frac{d}{dx}\left(\frac{x\sin x}{1+\cos x}\right)$$

$$= \frac{(1+\cos x)(\sin x + x\cos x) - (x\sin x)(0-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\sin x(1+\cos x) + x\cos x + x(\cos^2 x + \sin^2 x)}{(1+\cos x)^2}$$

$$= \frac{(x+\sin x)(1+\cos x)}{(1+\cos x)^2} = \frac{x+\sin x}{1+\cos x}$$

Theorem VI

Differentiation of a Composite function (Chain rule) :

If f (x) and g (x) are differentiable functions, then fog is also differentiable and

$(fog)'(x) = f'\{g(x)\}.\, g'(x)$

or, $$\frac{d}{dx}\{(fog)(x)\} = \frac{d}{d\,g(x)}\{(fog)(x)\}.\frac{d}{dx}(g(x))$$

Note :

(1) The above rule can also be restated as follows

If z = f (y) and y = g(x), then $\frac{dz}{dx} = \frac{dz}{dy}.\frac{dy}{dx}$

or (Derivative of z w.r.t. x = Derivative of z w.r.t.y × Derivative of y w.r.t.x)

(2) This chain rule can be extended further

Derivative of z w.r.t. x = (Derivative of z w.r.t. u) × (Derivative of u w.r.t v) × (Derivative of v w.r.t x)

Example 10 :

Find derivative of function $y = e^{ax}\cos bx$.

Sol. $y = e^{ax}\cos bx$.

$$\frac{dy}{dx} = e^{ax}\frac{d}{dx}(\cos bx) + \cos bx\frac{d}{dx}(e^{ax})$$

$$= e^{ax}(-\sin bx)\frac{d}{dx}(bx) + \cos bx\; e^{ax}\cdot a$$

$$= (a\cos bx - b\sin bx)\, e^{ax}$$

NOTE

* $$\frac{d}{dx}(e^{ax}\sin b x) = e^{ax}(a\sin b x + b\cos bx)$$

$$= \sqrt{a^2+b^2}\; e^{ax}\sin(bx + \tan^{-1}b/a)$$

* $$\frac{d}{dx}(e^{ax}\cos b x) = e^{ax}(a\cos b x - b\sin b x)$$

$$= \sqrt{a^2+b^2}\; e^{ax}\cos(bx + \tan^{-1}b/a)$$

Example 11 :

Find derivative of following functions

(i) $y = \frac{1}{(f(x))^n}$ (ii) $y = \sec^2(f^3(x))$

Sol. (i) $y = \frac{1}{(f(x))^n} = (f(x))^{-n}$

$$\frac{dy}{dx} = -n\,(f(x))^{-n-1}\frac{d}{dx}f(x) = \frac{-nf'(x)}{(f(x))^{n+1}}$$

(ii) $y = \sec^2(f^3(x))$

$$\frac{dy}{dx} = 2\sec(f^3(x))\frac{d}{dx}\sec(f^3(x))$$

$$= 2\sec(f^3(x))\cdot\sec(f^3(x))\tan(f^3(x))\frac{d}{dx}(f^3(x))$$

$$= 2\sec(f^3(x))\tan(f^3(x))\,3f^2(x)\frac{d}{dx}f(x)$$

$$= 6\sec^2(f^3(x))\tan(f^3(x))\,f^2(x)\cdot f'(x)$$

Example 12 :

Find the derivative of function $y = e^{\sqrt{\sin(\ln(x^2+7)^5)}}$

Sol. $y = e^{\sqrt{\sin(\ln(x^2+7)^5)}}$

$\therefore\ y = e^{\sqrt{\sin(5\ln(x^2+7))}}$

$$\therefore\ \frac{dy}{dx} = e^{\sqrt{\sin(5\ln(x^2+7))}}\cdot\frac{d}{dx}\sqrt{\sin(5\ln(x^2+7))}$$

$$= e^{\sqrt{\sin(5\ln(x^2+7))}}\cdot\frac{1}{2\sqrt{\sin(5\ln(x^2+7))}}\cdot\frac{d}{dx}\sin(5\ln(x^2+7))$$

$$= \frac{e^{\sqrt{\sin(5\ln(x^2+7))}}}{2\sqrt{\sin(5\ln(x^2+7))}}\cdot\cos(5\ln(x^2+7))\cdot\frac{5}{x^2+7}\cdot\frac{d}{dx}(x^2+7)$$

$$= \frac{5x\cos(5\ln(x^2+7))\,e^{\sqrt{\sin(5\ln(x^2+7))}}}{(x^2+7)\sqrt{\sin(5\ln(x^2+7))}}$$

Differentiation of logarithmic functions:

In differentiation of an expression or an equation is done after taking log on both sides, then it is called logarithmic differentiation. This method is useful for the function having following forms–

(i) When base and power both are the functions of x i.e. the functions is of the form $[f(x)]^{g(x)}$.

$y = [f(x)]^{g(x)}$; $\log y = g(x)\log[f(x)]$

$$\frac{1}{y}.\frac{dy}{dx} = \frac{d}{dx}\ g(x).\log[f(x)]$$

$$\frac{dy}{dx} = [f(x)^{g(x)}.\left\{\frac{d}{dx}[g(x)\log f(x)]\right\}$$

Example 13 :

Find $\frac{d}{dx}(x^{x^x})$

Sol. Let $y = x^{x^x}$; $\log y = x^x\log x$

$$\frac{1}{y}\frac{dy}{dx} = x^x.\frac{1}{x} + \log x\,[x^x(1+\log x)]$$

$$\frac{dy}{dx} = x^{x^x}\ [x^{x-1} + \log x\,\{x^x(1+\log x)\}]$$

Example 14 :

If $y = (\log x)^x + x^{\log x}$, then find $\frac{dy}{dx}$.

Sol. Let $y = (\log x)^x + x^{\log x}$

Put $u = (\log x)^x$, $v = x^{\log x}$

$\therefore\ y = u + v\ \therefore\ \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Now $u = (\log x)^x \Rightarrow\ \log u = x\log(\log x)$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = x.\frac{1}{\log x}.\frac{1}{x} + \log(\log x).1 = \frac{1}{\log x} + \log(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\log(\log x) + \frac{1}{\log x}\right]$$

$$= (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right]$$

$v = x^{\log x}$

$\therefore\ \log v = \log x.\log x = (\log x)^2$

$\therefore\ \frac{1}{v}\frac{dv}{dx} = 2\log x.\frac{1}{x}$ $\quad\therefore\ \frac{dv}{dx} = 2\frac{\log x}{x}, x^{\log x}$

$\therefore$ eq. (1) gives

$$\frac{dy}{dx} = (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right] + x^{\log x}.\left(\frac{2\log x}{x}\right)$$

Example 15 :

If $y = \sqrt{x\log_e x}$, then find $\frac{dy}{dx}$ at $x = e$.

Sol. $y = (x\log_e x)^{1/2}\ \therefore\ \log y = \frac{1}{2}[\log x + \log(\log x)]$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{\log x}.\frac{1}{x}\right)$$

$$\frac{1}{\sqrt{x\log x}}\frac{dy}{dx} = \frac{1}{2x}\left(1 + \frac{1}{\log x}\right)$$

At $x = e$, we have $\frac{1}{\sqrt{e\log e}}\frac{dy}{dx} = \frac{1}{2e}\left(1 + \frac{1}{\log e}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{e}}{2e}(1+1) = \frac{1}{\sqrt{e}}$$

Example 16 :

If $2^x + 2^y = 2^{x+y}$, then find the value of $\frac{dy}{dx}$ at $x = y = 1$.

Sol. $2^x + 2^y = 2^{x+y}$

$$\Rightarrow 2^x\log 2 + 2^y\log 2\frac{dy}{dx} = 2^{x+y}\log 2.\left[1 + \frac{dy}{dx}\right]$$

$$\Rightarrow 2^x + 2^y \frac{dy}{dx} = 2^{x+y} + 2^{x+y}\frac{dy}{dx}$$

$$\Rightarrow (2^y - 2^{x+y})\frac{dy}{dx} = 2^{x+y} - 2^x$$

$$\Rightarrow 2^y(1-2^x)\frac{dy}{dx} = 2^x(2^y - 1) \Rightarrow \frac{dy}{dx} = \frac{2^x(2^y-1)}{2^y(1-2^x)}$$

$$\left(\frac{dy}{dx}\right)_{x=y=1} = \frac{2^1(2^1-1)}{2^1(1-2^1)} = -1 = -y$$

Differentiation of Implicit functions :

If in a equation, x and y both occurs together i.e. f(x, y) = 0 and this equation can not be solved either for y or x, then y (or x) is called the implicit function of x (or y).

For example $x^3 + y^3 + 3xy = 2$, $x^y + y^x = a^b$, $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$

Working rule for finding the derivative :

First method :

(i) Differentiate every term of f(x, y) = 0 with respect to x.

(ii) Collect the coefficients of dy/dx and obtain the value of dy/dx.

Second Method :

f (x, y) = constant, then $\frac{dy}{dx} = \frac{\partial f/\partial x}{\partial f/\partial y}$ are partial differential coefficients of f(x, y) with respect to x and y respectively.

Note: Partial differential coefficient of f(x,y) with respect to x means the ordinary differential coefficient of f(x, y) with respect to x keeping y constant.

Example 17 :

If sin y = x sin (a + y) then find dy/dx.

Sol. sin y = x sin (a + y) $\quad\therefore\ x = \frac{\sin y}{\sin(a+y)}$

Differentiating the function with respect to y

$$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin^2(a+y)}$$

$$= \frac{\sin(a+y-y)}{\sin^2(a+y)} = \frac{\sin a}{\sin^2(a+y)} \quad \therefore \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Example 18 :

If $x^y + y^x = a^b$ then find $\frac{dy}{dx}$.

Sol. Here $f(x, y) = x^y + y^x - a^b$

so differentiating f (x, y) with respect to x keeping y constant, we get

$$\frac{\partial f}{\partial x} = y\,x^{y-1} + y^x \log y \text{ Similarly } \frac{\partial f}{\partial y} = x^y \log x + x.\,y^{x-1}$$

$$\text{Thus } \frac{dy}{dx} = -\frac{-\partial f/\partial x}{\partial f/\partial y} = -\frac{y\,x^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}$$

Example 19 :

If $x^3 + y^3 = 3xy$, then find $\frac{dy}{dx}$.

Sol. Differentiating w.r.t. x,

$$3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$$

$$\Rightarrow 3(x^2 - y) = 3\frac{dy}{dx}(x - y^2) \Rightarrow \frac{dy}{dx} = \frac{x^2 - y}{x - y^2}$$

Theorem VII :

Derivative of inverse function :

Let the function f (x) and g (x) be inverse of each other then

f (g (x) = g (f (x)) = x

$\therefore$ f ' (g (x) g' (x) = g' (f (x)) f ' (x) = 1

Relation between $\frac{dy}{dx}$ and $\frac{dx}{dy}$: $\frac{dy}{dx} = \frac{1}{dx/dy}\left[\frac{dx}{dy} \neq 0\right]$

Example 20 :

If $y = f(x) = x^3 + x^5$ and g is the inverse of f find g ' (2).

Sol. $y = f(x) = x^3 + x^5$

Let g (x) be the inverse of f (x) i.e., $g(x) = f^{-1}(x)$

f (g (x)) = g (f (x)) = x

Differentiating w.r.t. x

f ' (g (x)) . g '(x) = 1

$$\Rightarrow g'(x) = \frac{1}{f'(g(2))} \Rightarrow g(2) = f^{-1}(2) = y \text{ (say)}$$

$$\Rightarrow f(y) = 2 \Rightarrow y^3 + y^5 = 2 \Rightarrow y = 1$$

$$\Rightarrow g'(2) = \frac{1}{f'(1)} = \frac{1}{(3x^2 + 5x^4)_{x=1}} = \frac{1}{8}$$

Example 21 :

Let $f(x) = \exp(x^3 + x^2 + x)$ for any real number x, and let g be the inverse function for f. The value of $g'(e^3)$ is –

(A) $1/6e^3$ (B) 1/6

(C) $1/34e^{39}$ (D) 6

Sol. **(A).** f (g (x)) = x

$$\Rightarrow f'(g(x)).\,g'(x) = 1 \Rightarrow g'(x) = \frac{1}{f'(g(x))}$$

$$\Rightarrow g'(e^3) = \frac{1}{f'(g(e^3))} \Rightarrow e^{x^3+x^2+x} = e^3 \Rightarrow x = 1$$

$$\Rightarrow g'(e^3) = \frac{1}{f'(1)} = \frac{1}{[e^{x^3+x^2+x}(3x^2+2x+1)]_{x=1}}$$

$$= \frac{1}{e^3(3+2+1)} = \frac{1}{6e^3}$$

DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

1. $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$

2. $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$

3. $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

4. $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$

5. $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$; $|x| > 1$

6. $\frac{d}{dx}(\text{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$; $|x| > 1$

Differentiation by trigonometrical substitutions

Sometimes it comes very easy to differentiate a function by using trigonometrical transformations. Usually this is done in case of inverse trigonometrical functions.

Some suitable substitutions

	Functions	**Substitution**
(i)	$\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $a\cos\theta$
(ii)	$\sqrt{x^2+a^2}$	$x = \tan\theta$ or $a\cot\theta$
(iii)	$\sqrt{x^2-a^2}$	$x = a\sec\theta$ or $a\,\text{cosec}\,\theta$
(iv)	$\sqrt{\frac{a-x}{a+x}}$	$x = a\cos 2\theta$
(v)	$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$	$x^2 = a^2\cos 2\theta$
(vi)	$\sqrt{ax-x^2}$	$x = a\sin^2\theta$
(vii)	$\sqrt{\frac{x}{a+x}}$	$x = a\tan^2\theta$
(viii)	$\sqrt{\frac{x}{a-x}}$	$x = a\sin^2\theta$
(ix)	$\sqrt{(x-a)(x-b)}$	$x = a\sec^2\theta - b\tan^2\theta$
(x)	$\sqrt{(x-a)(b-x)}$	$x = a\cos^2\theta + b\sin^2\theta$

Example 22 :

If $y = \tan^{-1}\frac{x}{2} - \cot^{-1}\frac{x}{2}$, then find the value of $\frac{dy}{dx}$

Sol. $\frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1}\frac{x}{2}\right] - \frac{d}{dx}\left[\cot^{-1}\frac{x}{2}\right]$

$$= \frac{4}{4+x^2}\cdot\frac{1}{2} + \frac{4}{1+x^2}\cdot\frac{1}{2} = \frac{2.2}{4+x^2} = \frac{4}{1+x^2}$$

Example 23 :

If $y = x\sin^{-1}x + \sqrt{1-x^2}$ then find $\frac{dy}{dx}$

Sol. $\frac{dy}{dx} = 1.\sin^{-1}x + \frac{x}{\sqrt{1-x^2}} + \frac{1}{2}\frac{(-2x)}{\sqrt{1-x^2}}$

$$= \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \sin^{-1}x$$

Example 24 :

If $f(x) = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$, find $f'(0)$.

Sol. $f(x) = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$

Let $2^x = \tan\theta$

$$f(x) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$$

$$f(x) = 2\theta = 2\tan^{-1}(2^x) \Rightarrow f'(x) = \frac{1}{1+2^{2x}}(2^x \ln 2)$$

$$f'(0) = \frac{2}{1+1}(2^0 \ln 2) = \ln 2$$

Example 25 :

If $y = \sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}$, & $0 < x < 1$ then find $\frac{dy}{dx}$.

Sol. $y = \sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}$, where $0 < x < 1$

$$= \sin^{-1}[x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}] = \sin^{-1}x - \sin^{-1}\sqrt{x}$$

Differentiating w.r.t. x, we get

$$\frac{d}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x-x^2}}$$

Example 26 :

Let $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, $0 < x < 1$ and $0<y< \frac{\pi}{2}$, then find $\frac{dy}{dx}$

Sol. Put $x = \tan\theta$, $\frac{2x}{1+x^2} = \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$

$\therefore \quad y = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1} x$

$\therefore \quad \frac{dy}{dx} = \frac{2}{1+x^2}$

Example 27 :

Find $\frac{d}{dx}\left[\sin^{-1}\left(3x - 4x^3\right)\right]$ for $\frac{1}{2} < x < 1$.

Sol. If $\frac{1}{2} < x < 1$ then $3\sin^{-1} x = \pi - \sin^{-1}(3x - 4x^3)$

then $\sin^{-1}(3x - 4x^3) = \pi - 3\sin^{-1} x$

$\Rightarrow \frac{d}{dx}\left[\sin^{-1}(3x-4x^3)\right] = 0 - \frac{3d}{dx}(\sin^{-1} x) = \frac{-3}{\sqrt{1-x^2}}$

PARAMETRIC DIFFERENTIATION

In some situation curves are represented by the equations e.g. $x = \sin t$ & $y = \cos t$.

If $x = f(t)$ and $y = g(t)$ then $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{g'(t)}{f'(t)}$

Example 28 :

Find $\frac{dy}{dx}$ if $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

Sol. $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$

$\frac{dx}{dt} = a(-\sin t + \sin t + t\cos t) = at\cos t$

$\frac{dy}{dt} = a(\cos t - \sin t + t\sin t) = at\sin t \Rightarrow \frac{dy}{dx} = \tan t$

Example 29 :

Find $\frac{dy}{dx}$ if $x = a\sqrt{\cos 2t}\cos t$ and $y = a\sqrt{\cos 2t}\sin t$

then, $\left.\frac{dy}{dx}\right|_{t=\pi/6}$

Sol. $x = a\sqrt{\cos 2t}\cos t$ and $y = a\sqrt{\cos 2t}\sin t$

$\frac{dx}{dt} = a\left(\sqrt{\cos 2t}(-\sin t) + \frac{\cos t(-2\sin 2t)}{2\sqrt{\cos 2t}}\right)$

$\frac{dx}{dt} = a\left(\frac{\sin t\cos 2t + \cos t\sin 2t}{\sqrt{\cos 2t}}\right) = -a\frac{\sin 3t}{\sqrt{\cos 2t}}$

$\left.\frac{dx}{dt}\right|_{\pi/6} = -\sqrt{2}a$

$\frac{dy}{dt} = a\left(\sqrt{\cos 2t}\cos t + \frac{\sin t}{2\sqrt{\cos 2t}}(-2\sin 2t)\right)$

$= a\left(\frac{\cos 3t}{2\sqrt{\cos 2t}}\right) \Rightarrow \left.\frac{dy}{dt}\right|_{\pi/6} = 0 \quad ; \quad \left.\frac{dy}{dx}\right|_{\pi/6} = 0$

Theorem VIII :

Derivative of a function with respect to another function :

If f(x) and g(x) are two functions of a variable x, then

$\frac{d[f(x)]}{d[g(x)]} = \frac{d}{dx}[f(x)] / \frac{d}{dx}[g(x)]$

Example 30 :

Find the derivative of $\sin x^3$ w.r.t. $\cos x^3$.

Sol. Let $y = \sin x^3$, $\therefore \frac{dy}{dx} = \cos x^3 . 3x^2$; $z = \cos x^3$

$\frac{dz}{dx} = -\sin x^3 . 3x^2 \therefore \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = -\frac{\cos x^3}{\sin x^3} = -\cot x^3$

Example 31 :

Find the differential coefficient of $\log_{10} x$ w.r.t. $\log_x 10$.

Sol. Let $y = \log_{10} x$ and $z = \log_x 10$

$= \frac{1}{\log_{10} x} = \frac{1}{y} \quad \therefore y = \frac{1}{z} \quad \therefore \frac{dy}{dz} = -\frac{1}{z^2}$

$= -\frac{1}{(\log_x 10)^2} = -\frac{(\log x)^2}{(\log 10)^2}$

Differentiation of infinite series:

(i) If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + ...\infty}}}$

$\Rightarrow \; y = \sqrt{f(x) + y} \Rightarrow y^2 = f(x) + y$

$2y\frac{dy}{dx} = f'(x) + \frac{dy}{dx} \qquad \therefore \frac{dy}{dx} = \frac{f'(x)}{2y-1}$

Example 32 :

If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + ...}}}$ then find $\frac{dy}{dx}$

Sol. $y = \sqrt{\cos x + y} \;\Rightarrow\; y^2 = \cos x + y$

$\Rightarrow 2y.\frac{dy}{dx} = -\sin x + \frac{dy}{dx} \quad \therefore \quad (2y-1)\frac{dy}{dx} = -\sin x$

$\therefore \frac{dy}{dx} = \frac{\sin x}{1-2y}$

(ii) **If** $y = f(x)^{f(x)^{f(x)^{\dots\infty}}}$ then $y = f(x)^y$.

$\therefore \log y = y \log[f(x)]$

$$\frac{1}{y}\frac{dy}{dx} = \frac{y'.f'(x)}{f(x)} + \log f(x).\left(\frac{dy}{dx}\right)$$

$$\therefore \frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)[1 - y\log f(x)]}$$

Example 33 :

$y = x^{x^{x^{\dots\dots\infty}}}$ then find the value of $(1 - y\log x)\dfrac{dy}{dx}$.

Sol. $y = x^y$

$\log y = y \log x$

$$\frac{1}{y}\frac{dy}{dx} = \log x.\frac{dy}{dx} + \frac{y}{x}$$

$$\frac{dy}{dx}\left(\frac{1}{y} - \log x\right) = \frac{y}{x} \Rightarrow \frac{dy}{dx}\left(\frac{1 - y\log x}{y}\right) = \frac{y}{x}$$

$$\Rightarrow (1 - y\log x)\frac{dy}{dx} = \frac{y^2}{x}$$

(iii) **If** $y = f(x) + \cfrac{1}{f(x) + \cfrac{1}{f(x)\dots\dots}}$ then $\dfrac{dy}{dx} = \dfrac{y f'(x)}{2y - f(x)}$

Example 34 :

If $y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \dots\dots\dots}}}}$ then find $\dfrac{dy}{dx}$.

Sol. $y = x + \dfrac{1}{y}$

$y^2 = xy + 1$

Diff. w.r.t. to'x', $2y\dfrac{dy}{dx} = y + x\dfrac{dy}{dx} + 0$

$\dfrac{dy}{dx}(2y - x) = y$; $\dfrac{dy}{dx} = \dfrac{y}{2y - x}$

TRY IT YOURSELF

Q.1 Find the derivative of the function $y = \ln^2 x$ with respect to x using first principle.

Q.2 If $y = (1 + x^{1/4})(1 + x^{1/2})(1 - x^{1/4})$, then find $\dfrac{dy}{dx}$.

Q.3 If $f(x) = 1 + x + x^2 + \dots.. + x^{100}$ then $f'(1) =$

Q.4 Find the derivatie of function $y = \dfrac{1 - \ln x}{1 + \ln x}$.

Q.5 If $y = \dfrac{x^4 + x^2 + 1}{x^2 + x + 1}$ then $\dfrac{dy}{dx} = ax + b$ find a and b.

Q.6 If $f(x) = (1 + x)(3 + x2)^{1/2}(9 + x^3)^{1/3}$ then $f'(-1) =$

(A) 0 (B) $2\sqrt{2}$

(C) 4 (D) 6

Q.7 If $f(x) = (x + 1)(x + 2)(x + 3) \dots.. (x + n)$ then $f'(0)$ is

(A) n! (B) $\dfrac{n(n+1)}{2}$

(C) (n!) (ln n!) (D) $n!\left(1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dots + \dfrac{1}{n}\right)$

Q.8 Find the derivative of the function $y = \dfrac{(\ln x)^x}{2^{3x+1}}$

Q.9 Find the derivative of $\dfrac{\sqrt{x}(x + 4)^{3/2}}{(4x - 3)^{4/3}}$ w.r.t. x.

Q.10 If $y = \sin^{-1}\dfrac{2x}{1 + x^2}$, $z = \tan^{-1} x$, then find the value of dy/dz.

Q.11 If $y = \sqrt{x \log_e x}$, then find $\dfrac{dy}{dx}$ at $x = e$.

Q.12 Find $\dfrac{dy}{dx}$ for $y = \tan^{-1}\left\{\dfrac{1 - \cos x}{\sin x}\right\}$, where $-\pi < x < \pi$.

Q.13 If $f(x) = \log_x(\ln x)$, then at $x = e$, find the value of $f'(x)$

Q.14 Differentiating log sin x w.r.t. $\sqrt{\cos x}$

Q.15 Find the differential co-efficient of $\tan^{-1}\left(\dfrac{\sin x + \cos x}{\cos x - \sin x}\right)$ w.r.t 'x'.

Q.16 Differentiating $\tan^{-1}\left(\dfrac{\sqrt{1 + x^2} - 1}{x}\right)$ w.r.t. $\tan^{-1}x$.

Q.17 If $y = |\cos x| + |\sin x|$ then find $\dfrac{dy}{dx}$ at $x = \dfrac{2\pi}{3}$.

Q.18 Find derivative of $(\ln x)^{\tan x}$ w.r.t. x^x.

Q.19 If $y = \sec^{-1}\left(\dfrac{x + 1}{x - 1}\right) + \sin^{-1}\left(\dfrac{x - 1}{x + 1}\right)$, then find $\dfrac{dy}{dx}$.

ANSWERS

(1) $\dfrac{2\ln x}{x}$ **(2)** -1 **(3)** 5050

(4) $\dfrac{-2}{x(1 + \ln x)^2}$ **(5)** $a = 2, b = -1$ **(6)** (C)

(7) (D) **(8)** $y\left(\ln(\ln x)+\frac{1}{(\ln x)}-3\ln 2\right)$

(9) $\frac{\sqrt{x}(x+4)^{3/2}}{(4x-3)^{4/3}}\left\{\frac{1}{2x}+\frac{3}{2(x+4)}-\frac{16}{3(4x-3)}\right\}$

(10) 2 **(11)** $\frac{1}{\sqrt{e}}$ **(12)** 1/2

(13) 1/e **(14)** $-2\sqrt{\cos x}\cot x\,\mathrm{cosec}\,x$

(15) 1 **(16)** 1/2 **(17)** $\frac{\sqrt{3}-1}{2}$

(18) $\frac{(\ln x)^{\tan x}}{x^x}\left(\frac{x\ln x\ln(\ln x)\sec^2 x+\tan x}{(x\ln x)(\ln x+1)}\right)$

(19) 0

SUCCESSIVE DIFFERENTIATION

If y is a function of x and is differentiable with respect to x, then its derivative dy/dx can be found which is known as derivative of first order. If the first derivative dy/dx is also differentiable function, then it can be further differentiate with respect to x and this derivative is denoted by d^2y/dx^2 which is called the second derivative of y with respect to x further if d^2y/dx^2 is also differentiable then its derivative is called third derivative of y which is denoted by d^3y/dx^3. Similarly the n^{th} derivative of y is denoted by d^ny/dx^n. All these derivative are called as successive derivative of y and this process is known as successive differentiation. We also use the following symbols for the successive derivatives of y = f(x):

y_1,	y_2,	y_3,..........,	y_n,..................
y',	y'',	y''',..........,	y^n,.................
Dy,	D^2y,	D^3y,.........,	D^ny,.............

where (D = d/dx)

dy/dx,	d^2y/dx^2,	d^3y/dx^3........	d^ny/dx^n..........
$f'(x)$	$f''(x)$	$f'''(x)$,.......	$f^n(x)$,............

If y = f(x) then the value of the n^{th} order derivative at x = a is usually denoted by

$$\left(\frac{d^ny}{dx^n}\right)_{x=a} \text{ or } (y^n)_{x=a} \text{ or } (y_n)_{x=a} \text{ or } f^n(a)$$

Example 35 :

Let y = a sin mx + b cos mx, then find $\frac{d^2y}{dx^2}$

Sol. $\frac{dy}{dx}$ = am cos mx – bm sin mx

$\therefore\ \frac{d^2y}{dx^2} = -am^2\sin mx - bm^2\cos mx$

$= -m^2[a\sin mx + b\cos mx] = -m^2y$

Example 36 :

If $x = t^2$, $y = t^3$, then find $\frac{d^2y}{dx^2}$

Sol. $\frac{dx}{dt} = 2t,\ \frac{dy}{dt} = 3t^2$

$\therefore\ \frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3}{2}t$

$\therefore\ \frac{d^2y}{dx^2} = \frac{3}{2}\cdot\frac{dt}{dx} = \frac{3}{2}\frac{1}{dx/dt} = \frac{3}{2.2t} = \frac{3}{4t}$

Example 37 :

If $y = ax^{n+1} + bx^{-n}$, then find $x^2\frac{d^2y}{dx^2}$

Sol. $y = ax^{n+1} + bx^{-n}$

$\frac{dy}{dx} = (n+1)ax^n - n\,bx^{-n-1}$

$\frac{d^2y}{dx^2} = (n+1)n\,ax^{n-1} + n(n+1)\,bx^{-n-2}$

$\therefore\ x^2\frac{d^2y}{dx^2} = (n+1)\,na.\,x^{n+1} + n(n+1)\,b\,x^{-n}$

$= n(n+1)[ax^{n+1} + bx^{-n}] = n(n+1)y$

n^{th} DERIVATIVES OF SOME STANDARD FUNCTIONS :

(1) $\frac{d^n}{dx^n}\sin(ax+b) = a^n\sin\left(\frac{n\pi}{2}+ax+b\right)$

Proof :

y = sin (ax + b)

$\Rightarrow y_1 = a\cos(ax+b) = a\sin\left(\frac{\pi}{2}+ax+b\right)$

$\Rightarrow y_2 = -a^2\sin(ax+b) = a^2\sin\left(\frac{2\pi}{2}+ax+b\right)$

$\Rightarrow y_3 = -a^3\cos(ax+b) = a^3\sin\left(\frac{3\pi}{2}+ax+b\right)$

....

....

$\Rightarrow y_n = a^n\sin\left(\frac{n\pi}{2}+ax+b\right)$

Similarly $\frac{d^n}{dx^n}\cos(ax+b) = a^n\cos\left(\frac{n\pi}{2}+ax+b\right)$

Example 38 :

If y = sin mx + cos mx, then find y_n.

Sol. y = sin mx + cos mx

$$y_n = m^n\left[\sin\left(mx+\frac{n\pi}{2}\right)+\cos\left(mx+\frac{n\pi}{2}\right)\right]$$

$$= m^n\left[\begin{matrix}\sin^2\left(mx+\frac{n\pi}{2}\right)+\cos^2\left(mx+\frac{n\pi}{2}\right)\\ +2\cos\left(mx+\frac{n\pi}{2}\right)\sin\left(mx+\frac{n\pi}{2}\right)\end{matrix}\right]^{1/2}$$

$$= m^n\,[1+\sin(2mx+n\pi)]^{1/2} = m^n\,[1+(-1)^n\sin 2mx]^{1/2}$$

(2) $\dfrac{d^n}{dx^n}(ax+b)^m = \dfrac{m!}{(m-n)!}a^n(ax+b)^{m-n}$, where m < n

Example 39 :

If $y=\dfrac{1}{a-x}$ then find y_n.

Sol. $y_n = D^n\left(\dfrac{1}{a-x}\right) = D^n[-(x-a)^{-1}] = D^n(x-a)^{-1}$

$$= -\frac{(-1)^n n!}{(x-a)^{n+1}} = \frac{(-1)(-1)^n n!}{(-1)^{n+1}(a-x)^{n+1}} = \frac{n!}{(a-x)^{n+1}}$$

$$\left[\because D^n(ax+b)^{-1} = \frac{(-1)^n n!a^n}{(ax+b)^{n+1}}\right]$$

(3) $\dfrac{d^n}{dx^n}(\log(ax+b)) = \dfrac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$

Example 40 :

Find the n^{th} derivative of log {(ax + b). (cx + d)}.

Sol. $D^n \log\{(ax+b)(cx+d)\}$

$$= D^n\{\log(ax+b)\} + D^n\{\log(cx+d)\}$$

$$= \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n} + \frac{(-1)^{n-1}(n-1)!c^n}{(cx+d)^n}$$

$$= (-1)^n(n-1)!\left[\frac{a^n}{(ax+b)^n}+\frac{c^n}{(cx+d)^n}\right]$$

(4) $\dfrac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$

Example 41 :

Find the n^{th} derivative of e^{ax+b}

Sol. $y = e^{ax+b}$

$y_1 = ae^{ax+b}$

$y_2 = a^2 e^{ax+b}$

....

$y_n = a^n e^{ax+b}$

(5) $\dfrac{d^n(a^x)}{dx^n} = a^x(\log a)^n$

Example 42 :

Find the n^{th} derivative of 2^{3x+4}.

Sol. $y = 2^{3x+4}$

$y_1 = 2^{3x+4}.\log_e 2.3$

$y_2 = 2^{3x+4}(\log_e 2)^2.\,3^2$

....

$y_n = 2^{3x+4}(\log_e 2)^n.\,3^n$

$y_n = 2^{3x+4}(\log_e 2)^n.\,3^n$

(6) $\dfrac{d^n}{dx^n}(e^{ax}\sin(bx+c)) = r^n e^{ax}\sin(bx+c+n\phi)$

where $r=\sqrt{a^2+b^2}$; $\phi=\tan^{-1}\dfrac{b}{a}$

Example 43 :

Find $D^n(e^x \sin x)$

Sol. $D^n[e^{ax}\sin(bx+c)] = r^n e^{ax}\sin(bx+c+n\phi)$

where $r=\sqrt{a^2+b^2}$, $\phi=\tan^{-1}\left(\dfrac{b}{a}\right)$

Put a = 1, b = 1, c = 0

$$D^n[e^x\sin x] = (\sqrt{2})^n e^x \sin\left(x+\frac{n\pi}{4}\right)$$

n^{th} DERIVATIVE USING PARTIAL FRACTIONS

For finding the nth derivative of fractional expressions whose numerator and denominator are rational algebraic expression, firstly we resolve them into partial fractions and then we find nth derivative by using the formula giving the n^{th} derivative of $\dfrac{1}{ax+b}$.

Example 44 :

Find the n^{th} derivative of $\dfrac{1}{(1-5x+6x^2)}$

Sol. $y=\dfrac{1}{1-5x+6x^2}=\dfrac{1}{(2x-1)(3x-1)}=\dfrac{2}{2x-1}-\dfrac{3}{3x-1}$

$$\therefore\ y_n = (-1)^n\,n!\left[\frac{2^{n+1}}{(2x-1)^{n+1}}-\frac{3^{n+1}}{(3x-1)^{n+1}}\right]$$

Example 45 :

Find the n^{th} derivative of $\dfrac{x^4}{(x-1)(x-2)}$ (where n >2)

Sol. Let $y=\dfrac{x^4}{(x-1)(x-2)}=x^2+3x+7+\dfrac{16}{x-2}-\dfrac{1}{x-1}$

$$\therefore\ y_n=0+0+0+16.\frac{(-1)^n n!}{(x-2)^{n+1}}-\frac{(-1)^n n!}{(x-1)^{n+1}}$$

$$=(-1)^n n!\left[\frac{16}{(x-2)^{n+1}}-\frac{1}{(x-1)^{n+1}}\right]$$

LEIBNITZ'S THEOREM

G.W. Leibnitz, a German mathematician gave a method for evaluating the nth differential coefficient of the product of two functions. This method is known as Leibnitz's Theorem.

Statement of the theorem: If u and v are two functions of x such that their nth derivative exist, then

$D^n(uv)={}^nC_0\,(D^nu)v+{}^nC_1\,D^{n-1}.u.\,Dv+{}^nC_2\,D^{n-2}.\,u.\,D^2v+....+{}^nC_r\,D^{n-r}.\,u.\,D^rv+........+u.\,(D^nv)$

Note: The success in finding the nth derivative by this theorem lies in the proper selection of first and second function. Here first function should be selected whose nth derivative can be found by standard formulae. Second function should be such that on successive differentiation, at some stage, it becomes zero so that we need not to write further terms.

Example 46 :

Find n^{th} derivative of $x^2\ \sin 3x$

Sol. Let $u=\sin 3x$, $v=x^2$

$\therefore\ D^n(\sin 3x.x^2)=D^n(\sin 3x)x^2+{}^nC_1\,D^{n-1}(\sin3x).\,2x+{}^nC_2\,D^{\,n-2}(\sin3x).\,2$

$$=x^2\sin\left(3x+\frac{n\pi}{2}\right).\,3^n+2nx.\,3^{n-1}\sin\left[3x+\frac{1}{2}(n-1)\pi\right]$$

$$+\,3^{\,n-2}\,n\,(n-1)\,\sin\left\{3x+\frac{1}{2}(n-2)\pi\right\}$$

Example 47 :

Find n^{th} derivative of $x^3\,e^x$

Sol. $y=x^3\,e^x$

$D^n(e^x.\,x^3)=D^n(e^x).\,x^3+{}^nC_1\,D^{n-1}(e^x).\,3x^2+{}^nC_2\,D^{n-2}(e^x)\,6x+{}^nC_3\,D^{n-3}(e^x).\,6$

$=e^x x^3+3n\,e^x x^2+n\,(n-1)\,e^x.\,3x+n\,(n-1)\,(n-2)\,e^x$

ADDITIONAL EXAMPLES

Example 1 :

If $y=\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$, then find $\dfrac{dy}{dx}$.

Sol. Put $x=\tan\theta$

$\therefore\ \sqrt{1+x^2}=\sqrt{1+\tan^2\theta}=\sec\theta$

$$\therefore\ y=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)=\tan^{-1}\left(\tan\frac{\theta}{2}\right)=\frac{\theta}{2}=\frac{1}{2}\tan^{-1}x$$

$$\therefore\ \frac{dy}{dx}=\frac{1}{2}.\frac{1}{1+x^2}$$

Example 2 :

If $y=\cos^{-1}\left(\dfrac{5\cos x-12\sin x}{13}\right)$, $x\in\left(0,\dfrac{\pi}{2}\right)$, then find $\dfrac{dy}{dx}$

Sol. Let $\cos\alpha=\dfrac{5}{13}$. Then $\sin\alpha=\dfrac{12}{13}$.

So $y=\cos^{-1}\{\cos\alpha\cos x-\sin\alpha.\sin x\}$.

$\therefore\ y=\cos^{-1}\{\cos(x+\alpha)\}=x+\alpha$

($\therefore x+\alpha$ is the first or the second quadrant)

so $\dfrac{dy}{dx}=1$

Example 3 :

Evaluate $\dfrac{d}{dx}\left[\sin^2\left(\cot^{-1}\sqrt{\dfrac{1-x}{1+x}}\right)\right]$

Sol. $$\sin^2\left[\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right]=\frac{1}{\operatorname{cosec}^2\left[\cos^{-1}\sqrt{\frac{1-x}{1+x}}\right]}$$

$$=\frac{1}{1+\cot^2\left[\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right]}$$

$$=\frac{1}{1+\frac{1-x}{1+x}}=\frac{1+x}{1+x+1-x}=\frac{1+x}{2}$$

$$\therefore\ \frac{d}{dx}\left[\sin^2\left(\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right)\right]=\frac{1}{2}$$

Example 4 :

If f is an even function and f' exists, then f' (e)+ f' (–e) equals

(1) 0 (2) < 0

(3) > 0 (4) $\gtrless 0$

Sol. (1). Since f is an even function, $f(-x) = f(x)$, diff. both side w.r.t. x $\therefore f'(-x)(-1) = f'(x)$

$\Rightarrow -f'(-e) = f'(e)$ (By putting x = e)

$\Rightarrow f'(e) + f'(-e) = 0$

Example 5 :

Find the expression of $\frac{dy}{dx}$ of the function $y = a^{x^{x^{x\ldots\infty}}}$

Sol. We have $y = a^{x^y} \Rightarrow \log y = x^y \log a$ ...(1)

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log a\left[x^y \frac{d}{dx}(y\log x)\right] = (\log a)\,x^y\left(\frac{y}{x} + \log x\frac{dy}{dx}\right)$$

$$= (\log y)\left(\frac{y}{x} + \log x\frac{dy}{dx}\right) \quad \text{[By (1)]}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y} - \log x\log y\right) = \frac{y.\log y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2\log y}{x(1 - y\log x\log y)}$$

Example 6 :

Find the derivative of $(x^x)^x$

Sol. $\therefore (x^x)^x = x^{x^2}$

$$\therefore \frac{d}{dx}(x^x)^x = (x^{x^2})\frac{d}{dx}(x^2\log x)$$

$$= (x^x)^x[x^2(1/x) + 2x\log x] = x(x^x)^x\log(ex^2)$$

Example 7 :

If $y = f\left(\frac{2x-1}{x^2+1}\right)$ and $f'(x) = \sin x^2$, then find $\frac{dy}{dx}$

Sol. We have, $y = f\left(\frac{2x-1}{x^2+1}\right)$

$$\Rightarrow \frac{dy}{dx} = f'\left(\frac{2x-1}{x^2+1}\right).\left[\frac{(x^2+1)2 - (2x-1).2x}{(x^2+1)^2}\right]$$

$$= \sin\left(\frac{2x-1}{x^2+1}\right)^2.\left[\frac{2+2x-2x^2}{(x^2+1)}\right]$$

$$\left[\because f'(x) = \sin x^2, \therefore f'\left(\frac{2x-1}{x^2+1}\right) = \sin\left(\frac{2x-1}{x^2+1}\right)^2\right]$$

Example 8 :

If g is the inverse of f and $f'(x) = \frac{1}{1+x^3}$, then find g' (x).

Sol. We have, g = inverse of f = f^{-1}

$\Rightarrow g(x) = f^{-1}(x) \Rightarrow f[g(x)] = x$

Differentiating w.r.t.x, we get $f'[g(x)].\,g'(x) = 1$

$$\therefore g'(x) = \frac{1}{f'[g(x)]} = 1 + [g(x)]^3$$

$$\left[\because f'(x) = \frac{1}{1+x^3}, \therefore f'[g(x)] = \frac{1}{1+[g(x)]^3}\right]$$

Example 9 :

If y = cos 2x cos 3x, then find y_n.

Sol. We have, $y = \cos 2x\cos 3x = \frac{1}{2}[\cos 5x + \cos x]$

$$\therefore y_n = \frac{1}{2}\left[\frac{d^n}{dx^n}(\cos 5x) + \frac{d^n}{dx^n}(\cos x)\right]$$

$$= \frac{1}{2}\left[5^n\cos\left(\frac{n\pi}{2} + 5x\right) + \cos\left(\frac{n\pi}{2} + x\right)\right]$$

Example 10 :

Find the derivative of the function $f(x) = \log_5(\log_7 x)$, where x > 7 is

Sol. $f'(x) = \frac{1}{\log_7 x}, \log_5 e, \frac{1}{x}\log_7 e$

$$= \frac{1}{x\log_7 x(\log 5)(\log 7)} \quad \left[\because \log_5 e = \frac{1}{\log_e 5}, \log_7 e = \frac{1}{\log_e 7}\right]$$

Example 11 :

If $f(x) = x^{1/x}$, then find f" (e).

Sol. $\log y = \frac{1}{x}\log x \quad \therefore \frac{1}{y}\frac{dy}{dx} = \frac{x.\frac{1}{x} - 1.\log x}{x^2}$

$$\therefore f'(x) = f(x)\frac{(1-\log x)}{x^2} \quad \therefore f'(e) = 0 \text{ as } \log e = 1 \quad \text{..... (1)}$$

Differentiating by product

$$f''(x) = f'(x)\frac{(1-\log x)}{x^2} + f(x).\frac{x^2.\left(-\frac{1}{x}\right) - (1-\log x).2x}{x^4}$$

$$f''(x) = f'(x).\frac{1-\log x}{x^2} + \frac{f(x)}{x^3}[-1 - 2(1-\log x)]$$

Now put x = e, f' (e) = 0 and (1 – log x) = 0, by (1)

$$\therefore f''(e) = 0 - \frac{f(e)}{e^3} = -\frac{e^{1/e}}{e^3} = -e^{(1/e)-3}$$

Example 12 :

If $x = a\left(t+\frac{1}{t}\right), y = a\left(t-\frac{1}{t}\right)$, then find $\frac{dy}{dx}$.

Sol. $\frac{dx}{dt} = a\left(1-\frac{1}{t^2}\right) = a\frac{t^2-1}{t^2}$

$\frac{dy}{dt} = a\left(1+\frac{1}{t^2}\right) = a\left(\frac{t^2+1}{t^2}\right) \therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t^2+1}{t^2-1}$

Example 13 :

If $\sin y + e^{-x\cos y} = e$, then find $\frac{dy}{dx}$ at $(1, \pi)$.

Sol. Since $\sin y + e^{-x\cos y} = e$

$\Rightarrow \cos y \frac{dy}{dx} + e^{-x\cos y}\left(-\cos y + x\sin y\frac{dy}{dx}\right) = 0$

$\Rightarrow (\cos y + xe^{-x\cos y}.\sin y)\frac{dy}{dx} = \cos y.\, e^{-x\cos y}$

At $(1,\pi)$, we have

$(\cos\pi + e^{-\cos\pi}\sin\pi)\frac{dy}{dx} = \cos\pi e^{-\cos\pi}$

$\Rightarrow -1.\frac{dy}{dx} = (-1)\,e = -e \Rightarrow \frac{dy}{dx} = e$

Example 14 :

If $f(x) = x^n$ then find the value of

$f(1) + \frac{f'(1)}{1!} + \frac{f''(1)}{2!} + \frac{f'''(1)}{3!} + ... + \frac{f^n(1)}{n!}$

Sol. $f'(x) = nx^{n-1}$,

$f''(x) = n(n-1)x^{n-2}$, ... $f^n(x) = n!$. Putting $x = 1$ we have

$E = 1 + \frac{n}{1!} + \frac{n(n-1)}{2!} + ... + \frac{n!}{n!} = {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n$

Above is the sum of the binomial coefficients in the expansion of $(1 + x)^n$ which is $(1 + 1)^n = 2^n$

Example 15 :

If $y^2 = p(x)$, a polynomial of degree 3, then find

$2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$

Sol. $y^2 = p(x) \Rightarrow 2y\frac{dy}{dx} = p'(x) \Rightarrow 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = p''(x)$

$\therefore 2y\frac{d^3y}{dx^3} + 2\frac{dy}{dx}.\frac{d^2y}{dx^2} + 4\frac{dy}{dx}.\frac{d^2y}{dx^2} = p'''(x)$

$\Rightarrow 2yy_3 + 6y_1y_2 = p'''(x)$

Now $2\frac{d}{dx}(y^3 y_2) = 2[y^3 y_3 + 3y^2y_1y_2]$

$= y^2[2yy_3 + 6y_1y_2] = y^2p'''(x) = p(x)p'''(x)$

Example 16 :

If $y = x^3 \log x$, then find the value of y_4

Sol. Here $y = x^3 \log x$

$\Rightarrow y_1 = x^3.\frac{1}{x} + 3x^2\log x = x^2(1 + 3\log x)$

$\Rightarrow y_2 = 2x(1 + 3\log x) + x^2\left(0 + \frac{3}{x}\right) = x(5 + 6x\log x)$

$\Rightarrow y_3 = 1(5 + 6\log x) + x\left(0 + \frac{6}{x}\right) = 11 + 6\log x$

$\Rightarrow y_4 = 0 + \frac{6}{x} = \frac{6}{x}$

Example 17 :

If $y = \frac{x^4}{x^2 - 3x + 2}$ then for $n > 2$ the value of y_n = ?

Sol. Here $y = \frac{x^4}{x^2-3x+2} = x^2 + 3x + 7 + \frac{15x-14}{(x-1)(x-2)}$

$= x^2 + 3x + 7 + \frac{1}{x-1} + \frac{16}{x-2}$

$\therefore y_n = D^n(x^2) + D^n(3x) + D^n(7) + D^n[(x-1)^{-1}] + 16D^n[(x-2)^{-1}]$

$= (-1)^n n! [(x-1)^{-n-1} + 16(x-2)^{-n-1}]$

Example 18 :

If $(x-a)^2 + (y-b)^2 = c^2$ $(c > 0)$ then $\left[\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}\right] =$

(A) c
(B) c^2
(C) c^3
(D) c^4

Sol. **(A).** $(x-a)^2 + (y-b)^2 = c^2$ $(c > 0)$ .... (1)

Now differentiating w.r.t. to x.

$2(x-a) + 2(y-b)y_1 = 0$

$\Rightarrow (x-a) + (y-b)y_1 = 0$ .... (2)

$\Rightarrow 1 + (y-b)y_2 + (y_1)^2 = 0$ .... (3)

$1 + (y_1)^2 = 1 + \frac{(x-a)^2}{(y-b)^2} = \frac{c^2}{(y-b)^2}$ [From eq. (2)]

$\frac{(1+(y_1)^2)^{3/2}}{y_2} = \frac{\left(\frac{c^2}{(y-b)^2}\right)^{3/2}}{\frac{-(1+(y_1)^2)}{(y-b)}} = \frac{|c^3/(y-b)^3|}{-c^2/(y-b)^3} = \pm c$

[From eq. (3)]

QUESTION BANK

CHAPTER 5 : DIFFERENTIATION

EXERCISE - 1 [LEVEL-1]

Q.1 If $x = a\cos^3\theta$, $y = a\sin^3\theta$, then $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ equals-

(A) $\tan^2\theta$ (B) $\sec^2\theta$
(C) $\sec\theta$ (D) $|\sec\theta|$

Q.2 If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$, then $(1-x^2)\frac{dy}{dx}$ equals-

(A) $x + y$ (B) $1 + xy$
(C) $1 - xy$ (D) $xy - 2$

Q.3 If $y = \frac{1}{x^2 - a^2}$, then $\frac{d^2y}{dx^2}$ equals-

(A) $\frac{3x^2+a^2}{(x^2-a^2)^3}$ (B) $\frac{3x^2+a^2}{(x^2-a^2)^4}$

(C) $\frac{2(3x^2+a^2)}{(x^2-a^2)^3}$ (D) $\frac{2(3x^2+a^2)}{(x^2-a^2)^4}$

Q.4 If $y = \frac{\sec x - \tan x}{\sec x + \tan x}$, then $\frac{dy}{dx}$ equals-

(A) $2\sec x(\sec x - \tan x)^2$ (B) $-2\sec x(\sec x - \tan x)^2$
(C) $2\sec x(\sec x + \tan x)^2$ (D) $-2\sec x(\sec x + \tan x)^2$

Q.5 If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, then $\frac{dy}{dx}$ equals-

(A) $\frac{1}{(1+x)^2}$ (B) $-\frac{1}{(1+x)^2}$

(C) $\frac{1}{1+x^2}$ (D) None of these

Q.6 If $y = \sin^{-1}\sqrt{\sin x}$, then $\frac{dy}{dx}$ equals-

(A) $\frac{2\sqrt{\sin x}}{\sqrt{1+\sin x}}$ (B) $\frac{\sqrt{\sin x}}{\sqrt{1-\sin x}}$

(C) $\frac{1}{2}\sqrt{1+\operatorname{cosec} x}$ (D) $\frac{1}{2}\sqrt{1-\operatorname{cosec} x}$

Q.7 If y is a function of x then $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$. If x is a function of y then the equation becomes

(A) $\frac{d^2x}{dy^2} + x\frac{dx}{dy} = 0$ (B) $\frac{d^2x}{dy^2} + y\left(\frac{dx}{dy}\right)^3 = 0$

(C) $\frac{d^2x}{dy^2} - y\left(\frac{dx}{dy}\right)^2 = 0$ (D) $\frac{d^2x}{dy^2} - x\left(\frac{dx}{dy}\right)^2 = 0$

Q.8 If $\cos(xy) = x$, then $\frac{dy}{dx}$ is equal to -

(A) $\frac{y + \operatorname{cosec}(xy)}{x}$ (B) $\frac{y + \sin(xy)}{x}$

(C) $\frac{y + \cos(xy)}{x}$ (D) $-\frac{y + \operatorname{cosec}(xy)}{x}$

Q.9 If $x^2e^y + 2xye^x + 13 = 0$, then dy/dx equals-

(A) $-\frac{2xe^{y-x} + 2y(x+1)}{x(xe^{y-x}+2)}$ (B) $\frac{2xe^{y-x} + 2y(x+1)}{x(xe^{y-x}+2)}$

(C) $-\frac{2xe^{x-y} + 2y(x+1)}{x(xe^{x-y}+2)}$ (D) None of these

Q.10 Differentiate ln tanx with respect to $\sin^{-1}(e^x)$

(A) $\frac{e^{-x}\sqrt{1-e^{2x}}}{\sin x.\cos x}$ (B) $\frac{e^{-x}\sqrt{1-e^{2x}}}{\sin x.\cot x}$

(C) $\frac{e^{-x}\sqrt{1+e^{x}}}{\sin x.\cos x}$ (D) $\frac{e^{x}\sqrt{1+e^{2x}}}{\sin x.\sec x}$

Q.11 If $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$, then dy/dx equals-

(A) $\tan\theta$ (B) $\cot\theta$

(C) $\tan\frac{1}{2}\theta$ (D) $\cot\frac{1}{2}\theta$

Q.12 If $y = \log\left(\frac{e^x}{e^x+1}\right)$, then dy/dx equals-

(A) $\frac{1}{e^x+1}$ (B) $\frac{1}{(e^x+1)^2}$

(C) $\frac{e^x-1}{e^x+1}$ (D) None

Q.13 If $y = (\sin x)^{\ln x}\operatorname{cosec}(e^x(a+bx))$ and $a + b = \frac{\pi}{2e}$ then the value of $\frac{dy}{dx}$ at $x = 1$ is

(A) $(\sin 1)\ln\sin 1$ (B) 0
(C) $\ln\sin 1$ (D) indeterminate

Q.14 If $x = e^{\tan^{-1}\left(\frac{y-x^2}{x^2}\right)}$,then dy/dx equals-

(A) $x[1+\tan(\log x)] + \sec^2(\log x)$
(B) $2x[1+\tan(\log x)] + x\sec^2(\log x)$
(C) $2x[1+\tan(\log x)] + x\sec(\log x)$
(D) None of these

Q.15 If $y = \tan^{-1}\frac{3x-x^3}{1-3x^2}$, then dy/dx equals-

(A) 3x (B) tan 3x
(C) $\frac{3}{1+x^2}$ (D) $3\tan^{-1}x$

Q.16 If $f(x) = 3x^{10} - 7x^8 + 5x^6 - 21x^3 + 3x^2 - 7$ then the value of $\lim_{x\to1^+}\frac{f(1-h)-f(1)}{h^3+3h}$ is

(A) – 53/3 (B) – 22/3
(C) 53/3 (D) 22/3

Q.17 If $y = \tan^{-1}\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$, then $\frac{dy}{dx}$ equals-

(A) $-\frac{1}{2\sqrt{1-x^2}}$ (B) $-\frac{1}{\sqrt{1-x^4}}$
(C) $-\frac{x}{\sqrt{1-x^4}}$ (D) $-\frac{x}{2\sqrt{1-x^4}}$

Q.18 If $\sqrt{1-x^2}+\sqrt{1-y^2} = a(x-y)$, then the value of $\frac{dy}{dx}$ is-

(A) $\frac{\sqrt{1-x^2}}{\sqrt{1-y^2}}$ (B) $\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$
(C) $-\frac{\sqrt{1-x^2}}{\sqrt{1-y^2}}$ (D) $-\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$

Q.19 If $y = \frac{1}{\sqrt{a^2-b^2-c^2}}\cos^{-1}\left\{\frac{a\theta - a^2+b^2+c^2}{\theta\sqrt{b^2+c^2}}\right\}$ &

$\theta = a + b\cos x + c\sin x$; find $\frac{dy}{dx}$.

(A) $\frac{1}{\theta}$ (B) $\frac{1}{\sqrt{\theta}}$
(C) $\frac{1}{\theta^2}$ (D) None

Q.20 If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + ...\infty}}}$, then $\frac{dy}{dx}$ equals-

(A) $\frac{\sin x}{2y+1}$ (B) $\frac{\cos x}{2y-1}$
(C) $\frac{\cos x}{2y+1}$ (D) None

Q.21 If $y = \cfrac{x}{a+\cfrac{x}{b+\cfrac{x}{a+\cfrac{x}{b+...\infty}}}}$ ..... ∞, then $\frac{dy}{dx}$ equals-

(A) $\frac{b}{a(b+2y)}$ (B) $\frac{a}{b(a+2y)}$
(C) $\frac{a}{b(b+2y)}$ (D) None

Q.22 If $e^{x+e^{x+e^{x+.....\infty}}}$, then dy/dx is -

(A) $\frac{y}{1+y}$ (B) $\frac{y}{y-1}$
(C) $\frac{y}{1-y}$ (D) None of these

Q.23 If $(a+bx)e^{y/x} = x$, then the value of $x^3\frac{d^2y}{dx^2}$ is-

(A) $\left(y\frac{dy}{dx}-x\right)^2$ (B) $\left(x\frac{dy}{dx}-y\right)^2$
(C) $x\frac{dy}{dx}-y$ (D) None of these

Q.24 Let $f(x) = \frac{\tan^6 x + 9\tan^4 x - 9\tan^2 x - 1}{3\tan^3 x}$, if $f'(x) = \lambda\,\text{cosec}^4(2x)$ then the value of λ equals

(A) 16 (B) 12
(C) 18 (D) 14

Q.25 Suppose $F(x) = f(g(x))$ and $g(3) = 5$, $g'(3) = 3$, $f'(3) = 1$, $f'(5) = 4$. Then the value of $F'(3)$, is

(A) 16 (B) 12
(C) 18 (D) 14

Q.26 If $f(x) = |x-2|$ and $g(x) = f[f(x)]$, then for $x > 20$, $g'(x) =$

(A) 1 (B) –1
(C) 0 (D) None of these

Q.27 f(x) is a function such that $f''(x) = -f(x)$ and $f'(x) = g(x)$ and h(x) is a function such that $h(x) = [f(x)]^2 + [g(x)]^2$ and $h(5) = 11$, then the value of h (10) is-

(A) 0 (B) 1
(C) 10 (D) None of these

Q.28 If $x = (\sec\theta - \cos\theta)$ and $y = \sec^n\theta - \cos^n\theta$, then $(dy/dx)^2 =$

(A) $\frac{y^2+4}{n^2(x^2+4)}$ (B) $\frac{y^2+4}{n(x^2+4)}$

(C) $\frac{n^2(y^2+4)}{x^2+4}$ (D) $\frac{n(y^2+4)}{x^2+4}$

Q.29 The value of the derivative of $|x-1|+|x-3|$ at $x = 2$ is-

(A) –2 (B) 0
(C) 2 (D) Not defined

Q.30 Let $f(x) = \frac{\sin x}{x}$ if $x \neq 0$ and $f(0) = 1$ then define the function $f'(x)$ for all x

(A) $\begin{cases} \frac{x\cos x - \sin x}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(B) $\begin{cases} \frac{x\cos x + \sin x}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(C) $\begin{cases} \frac{x\cos x - \sin x}{x^3} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(D) None of these

Q.31 The first derivative of the function $(\sin 2x \cos 2x \cos 3x + \log_2 2^{x+3})$ w.r.t. x at $x = \pi$ is -

(A) 2 (B) –1
(C) $-2+2\pi\log_e 2$ (D) $-2+\log_e 2$

Q.32 If $f(x) = \cos^{-1}\left(\sin\sqrt{\frac{1+x}{2}}\right) + x^x$, then at $x = 1$, $f'(x) =$

(A) 0 (B) 1/2
(C) –1/2 (D) 3/4

Q.33 If $y = \sin^{-1}\frac{2x}{1+x^2} + \sec^{-1}\frac{1+x^2}{1-x^2}$, then $\frac{dy}{dx} =$

(A) $\frac{4}{1-x^2}$ (B) $\frac{1}{1+x^2}$

(C) $\frac{4}{1-x^2}$ (D) $\frac{-4}{1+x^2}$

Q.34 $\frac{d}{dx}\tan^{-1}\left[\frac{3a^2x - x^3}{a(a^2-3x^2)}\right]$ at $x = 0$ is

(A) 1/a (B) 3/a
(C) 3a (D) 3

Q.35 The differential coefficient of $\cos^{-1}\left\{\sqrt{\frac{1+x}{2}}\right\}$ wrt x is

(A) $-\frac{1}{2\sqrt{1-x^2}}$ (B) $\frac{1}{2\sqrt{1-x^2}}$

(C) $\frac{1}{\sqrt{1-x}}$ (D) $\sin^{-1}\left\{\sqrt{\frac{1+x}{2}}\right\}$

Q.36 If $y = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, then $\frac{dy}{dx} =$

(A) $-\frac{1}{\sqrt{1-x^2}}$ (B) $\frac{x}{\sqrt{1-x^2}}$

(C) $\frac{1}{\sqrt{1-x^2}}$ (D) $\frac{\sqrt{1-x^2}}{x}$

Q.37 The differential coefficient of $\tan^{-1}\sqrt{x}$ with respect to $\sqrt{x}$ is –

(A) $\frac{1}{\sqrt{1+x}}$ (B) $\frac{1}{2x\sqrt{1+x}}$

(C) $\frac{1}{2\sqrt{x(1+x)}}$ (D) $\frac{1}{1+x}$

Q.38 $\frac{d^n}{dx^n}(e^{2x}+e^{-2x}) =$

(A) $e^{2x}+(-1)^n e^{-2x}$ (B) $2^n(e^{2x}-e^{-2x})$

(C) $2^n[e^{2x}+(-1)^n e^{-2x}]$ (D) None of these

Q.39 If $y = x^3 \log\log_e(1+x)$, then $y''(0)$ equals

(A) 0 (B) – 1
(C) $6\log_e 2$ (D) 6

Q.40 If $x = A\cos 4t + B\sin 4t$, then $\frac{d^2x}{dt^2} =$

(A) – 16x (B) 16x
(C) x (D) – x

Q.41 If $y = b\cos\log\left(\frac{x}{n}\right)^n$, then $\frac{dy}{dx} =$

(A) $-n\ b\sin\log\left(\frac{x}{n}\right)^n$ (B) $n\ b\sin\log\left(\frac{x}{n}\right)^n$

(C) $\frac{-nb}{x}\sin\log\left(\frac{x}{n}\right)^n$ (D) None of these

Q.42 $\frac{d}{dx}e^{x+3\log x} =$

(A) $e^x.x^2(x+3)$ (B) $e^x.x(x+3)$

(C) $e^x+\frac{3}{x}$ (D) None of these

Q.43 If $f(x) = (\cos x + i\sin x)(\cos 3x + i\sin 3x)$............ $\{\cos(2n-1)x + i\sin(2n-1)x\}$ then $f''(x)$ is equal to –

(A) $n^2 f(x)$ (B) $-n^4 f(x)$

(C) $-n^2 f(x)$ (D) 0

Q.44 $\frac{d}{dx}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 =$

(A) $1-\frac{1}{x^2}$ (B) $1+\frac{1}{x^2}$

(C) $1-\frac{1}{2x}$ (D) None of these

Q.45 If $x^m y^n = 2(x+y)^{m+n}$, the value of $\frac{dy}{dx}$ is

(A) $x+y$ (B) x/y

(C) y/x (D) $x-y$

Q.46 If $y = \sin^{-1}(x\sqrt{1-x}+\sqrt{x}\sqrt{1-x^2})$, then $\frac{dy}{dx} =$

(A) $\frac{-2x}{\sqrt{1-x^2}}+\frac{1}{2\sqrt{x-x^2}}$ (B) $\frac{-1}{\sqrt{1-x^2}}-\frac{1}{2\sqrt{x-x^2}}$

(C) $\frac{1}{\sqrt{1-x^2}}+\frac{1}{2\sqrt{x-x^2}}$ (D) None of these

Q.47 $\frac{d}{dx}\tan^{-1}\left(\frac{ax-b}{bx+a}\right) =$

(A) $\frac{1}{1+x^2}-\frac{a^2}{a^2+b^2}$ (B) $\frac{-1}{1+x^2}-\frac{a^2}{a^2+b^2}$

(C) $\frac{1}{1+x^2}+\frac{a^2}{a^2+b^2}$ (D) None of these

Q.48 $\frac{d}{dx}\log_7(\log_7 x) =$

(A) $\frac{1}{x\log_e x}$ (B) $\frac{\log_e 7}{x\log_e x}$

(C) $\frac{\log_7 e}{x\log_e x}$ (D) $\frac{\log_7 e}{x\log_7 x}$

Q.49 $\frac{d}{dx}\cos^{-1}\sqrt{\cos x} =$

(A) $\frac{1}{2}\sqrt{1+\sec x}$ (B) $\sqrt{1+\sec x}$

(C) $-\frac{1}{2}\sqrt{1+\sec x}$ (D) $-\sqrt{1+\sec x}$

Q.50 If $y = (x\cot^3 x)^{3/2}$, then dy/dx =

(A) $\frac{3}{2}(x\cot^3 x)^{1/2}[\cot^3 x - 3x\cot^2 x\,\mathrm{cosec}^2 x]$

(B) $\frac{3}{2}(x\cot^3 x)^{1/2}[\cot^2 x - 3x\cot^2 x\,\mathrm{cosec}^2 x]$

(C) $\frac{3}{2}(x\cot^3 x)^{1/3}[\cot^3 x - 3x\,\mathrm{cosec}^2 x]$

(D) $\frac{3}{2}(x\cot^3 x)^{3/2}[\cot^3 x - 3x\,\mathrm{cosec}^2 x]$

Q.51 $\frac{d}{dx}\sqrt{x\sin x} =$

(A) $\frac{\sin x + x\cos x}{2\sqrt{x\sin x}}$ (B) $\frac{\sin x + x\cos x}{\sqrt{x\sin x}}$

(C) $\frac{x\sin x + \cos x}{\sqrt{2\sin x}}$ (D) $\frac{\sin x + x\cos x}{2\sqrt{x\sin x}}$

Q.52 The differential coefficient of the given function $\log_e\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$ with respect to x is

(A) cosec x (B) tan x

(C) cos x (D) sec x

Q.53 If $y = \log x.e^{(\tan x + x^2)}$, then $\frac{dy}{dx} =$

(A) $e^{(\tan x+x^2)}\left[\frac{1}{x}+(\sec^2 x + x)\log x\right]$

(B) $e^{(\tan x+x^2)}\left[\frac{1}{x}+(\sec^2 x - x)\log x\right]$

(C) $e^{(\tan x+x^2)}\left[\frac{1}{x}+(\sec^2 x + 2x)\log x\right]$

(D) $e^{(\tan x+x^2)}\left[\frac{1}{x}+(\sec^2 x - 2x)\log x\right]$

Q.54 At $x = \sqrt{\frac{\pi}{2}}, \frac{d}{dx}\cos(\sin x^2) =$

(A) –1 (B) 1

(C) 0 (D) None of these

Q.55 If $y = \log_{\sin x}(\tan x)$, then $\left(\frac{dy}{dx}\right)_{\pi/4} =$

(A) $\frac{4}{\log 2}$ (B) – 4 log 2 (C) $\frac{-4}{\log 2}$ (D) None

Q.56 If $y = (\sin x)^{\tan x}$, then $\frac{dy}{dx}$ is equal to

(A) $(\sin x)^{\tan x}.(1+\sec^2 x.\log \sin x)$

(B) $\tan x.(\sin x)^{\tan x-1}.\cos x$

(C) $(\sin x)^{\tan x}.\sec^2 x.\log \sin x$

(D) $\tan x.(\sin x)^{\tan x-1}$

Q.57 The first derivative of the function $(\sin 2x \cos 2x \cos 3x + \log_2 2^{x+3})$ with respect to x at $x=\pi$ is

(A) 2 (B) –1

(C) $-2+2^{\pi}\log_e 2$ (D) $-2+\log_e 2$

Q.58 If $f(1)=3, f'(1)=2$, then $\frac{d}{dx}\{\log f(e^x+2x)\}$ at $x=0$ is

(A) 2 / 3 (B) 3 / 2

(C) 2 (D) 0

Q.59 If $y=(1+x^2)\tan^{-1}x - x$, then $\frac{dy}{dx}=$

(A) $\tan^{-1}x$ (B) $2x\tan^{-1}x$

(C) $2x\tan^{-1}x-1$ (D) $\frac{2x}{\tan^{-1}x}$

Q.60 $\frac{d}{dx}\left[\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}\right]$ equals to

(A) 1 (B) $\frac{x^2+1}{x^2-4}$

(C) $\frac{x^2-1}{x^2-4}$ (D) $e^x\frac{x^2-1}{x^2-4}$

Q.61 If $f(x)=\sqrt{ax}+\frac{a^2}{\sqrt{ax}}$, then $f'(a)=$

(A) – 1 (B) 1

(C) 0 (D) a

Q.62 If $y=\tan^{-1}(\sec x-\tan x)$ then $\frac{dy}{dx}=$

(A) 2 (B) –2

(C) 1/2 (D) –1/2

Q.63 If $f(x)=\cos x\cos 2x\cos 4x\cos 8x\cos 16x$, then $f'\left(\frac{\pi}{4}\right)$

(A) $\sqrt{2}$ (B) $\frac{1}{\sqrt{2}}$

(C) 1 (D) $\frac{\sqrt{3}}{2}$

Q.64 If $x=at^2, y=2at$, then $\frac{d^2y}{dx^2}=$

(A) $-\frac{1}{t^2}$ (B) $\frac{1}{2at^3}$

(C) $-\frac{1}{t^3}$ (D) $-\frac{1}{2at^3}$

Q.65 If $\sin y = x\sin(a+y)$, then $\frac{dy}{dx}=$

(A) $\frac{\sin^2(a+y)}{\sin(a+2y)}$ (B) $\frac{\sin^2(a+y)}{\cos(a+2y)}$

(C) $\frac{\sin^2(a+y)}{\sin a}$ (D) $\frac{\sin^2(a+y)}{\cos a}$

Q.66 If $ax^2+2hxy+by^2+2gx+2fy+c=0$, then $\frac{dy}{dx}=$

(A) $-\frac{ax+hy+g}{hx-by+f}$ (B) $\frac{ax+hy+g}{hx-by+f}$

(C) $\frac{ax-hy-g}{hx-by-f}$ (D) None of these

Q.67 If $x=a\sin\theta$ and $y=b\cos\theta$, then $\frac{d^2y}{dx^2}$ is

(A) $\frac{a}{b^2}\sec^2\theta$ (B) $\frac{-b}{a}\sec^2\theta$

(C) $\frac{-b}{a^2}\sec^3\theta$ (D) $\frac{-b}{a^2}\sec^3\theta$

Q.68 If $x=\frac{2t}{1+t^2}$, $y=\frac{1-t^2}{1+t^2}$, then $\frac{dy}{dx}$ equals

(A) $\frac{2t}{t^2+1}$ (B) $\frac{2t}{t^2-1}$

(C) $\frac{2t}{1-t^2}$ (D) None

Q.69 If $x=\sin^{-1}(3t-4t^3)$ & $y=\cos^{-1}\sqrt{(1-t^2)}$, then $\frac{dy}{dx}$

(A) 1/2 (B) 2/5

(C) 3/2 (D) 1/3

Q.70 If $x^y=e^{x-y}$, then $\frac{dy}{dx}=$

(A) $\log x.[\log(ex)]^{-2}$ (B) $\log x.[\log(ex)]^2$

(C) $\log x.(\log x)^2$ (D) None of these

Q.71 If $y=(1+x)^x$, then $\frac{dy}{dx}=$

(A) $(1+x)^x\left[\frac{x}{1+x}+\log ex\right]$ (B) $\frac{x}{1+x}+\log(1+x)$

(C) $(1+x)^x\left[\frac{x}{1+x}+\log(1+x)\right]$ (D) None of these

Q.72 $\frac{d}{dx}(x^{\log_e x})=$

(A) $2x^{(\log_e x-1)}.\log_e x$ (B) $x^{(\log_e x-1)}$

(C) $\frac{2}{x}\log_e x$ (D) $x^{(\log_e x-1)}.\log_e x$

Q.73 If $y=x^{\sin x}$, then $\frac{dy}{dx}=$

(A) $\frac{x\cos x.\log x+\sin x}{x}.x^{\sin x}$

(B) $\frac{y[x\cos x.\log x+\cos x]}{x}$

(C) $y[x\sin x.\log x+\cos x]$

(D) None of these

Q.74 $\frac{d}{dx}\{(\sin x)^{\log x}\}=$

(A) $(\sin x)^{\log x}\left[\frac{1}{x}\log\sin x+\cot x\right]$

(B) $(\sin x)^{\log x}\left[\frac{1}{x}\log\sin x+\cot x\log x\right]$

(C) $(\sin x)^{\log x}\left[\frac{1}{x}\log\sin x+\log x\right]$

(D) None of these

Q.75 **Statement 1 :** $\frac{d}{dx}\{\tan^{-1}(\sec x+\tan x)\}$

$=\frac{d}{dx}\{\cot^{-1}(\operatorname{cosec} x+\cot x)\}$] $x\in\left(0,\frac{\pi}{4}\right)$

Statement 2 : $\sec^2 x-\tan^2 x=1=\operatorname{cosec}^2 x-\cot^2 x$

(A) Statement- 1 is True, Statement-2 is true, statement-2 is a correct explanation for Statement -1

(B) Statement -1 is True, Statement -2 is true; statement-2 is NOT a correct explanation for Statement - 1

(C) Statement - 1 is True, Statement- 2 is False

(D) Statement -1 is False, Statement -2 is True

Passage (Q.76-Q.77)

In certain problem the differentiation of {f (x) . g (x)} appears. One student commits mistake and differentiate as $\frac{df}{dx}.\frac{dg}{dx}$ but he gets correct result if $f(x)=x^3+g(x)$ is a decreasing function for which g (0) = 1/3.

Q.76 The function g (x) is –

(A) $\frac{3}{(x-3)^3}$ (B) $\frac{4}{(x-3)^3}$

(C) $\frac{9}{(x-3)^3}$ (D) $\frac{27}{(x-3)^3}$

Q.77 Derivative of { f (x – 3) . g (x) } with respect to x at x = 100 is

(A) 0 (B) 1

(C) – 1 (D) 2

NOTE : The answer to each question is a NUMERICAL VALUE.

Q.78 Suppose the function $f(x)-f(2x)$ has the derivative 5 at x = 1 and derivative 7 at x = 2. The derivative of the function $f(x)-f(4x)$ at x = 1, has the value equal to

Q.79 The equation $y^2e^{xy}=9e^{-3}\cdot x^2$ defines y as a differentiable function of x. The value of $\frac{dy}{dx}$ for x = – 1 and y = 3 is

Q.80 The derivative of the function,

$$f(x)=\cos^{-1}\left\{\frac{1}{\sqrt{13}}(2\cos x-3\sin x)\right\}$$

$$+\sin^{-1}\left\{\frac{1}{\sqrt{13}}(2\cos x+3\sin x)\right\} \text{ w.r.t. } \sqrt{1+x^2}$$

at $x=\frac{3}{4}$ is 10/A. Find the value of A.

Q.81 The graph of function f contains the point P (1, 2) and Q(s, r). The equation of the secant line through P and Q is $y=\left(\frac{s^2+2s-3}{s-1}\right)x-1-s$. The value of $f'(1)$, is

Q.82 If $y=\frac{\sin^2 x}{1+\cot x}+\frac{\cos^2 x}{1+\tan x}$ then $\frac{dy}{dx}$ at $x=\frac{\pi}{4}$ is

Q.83 Let $f(x)=1+x^3$. If $g(x)=f^{-1}(x)$, i.e. if g is the inverse f, then g'(9) equal to 1/A. Find the value of A.

Q.84 Suppose f is a differentiable function such that for every real number x, $f(x)+2f(-x)=\sin x$, then $f'\left(\frac{\pi}{4}\right)$ has the value equal to $-\frac{1}{\sqrt{A}}$.

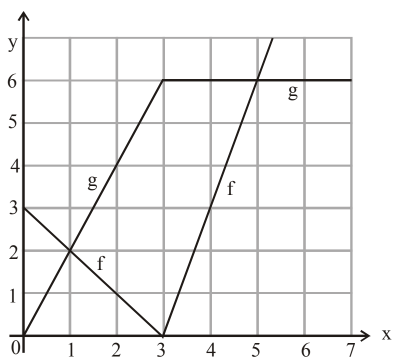
Q.85 If sin(x + 2y) = 2x cos y, the value of $\frac{dy}{dx}$ at the point (0, π) must be –3/A. Find the value of A.

Q.86 Let f, g and h are differentiable functions. If $f(0) = 1$; $g(0) = 2$; $h(0) = 3$ and the derivatives of their pair wise products at $x = 0$ are $(fg)'(0) = 6$; $(gh)'(0) = 4$ and $(hf)'(0) = 5$ then compute the value of $(fgh)'(0)$.

Q.87 Let u (x) and v (x) are differentiable functions such that $\frac{u(x)}{v(x)} = 7$. If $\frac{u'(x)}{v'(x)} = p$ and $\left(\frac{u(x)}{v(x)}\right)' = q$, then $\frac{p+q}{p-q}$ has the value equal to

Q.88 If f and g are the functions whose graphs are shown, let $P(x) = f(x)\,g(x)$, $Q(x) = \frac{f(x)}{g(x)}$ and $C(x) = f(g(x))$.

The value of $(P'(2) - C'(2))\,Q'(2)$ equals –



Q.89 Let $g(x) = f(x / f(x))$ where f (x) is a differentiable positive function on $(0, \infty)$ such that $f(1) = f'(1) = 2$, then $g'(1)$ equals –

Q.90 A student forgot the product rule for differentiation and made the mistake of thinking that $(f \cdot g)' = f'g'$. However he was lucky to get the correct answer. The function f that he used was $f(x) = e^{x^2}$. If the domain of g (x) was the interval $\left(\frac{1}{2}, \infty\right)$ with $g(1) = e$ and the value of g (5) is ke^c, find $(k + c)$.

Q.91 If the function $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is

Q.92 Let $y(x) = \cos(3\cos^{-1}x)$, $x \in [-1, 1]$; $x \neq \pm\frac{\sqrt{3}}{2}$ then find

$$\frac{1}{y(x)}\left\{(x^2-1)\frac{d^2y(x)}{dx^2} + x\frac{dy(x)}{dx}\right\}$$

EXERCISE - 2 [PREVIOUS YEARS JEE MAIN QUESTIONS]

Q.1 If $y = \log_y x$, then $\frac{dy}{dx} =$ **[AIEEE 2002]**

(A) $\frac{1}{x + \log y}$ (B) $\frac{1}{\log x(1+y)}$

(C) $\frac{1}{x(1+\log y)}$ (D) $\frac{1}{y + \log x}$

Q.2 If $x = 3\cos\theta - 2\cos^3\theta$ and $y = 3\sin\theta - 2\sin^3\theta$, then $\frac{dy}{dx} =$ **[AIEEE 2002]**

(A) $\sin\theta$ (B) $\cos\theta$

(C) $\tan\theta$ (D) $\cot\theta$

Q.3 If $y = (x + \sqrt{1+x^2})^n$ then $(1+x^2)y_2 + xy_1 =$ **[AIEEE-2002]**

(A) ny^2 (B) n^2y

(C) n^2y^2 (D) None of these

Q.4 If for all values of x and y $f(x+y) = f(x).f(y)$ and $f(5) = 2f'(0) = 3$ then $f'(5)$ is – **[AIEEE-2002]**

(A) 3 (B) 4

(C) 5 (D) 6

Q.5 If $f(x) = x^n$, then the value of

$f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + ... + \frac{(-1)^n f^n(1)}{n!}$ is-

(A) 1 (B) 2^n **[AIEEE 2003]**

(C) 2^{n-1} (D) 0

Q.6 Let f(x) be a polynomial function of second degree. If $f(A) = f(-1)$ and a, b, c are in A.P. then $f'(a)$, $f'(b)$ and $f'(c)$ are in- **[AIEEE 2003]**

(A) Arithmetic - Geometric Progressio

(B) A.P.

(C) G.P.

(D) H.P.

Q.7 If $x = e^{y+e^{y+.....to\ \infty}}$, $x > 0$, then $\frac{dy}{dx}$ is **[AIEEE 2004]**

(A) $\frac{x}{1+x}$ (B) $\frac{1}{x}$ (C) $\frac{1-x}{x}$ (D) $\frac{1+x}{x}$

Q.8 If $x^m.y^n = (x+y)^{m+n}$, then $\frac{dy}{dx}$ is **[AIEEE 2006]**

(A) $\frac{x+y}{xy}$ (B) xy

(C) $\frac{x}{y}$ (D) $\frac{y}{x}$

Q.9 Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then $y'(1)$ equals - **[AIEEE 2009]**

(A) –1 (B) 1

(C) log 2 (D) – log 2

Q.10 Let $f : (-1, 1) \to R$ be a differentiable function with $f(0) = -1$ and $f'(0) = 1$. Let $g(x) = [f(2f(x) + 2)]^2$. Then $g'(0) =$ **[AIEEE 2010]**

(A) – 4 (B) 0

(C) –2 (D) 4

Q.11 $\frac{d^2x}{dy^2}$ equals – **[AIEEE 2011]**

(A) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (B) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

(C) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (D) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

Q.12 If $y = \sec(\tan^{-1}x)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to –

(A) $1/\sqrt{2}$ (B) 1/2 **[JEE MAIN 2013]**

(C) 1 (D) $\sqrt{2}$

Q.13 If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^5}$, then $g'(x)$ is equal to – **[JEE MAIN 2014]**

(A) $1 + x^5$ (B) $5x^4$

(C) $\frac{1}{1+\{g(x)\}^5}$ (D) $1 + \{g(x)\}^5$

Q.14 If for $x \in \left(0, \frac{1}{4}\right)$, the derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x}\cdot g(x)$, then g (x) equals – **[JEE MAIN 2017]**

(A) $\frac{3x}{1-9x^3}$ (B) $\frac{3}{1+9x^3}$

(C) $\frac{9}{1+9x^3}$ (D) $\frac{3x\sqrt{x}}{1-9x^3}$

Q.15 If $x = 3\tan t$ and $y = 3\sec t$, then the value of $\frac{d^2y}{dx^2}$ at $t = \pi/4$, is: **[JEE MAIN 2019 (Jan)]**

(A) $\frac{3}{2\sqrt{2}}$ (B) $\frac{1}{3\sqrt{2}}$ (C) $\frac{1}{6}$ (D) $\frac{1}{6\sqrt{2}}$

Q.16 If $f(1) = 1$, $f'(1) = 3$, then the derivative of $f(f(f(x))) + (f(x))^2$ at $x = 1$ is : **[JEE MAIN 2019 (April)]**

(A) 12 (B) 33

(C) 9 (D) 15

Q.17 Let $(x)^k + (y)^k = (a)^k$ where a, k > 0 and

$\frac{dy}{dx} + \left(\frac{y}{x}\right)^{1/3} = 0$, then find k **[JEE MAIN 2020 (Jan)]**

(A) 1/3 (B) 2/3
(C) 4/3 (D) 2

Q.18 If $y = \sqrt{\frac{2(\tan\alpha + \cot\alpha)}{1+\tan^2\alpha} + \frac{1}{\sin^2\alpha}}$ when

$\alpha \in \left(\frac{3\pi}{4}, \pi\right)$ then find $\frac{dy}{d\alpha}$ at $\alpha = \frac{5\pi}{6}$

[JEE MAIN 2020 (Jan)]

(A) 4 (B) 2
(C) 3 (D) – 4

Q.19 If x = 2sin θ – sin 2θ and y = 2cos θ – cos 2θ,

$\theta \in [0, 2\pi]$, then $\frac{d^2y}{dx^2}$ at $\theta = \pi$ is :

[JEE MAIN 2020 (JAN)]

(A) 3/2 (B) –3/4
(C) 3/4 (D) 3/8

Q.20 Let f and g be differentiable functions on R such that fog is the identity function. If for some a,b ∈R, g'(a) = 5 and g(a) = b, then f '(b) is equal to : **[JEE MAIN 2020 (JAN)]**

(A) 2/5 (B) 1
(C) 1/5 (D) 5

ANSWER KEY

EXERCISE - 1

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
A	D	B	C	B	B	C	C	D	A	A	C	A	C	B	C	C	C	B	A	B	A	C	B	A	B	A	D	C	B	A
Q	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
A	B	D	C	B	A	C	D	C	A	A	C	A	B	A	C	C	D	C	A	A	A	D	C	C	C	A	B	C	B	C
Q	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90
A	C	B	A	D	C	A	C	B	D	A	C	A	A	B	B	C	A	19	15	3	4	0	12	2	2	16	1	6	0	8
Q	91	92																												
A	2	9																												

EXERCISE - 2

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
A	C	D	B	D	D	B	C	D	A	A	D	A	D	C	D	B	B	A	D	C

CHAPTER- 5:
DIFFERENTIATION
TRY IT YOURSELF

(1) $y = f(x) = \ln^2 x$

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\ln^2(x+h)-\ln^2 x}{h}$$

$$= \lim_{h\to 0}\left(\frac{\ln(x+h)-\ln x}{h}\right)(\ln(x+h)+\ln x)$$

$$= \lim_{h\to 0}\frac{\ln\left(1+\frac{x}{h}\right)}{h}(\ln(x+h)+\ln x) = \frac{2\ln x}{x}$$

(2) $y = (1+x^{1/4})(1+x^{1/2})(1-x^{1/4})$
$= (1+x^{1/4})(1-x^{1/4})(1+x^{1/2}) = (1-x^{1/2})(1+x^{1/2})$

$$= 1-x \Rightarrow \frac{dy}{dx} = -1$$

(3) $f(x) = 1+x+x^2+.....+x^{100}$
$f'(x) = 0+1+2x+3x^2+.......+100x^{99}$

$$\therefore\ f'(1) = 1+2+3+.....+100 = \frac{100(100+1)}{2} = 5050$$

(4) $y = \dfrac{1-\ln x}{1+\ln x}$.

$$\frac{dy}{dx} = \frac{(1+\ln x)\left(\frac{-1}{x}\right)-\frac{(1-\ln x)}{x}}{(1+\ln x)^2}$$

$$= \frac{-(1+\ln x+1-\ln x)}{x(1+\ln x)^2} = \frac{-2}{x(1+\ln x)^2}$$

(5) $y = \dfrac{x^4+x^2+1}{x^2+x+1} = \dfrac{(x^4+2x^2+1)-x^2}{x^2+x+1} = x^2-x+1$

$$\frac{dy}{dx} = 2x-1 = ax+b \Rightarrow a = 2,\ b = -1$$

(6) **(C).** $f(x) = (1+x)(3+x2)^{1/2}(9+x^3)^{1/3}$

$$f'(x) = (3+x^2)^{1/2}(9+x^3)^{1/3}+(1+x)\,\frac{1}{2}\,(3+x^2)^{-1/2}$$

$$.2x\,(9+x^3)^{1/3}+(1+x)\,(9+x^2)^{1/2}+\left(\frac{1}{3}(9+x^3)^{-2/3}\cdot 3x^2\right)$$

$f'(-1) = (3+1)^{1/2}(9-1)^{1/3}+0+0 = 2\times 2 = 4$

(7) **(D).** $f(x) = (x+1)(x+2)(x+3)\,.....\,(x+n)$
$\ln f(x) = \ln(x+1)+\ln(x+2)+\ln(x+3)\,.....+\ln(x+n)$

$$\frac{f'(x)}{f(x)} = \frac{1}{(x+1)}+\frac{1}{(x+2)}+\frac{1}{(x+3)}+.....+\frac{1}{(x+n)}$$

$f(0) = n!$

$$f'(0) = n!\left(1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}\right)$$

(8) $y = \dfrac{(\ln x)^x}{2^{3x+1}}$

$\ln y = x\ln(\ln x)-(3x+1)\ln 2$

$$\frac{1}{y}\frac{dy}{dx} = \ln(\ln x)+\frac{x}{(\ln x)}\frac{1}{x}-3\ln 2$$

$$\frac{dy}{dx} = y\left(\ln(\ln x)+\frac{1}{(\ln x)}-3\ln 2\right)$$

(9) Let $y = \dfrac{\sqrt{x}(x+4)^{3/2}}{(4x-3)^{4/3}}$

Taking log of both sides, we get

$$\log y = \frac{1}{2}\log x+\frac{3}{2}\log(x+4)-\frac{4}{3}\log(4x-3)$$

Diff. both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2x}+\frac{3}{2x+4}\times 1-\frac{4}{3}\times\frac{1}{4x-3}\times 4$$

$$\Rightarrow \frac{dy}{dx} = y\left\{\frac{1}{2x}+\frac{3}{2(x+4)}-\frac{16}{3(4x-3)}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x}(x+4)^{3/2}}{(4x-3)^{4/3}}\left\{\frac{1}{2x}+\frac{3}{2(x+4)}-\frac{16}{3(4x-3)}\right\}$$

(10) $y = \sin^{-1}\dfrac{2x}{1+x^2} = 2\tan^{-1}x$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$$

and $z = \tan^{-1}x \Rightarrow \dfrac{dz}{dx} = \dfrac{1}{1+x^2}$

$$\therefore \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{2}{1+x^2}\cdot\frac{1+x^2}{1} = 2$$

(11) $$\frac{dy}{dx} = \frac{1}{2\sqrt{x\log_e x}}\frac{d}{dx}[x\log_e x]$$

$$= \frac{1}{2\sqrt{x\log_e x}}\left[x\cdot\frac{1}{x}+1\cdot\log_e x\right]$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{x=e} = \frac{1}{2\sqrt{e\cdot 1}}(1+1) = \frac{1}{\sqrt{e}} \qquad (\because \log_e e = 1)$$

(12) $y=\tan^{-1}\left\{\frac{1-\cos x}{\sin x}\right\}=\tan^{-1}\left\{\frac{2\sin\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right\}$

$=\tan^{-1}\left(\tan\frac{x}{2}\right)=\frac{x}{2}\left(\because -\pi<x<\pi\Rightarrow -\frac{\pi}{2}<\frac{x}{2}<\frac{\pi}{2}\right)$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2}$

(13) $\therefore \ln x=\log_e x$, so

$f(x)=\log_x(\log_e x)=\frac{\log(\log x)}{\log x}$

$\Rightarrow f'(x)=\frac{\log x\left(\frac{1}{x\log x}\right)-\log(\log x).\frac{1}{x}}{(\log x)^2}$

$\therefore f'(e)=\frac{1/e-0}{(1)^2}=\frac{1}{e}$

(14) Let $u=\log\sin x$ and $v=\sqrt{\cos x}$

Then, $\frac{du}{dx}=\cot x$ and $\frac{dv}{dx}=-\frac{\sin x}{2\sqrt{\cos x}}$

$\frac{du}{dx}=\frac{du/dx}{dv/dx}=\frac{\cot x}{-\frac{\sin x}{2\sqrt{\cos x}}}=-2\sqrt{\cos x}\cot x\,\mathrm{cosec}\,x$

(15) Let $y=\tan^{-1}\left(\frac{\sin x+\cos x}{\cos x-\sin x}\right)$

$=\tan^{-1}\left(\frac{1+\tan x}{1-\tan x}\right)=\tan^{-1}\left(\tan\left(\frac{\pi}{4}+x\right)\right)=\frac{\pi}{4}+x$

$\therefore \frac{dy}{dx}=1$

(16) Let $u=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v=\tan^{-1}x$

Putting $x=\tan\theta$

We get $u=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)=\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$

$=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)=\tan^{-1}\left(\tan\frac{\theta}{2}\right)=\frac{1}{2}\theta=\frac{1}{2}\tan^{-1}x$

Thus, we have $u=\frac{1}{2}\tan^{-1}x$ and $v=\tan^{-1}x$

$\Rightarrow \frac{du}{dx}=\frac{1}{2}\times\frac{1}{1+x^2}$ and $\frac{dv}{dx}=\frac{1}{1+x^2}$

$\Rightarrow \frac{du}{dx}=\frac{du/dx}{dv/dx}=\frac{1}{2(1+x^2)}(1+x^2)=\frac{1}{2}$

(17) Around $x=\frac{2\pi}{3}$, $|\cos x|=-\cos x$ and $|\sin x|=\sin x$.

$\therefore y=-\cos x+\sin x \quad \therefore \frac{dy}{dx}=\sin x+\cos x$

At $x=\frac{2\pi}{3}$, $\frac{dy}{dx}=\sin\frac{2\pi}{3}+\cos\frac{2\pi}{3}=\frac{\sqrt{3}}{2}-\frac{1}{2}=\frac{\sqrt{3}-1}{2}$

(18) $u=(\ln x)^{\tan x}$; $v=x^3$

$\ln u=\tan x\ln(\ln x)$

$\Rightarrow \frac{1}{u}\left(\frac{du}{dx}\right)=(\sec^2 x)\ln(\ln x)+\tan x\left(\frac{1}{\ln x}\cdot\frac{1}{x}\right)$

$\frac{du}{dx}=\frac{u((x\ln x)\ln(\ln x)\sec^2 x+\tan x)}{(x\ln x)}$

$\ln v=x\ln x$

$\Rightarrow \frac{1}{v}\left(\frac{dv}{dx}\right)=(\ln x+1)$

$\frac{du}{dx}=\frac{(\ln x)^{\tan x}}{x^x}\left(\frac{x\ln x\ln(\ln x)\sec^2 x+\tan x}{(x\ln x)(\ln x+1)}\right)$

(19) $y=\sec^{-1}\left(\frac{x+1}{x-1}\right)+\sin^{-1}\left(\frac{x-1}{x+1}\right)$

$=\cos^{-1}\left(\frac{x-1}{x+1}\right)+\sin^{-1}\left(\frac{x-1}{x+1}\right)=\frac{\pi}{2} \quad \therefore \frac{dy}{dx}=0$

CHAPTER-5 : DIFFERENTIATION
EXERCISE-1

(1) (D). $\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{3a\ \sin^2\theta, \cos\theta}{-3a\ \cos^2\theta\ \sin\theta} = -\tan\theta$

$\therefore$ exp. $= \sqrt{1+\tan^2\theta} = |\sec\theta|$

(2) (B). From the given equation, we have
$y^2(1-x^2) = (\sin^{-1}x)^2$

$\Rightarrow (1-x^2)\,2y\frac{dy}{dx} - 2xy^2 = 2\frac{\sin^{-1}x}{\sqrt{1-x^2}}$

$\Rightarrow 2(1-x^2)\,y\frac{dy}{dx} - 2xy^2 = 2y \Rightarrow (1-x^2)\frac{dy}{dx} = 1+xy$

(3) (C). $\frac{dy}{dx} = \frac{-2x}{(x^2-a^2)^2} \Rightarrow \frac{d^2y}{dx^2}$

$= -\frac{(x^2-a^2)^2.2 - 2x.2(x^2-a^2).2x}{(x^2-a^2)^4} = \frac{2(3x^2+a^2)}{(x^2-a^2)^3}$

(4) (B). $y = \frac{\sec x - \tan x}{\sec x + \tan x}\cdot\frac{\sec x - \tan x}{\sec x - \tan x}$

$= (\sec x - \tan x)^2/1$

$\therefore \frac{dy}{dx} = 2(\sec x - \tan x)(\sec x\tan x - \sec^2 x)$

$= -2\sec x(\sec x - \tan x)^2$

(5) (B). Let us first express y in terms of x because all alternatives are in terms of x. So

$x\sqrt{1+y} = -y\sqrt{1+x} \Rightarrow x^2(1+y) = y^2(1+x)$

$\Rightarrow x^2 - y^2 + x^2y - y^2x = 0$

$\Rightarrow (x-y)(x+y+xy) = 0$

$\Rightarrow x+y+xy = 0 \qquad (\because x \neq y)$

$\Rightarrow y = -\frac{x}{1+x} \therefore \frac{dy}{dx} = -\frac{(1+x)1 - x.1}{(1+x)^2} = -\frac{1}{(1+x)^2}$

(6) (C). $\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin x}}\cdot\frac{1}{2\sqrt{\sin x}}\cos x$

$= \frac{\sqrt{1+\sin x}}{2\sqrt{\sin x}} = \frac{1}{2}\sqrt{1+\operatorname{cosec} x}$

(7) (C). Given $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$

now $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{\frac{dx}{dy}}\right)$

$= \frac{d}{dy}\left(\frac{1}{\frac{dx}{dy}}\right)\cdot\frac{dy}{dx} = -\frac{1}{\left(\frac{dx}{dy}\right)^2}\cdot\frac{d^2x}{dy^2}\cdot\frac{1}{\frac{dx}{dy}}$

$\frac{d^2y}{dx^2} = -\frac{\frac{d^2x}{d^2y}}{\left(\frac{dx}{dy}\right)^3}$ (putting in (1))

$-\frac{\frac{d^2x}{d^2y}}{\left(\frac{dx}{dy}\right)^3} + y\frac{dy}{dx} = 0 \qquad \Rightarrow y\left(\frac{dy}{dx}\right)^2 - \frac{d^2x}{dy^2} = 0$

(8) (D). $\because \cos(xy) - x = 0$

$\therefore \frac{dy}{dx} = -\frac{-y\sin(xy) - 1}{-x\sin(xy)} = -\frac{y + \operatorname{cosec}(xy)}{x}$

(9) (A). Let $f(x,y) = x^2e^y + 2xye^x + 13$

$\therefore \frac{dy}{dx} = -\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y} = -\frac{2xe^y + 2ye^x + 2xye^x}{x^2e^y + 2xe^x}$

Dividing Num^r and Den^r by e^x

$\frac{dy}{dx} = -\frac{2xe^{y-x} + 2y(x+1)}{x(xe^{y-x}+2)}$

(10) (A). $\frac{d(\ln\tan x)}{d(\sin^{-1}(e^x))} = \frac{\frac{d}{dx}(\ln\tan x)}{\frac{d}{dx}\sin^{-1}(e^x)} = \frac{\cot x\sec^2 x}{e^x.\frac{1}{\sqrt{1-e^{2x}}}}$

$= \frac{e^{-x}\sqrt{1-e^{2x}}}{\sin x.\cos x}$

(11) (C). $\frac{dx}{d\theta} = a(1+\cos\theta),\ \frac{dy}{d\theta} = a\sin\theta$

$\therefore \frac{dy}{d\theta} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\ \sin\theta}{a\ (1+\cos\theta)} = \tan\frac{1}{2}\theta$

(12) (A). $y = \log e^x - \log(e^x+1) = x - \log(e^x+1)$

$\therefore \frac{dy}{dx} = 1 - \frac{e^x}{e^x+1} = \frac{1}{e^x+1}$

(13) **(C).** $y = (\sin x)^{\ln x} \operatorname{cosec}(e^x(a+bx))$; $(a+b) = \dfrac{\pi}{2e}$

$$\frac{dy}{dx} = (\sin)^{\ln x}(-\operatorname{cosec}E.\cot E)\{e^x(b)+(a+bx)e^x\}$$

$$+\operatorname{cosec}(e^x(a+bx))e^{\ln x \ln(\sin x)}.\left\{\ln x \cot x + \frac{\ln \sin x}{x}\right\}$$

$$\left.\frac{dy}{dx}\right]_{x=1} = (1)(-1).(0)\{be+(a+b)e\}+(1)(1)[\ln \sin 1]$$

$= \ln(\sin 1)]$

(14) **(B).** $x = e^{\tan^{-1}\left(\frac{y-x^2}{x^2}\right)}$

Taking logarithm of both the sides, we get

$$\log x = \tan^{-1}\left(\frac{y-x^2}{x^2}\right) \Rightarrow y = x^2 + x^2\tan(\log x)$$

$$\therefore \frac{dy}{dx} = 2x + 2x\tan(\log x) + x^2\sec^2(\log x).\frac{1}{x}$$

$$= 2x[1+\tan(\log x)] + x\sec^2(\log x)$$

(15) **(C).** $y = \tan^{-1}\dfrac{3x-x^3}{1-3x^2} = 3\tan^{-1}x \therefore \dfrac{dy}{dx} = \dfrac{3}{1+x^2}$

(16) **(C).** $f(x) = 3x^{10} - 7x^8 + 5x^6 - 21x^3 + 3x^2 - 7$

$$l = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}\cdot\frac{-h}{h(h^2+3)}$$

$$l = \lim_{h\to 0} -f'(1)\cdot\frac{1}{h^2+3} = -\frac{1}{3}f'(1) \qquad \text{....(1)}$$

now $\quad f'(x) = 30x^9 - 56x^7 + 30x^5 - 63x^2 + 6x$

$f'(1) = 30 - 56 + 30 - 63 + 6 = -53$

$$\therefore \qquad l = -\frac{1}{3}(-53) = \frac{53}{3}$$

(17) **(C).** $y = \tan^{-1}\left(\dfrac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right)$,

where $x^2 = \cos\theta$

$$= \tan^{-1}\left(\frac{\cos\theta/2+\sin\theta/2}{\cos\theta/2-\sin\theta/2}\right) = \tan^{-1}\left(\frac{1+\tan\theta/2}{1-\tan\theta/2}\right)$$

$$= \tan^{-1}[\tan(\pi/4+\theta/2)] = \pi/4+\theta/2$$

$$= \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2}\frac{1}{\sqrt{1-x^4}}2x = -\frac{x}{\sqrt{1-x^4}}$$

(18) **(B).** Substituting $x = \sin\theta$ and $y = \sin\phi$ in the given equation, we get

$\cos\theta + \cos\phi = a(\sin\theta - \sin\phi)$

$$\Rightarrow 2\cos\frac{\theta+\phi}{2}.\cos\frac{\theta-\phi}{2} = 2a\cos\frac{\theta+\phi}{2}.\sin\frac{\theta-\phi}{2}$$

$$\Rightarrow \cot\frac{\theta-\phi}{2} = a \Rightarrow \theta-\phi = 2\cot^{-1}a$$

$$\Rightarrow \sin^{-1}x - \sin^{-1}y = 2\cot^{-1}a$$

Differentiating with respect to x, we get

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

(19) **(A).** $\dfrac{1}{A}\cos^{-1}\left(\dfrac{a\theta - A^2}{B\theta}\right)$, where

$$A = \sqrt{a^2-b^2-c^2}, B = \sqrt{b^2+c^2}$$

$$\frac{dy}{d\theta} = \frac{1}{A}\left[\frac{-1}{\sqrt{1-\left(\frac{a\theta-A^2}{B\theta}\right)^2}}.\frac{B\theta a-(a\theta-A^2)B}{B^2\theta^2}\right]$$

$$= -\frac{1}{A}\left[\frac{B\theta}{\sqrt{B^2\theta^2-(a\theta-A^2)^2}}.\frac{A^2B}{B^2\theta^2}\right]$$

$$\frac{dy}{d\theta} = \frac{-A}{\theta\sqrt{B^2\theta^2-(a\theta-A^2)^2}} \qquad \text{.....(1)}$$

$$\frac{d\theta}{dx} = -b\sin x + c\cos x$$

.....(2)

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{A(c\cos x - b\sin x)}{\theta\sqrt{B^2\theta^2-(a\theta-A^2)^2}}$$

[from (1) and (2)]

$$= \left[\frac{A(-c\cos x + b\sin x)}{\sqrt{B^2\theta^2 - a^2\theta^2 - A^4 + 2aA^2\theta}}\right]\frac{1}{\theta}$$

$$\Rightarrow \left[\frac{A(-c\cos x + b\sin x)}{\sqrt{(b^2+c^2)\theta^2 - a^2\theta^2 - (a^2-b^2-c^2) + 2a\theta(a^2-b^2-c^2)}}\right]\frac{1}{\theta}$$

$$\Rightarrow \left[\frac{A(-c\cos x + b\sin x)}{\sqrt{-(a^2+b^2-c^2)(\theta^2+a^2-b^2-c^2) + 2a\theta(a^2-b^2-c^2)}}\right]\frac{1}{\theta}$$

$$\Rightarrow \left[\frac{A(-c\cos x + b\sin x)}{\sqrt{a^2-b^2-c^2}\sqrt{(b^2+c^2)-(\theta^2-2a\theta+a^2)}}\right]\frac{1}{\theta}$$

$$\Rightarrow \left[\frac{A(-c\cos x + b\sin x)}{\sqrt{a^2-b^2-c^2}\sqrt{(b^2+c^2)-(\theta-a)^2}}\right]\frac{1}{\theta}$$

$= (\theta - a)^2 = b^2\cos^2 x + c^2\sin^2 x + 2bc\sin x\cos x$

$\therefore (b^2+c^2) - (\theta-a)^2 = b^2\sin^2 x + c^2\cos^2 x - 2bc\sin x\cos x$

$= (c\cos x - b\sin x)^2$

$$\therefore \frac{dy}{dx} = \left\{\frac{A(-c\cos x + b\sin x)}{A(-c\cos x + b\sin x)}\right\}\frac{1}{\theta};\ \frac{dy}{dx} = \frac{1}{\theta}$$

(20) **(B).** Here $y = \sqrt{\sin x + y}$

$\Rightarrow y^2 = \sin x + y$

$$\therefore 2y\frac{dy}{dx} = \cos x + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y-1}$$

(21) **(A).** Here $y = \dfrac{x}{a+\dfrac{x}{b+y}} = \dfrac{x(b+y)}{a(b+y)+x}$

$\Rightarrow aby + ay^2 + xy = bx + xy$

$\Rightarrow ay^2 + aby = bx$

$$\Rightarrow 2ay\frac{dy}{dx} + ab\frac{dy}{dx} = b \Rightarrow \frac{dy}{dx} = \frac{b}{a(b+2y)}$$

(22) **(C).** $y = e^{x+y} \Rightarrow \log y = x + y$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{y}{1-y}$$

(23) **(B).** Taking logarithm of both the sides

$$\log(a+bx) + \frac{y}{x} = \log x$$

Now differentiating with respect to x, we get

$$\frac{b}{a+bx} + \frac{x\frac{dy}{dx} - y}{x^2} = \frac{1}{x}$$

$$\Rightarrow x\frac{dy}{dx} - y = x^2\left(\frac{a+bx-bx}{x(a+bx)}\right) = \frac{ax}{(a+bx)}$$

Again differentiating with respect to x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx} = \frac{(a+bx)a - ax(b)}{(a+bx)^2}$$

$$x^3\frac{d^2y}{dx^2} = \left(\frac{ax}{a+bx}\right)^2 = \left(x\frac{dy}{dx} - y\right)^2$$

(24) **(A).** $f(x) = \dfrac{1}{3}\tan^3 x + 3\tan x - 3\cot x - \dfrac{1}{3}\cot^3 x$

$f'(x) = \sec^2 x(\sec^2 x + 2) + \operatorname{cosec}^2 x(\operatorname{cosec}^2 x + 2)$

$= (\sec^2 x + \operatorname{cosec}^2 x)^2$

$$= (\sec^2 x\operatorname{cosec}^2 x)^2 = \frac{16}{\sin^4 x\cos^4 x}$$

$= 16\operatorname{cosec}^4(2x)$

(25) **(B).** $F(x) = f(g(x)) \Rightarrow f'(g(x))g'(x) = F'(x)$

$F'(3) = f'(g(3))g'(3) = f'(5)\cdot g'(3) = 4\times 3 = 12$

(26) **(A).** $\because g(x) = f[f(x)] = f\{|x-2|\} = ||x-2|-2|$

But $x > 20 \Rightarrow |x-2| = x-2$

$\Rightarrow g(x) = |x-2-2| = x-4$

$\therefore g'(x) = 1$

(27) **(D).** $h'(x) = 2f(x)f'(x) + 2g(x)g'(x)$

$= 2f(x)g(x) + 2g(x)f''(x)$

$= 2f(x)g(x) - 2f(x)g(x)$

$= 0$ $\quad [\because f''(x) = -f(x)]$

$\Rightarrow h(x) = c \Rightarrow h(10) = h(5) = 11$

(28) **(C).** Here $\dfrac{dx}{d\theta}\sec\theta\tan\theta + \sin\theta = \tan\theta(\sec\theta + \cos\theta)$

$$= \tan\theta\sqrt{(\sec\theta - \cos\theta)^2 + 4} = \tan\theta\sqrt{x^2+4}$$

and $\dfrac{dy}{d\theta} = n\sec^n\theta\tan\theta + n\cos^{n-1}\theta\sin\theta$

$= n\tan\theta(\sec^n\theta + \cos^n\theta)$

$$= n\tan\theta\sqrt{(\sec^n\theta - \cos^n\theta)^2 + 4} = n\tan\theta\sqrt{y^2+4}$$

$$\therefore \frac{dy}{dx} = \frac{n\tan\theta\sqrt{y^2+4}}{\tan\theta\sqrt{x^2+4}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{n^2(y^2+4)}{x^2+4}$$

(29) **(B).** When $1 < x \le 3$,

$f(x) = (x-1) - (x-3) = 2$

$\Rightarrow f'(2-0) = 0,\ f'(2+0) = 0 \quad \therefore f'(2) = 0$

(30) **(A).** $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$

$$f'(0^+) = \lim_{h\to 0}\frac{\frac{\sin h}{h} - 1}{h} = \lim_{h\to 0}\frac{\sin h - h}{h^2} = 0$$

(using Lopital rule)

$$f'(0^-) = \lim_{h\to 0}\frac{\frac{\sin h}{h}-1}{-h} = \lim_{h\to 0}\frac{\sin h - h}{-h^2} = 0$$

(using Lopital rule)

$$\text{Hence } f'(x) = \begin{cases} \dfrac{x\cos x - \sin x}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(31) **(B).** Let $y = \sin 2x \cos 2x \cos 3x + \log_2 2^{x+3}$

$= \frac{1}{2}\sin 4x \cos 3x + (x+3)\log_2 2 = \frac{1}{4}[\sin 7x + \sin x] + x + 3$

$\therefore \frac{dy}{dx} = \frac{1}{4}[7\cos 7x + \cos x] + 1$

$\left(\frac{dy}{dx}\right)_{x=\pi} = \frac{1}{4}[7\cos 7\pi + \cos \pi] + 1 = \frac{1}{4}[-8] + 1 = -1$

(32) **(D).** $y = \cos^{-1}\left(\sin\sqrt{\frac{1+x}{2}}\right) + x^x$

$= \cos^{-1}\cos\left(\frac{\pi}{2} - \sqrt{\frac{1+x}{2}}\right) + x^x = \frac{\pi}{2} - \sqrt{\frac{1+x}{2}} + x^x$

$\frac{dy}{dx} = 0 - \frac{1}{2\sqrt{\frac{1+x}{2}}} \times \frac{1}{2} + x^x(1+\log x)$

at $x = 1$ $\quad = -\frac{1}{2\sqrt{\frac{1+1}{2}}} \times \frac{1}{2} + 1(1+0) = -\frac{1}{4} + 1 = 3/4$

(33) **(C).** Putting $x = \tan\theta$

$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) + \sec^{-1}\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right)$

$= 2\theta + 2\theta = 4\tan^{-1}x$.

$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) + \sec^{-1}\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(4\tan^{-1}x) = \frac{4}{1+x^2}$

(34) **(B).** $\frac{d}{dx}\tan^{-1}\left[\frac{3a^2x - x^3}{a(a^2-3x^2)}\right]$

Put $x = a\tan\theta$ $\Rightarrow \frac{d}{dx}\tan^{-1}\left[\frac{3a^3\tan\theta - a^3\tan^3\theta}{a^3 - 3a^3\tan^2\theta}\right]$

$= \frac{d}{dx}\tan^{-1}(\tan 3\theta) = \frac{d}{dx}(3\theta) = \frac{3a}{x^2+a^2}$

If $x = 0$, then $\frac{d}{dx}\tan^{-1}\left[\frac{3a^2x - x^3}{a(a^2-3x^2)}\right] = \frac{3}{a}$.

(35) **(A).** $y = \cos^{-1}\left\{\sqrt{\frac{1+x}{2}}\right\}$

Let $\sqrt{\frac{1+x}{2}} = \cos\theta$ or $x = 2\cos^2\theta - 1 = \cos 2\theta$;

$\therefore \theta = \frac{1}{2}\cos^{-1}x$. So, $y = \frac{1}{2}\cos^{-1}x \Rightarrow -\frac{1}{2\sqrt{1-x^2}}$.

(36) **(C).** $y = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Put $x = \sin\theta \;\therefore\; dx = \cos\theta\, d\theta$, $\frac{d\theta}{dx} = \frac{1}{\cos\theta}$

$\therefore y = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) \Rightarrow y = \theta$

$\therefore dy = d\theta$

Now $\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx} = 1.\frac{1}{\cos\theta} = \sec\theta = \frac{1}{\sqrt{1-x^2}}$.

(37) **(D).** Let $y_1 = \tan^{-1}\sqrt{x}$ and $y_2 = \sqrt{x}$

Differentiating w.r.t. x of y_1 and y_2, we get

$\frac{dy_1}{dx} = \frac{1}{(1+x)}.\frac{1}{2\sqrt{x}}$ and $\frac{dy_2}{dx} = \frac{1}{2\sqrt{x}}$

Hence $\frac{dy_1}{dy_2} = \frac{1}{1+x}$

(38) **(C).** $\frac{d}{dx}[e^{2x} + e^{-2x}] = 2e^{2x} + 2e^{-2x} = 2^1[e^{2x} - e^{-2x}]$

$\frac{d^2}{dx^2}[e^{2x} + e^{-2x}] = 2^2[e^{2x} + e^{-2x}]$

$\frac{d^2}{dx^2}[e^{2x} + e^{-2x}] = 2^2[e^{2x} - e^{-2x}]$

....

....

$\frac{d^n}{dx^n}[e^{2x} + e^{-2x}] = 2^n[e^{2x} + (-1)^n e^{-2x}]$.

(39) **(A).** $y = x^3 \log\log_e(1+x)$

$\Rightarrow\ y' = 3x^2 \log\log_e(1+x) + \frac{x^3}{1+x}\cdot\frac{1}{\log_e(1+x)}$

$\Rightarrow\ y'' = 6x\log\log_e(1+x) + \frac{3x^2}{\log_e(1+x)}\cdot\frac{1}{(1+x)}$

$\Rightarrow\ y''(0) = 0$

(40) **(A).** $x = A\cos 4t + B\sin 4t$

Differentiate w.r.t. t, $\frac{dx}{dt} = -4A\sin 4t + 4B\cos 4t$

Again, differentiate w.r.t. t,

$\frac{d^2x}{dt^2} = -16A\cos 4t - 16B\sin 4t$;

$\frac{d^2x}{dt^2} = -16[A\cos 4t + B\sin 4t]$. Hence, $\frac{d^2x}{dt^2} = -16x$

(41) **(C).** $\frac{dy}{dx} = -b\sin\log\left(\frac{x}{n}\right)^n \frac{1}{(x/n)^n}\frac{n}{n}\left(\frac{x}{n}\right)^{n-1}$

$= -\frac{nb}{x}\sin\log\left(\frac{x}{n}\right)^n$.

(42) **(A).** $e^{x+3\log x} = e^x.e^{3\log x} = e^x.e^{\log x^3} = e^x.x^3$

Therefore, $y = e^x.x^3$

$\Rightarrow \frac{dy}{dx} = e^x.3x^2 + x^3.e^x = e^x x^2(3+x)$

(43) **(B).** $f(x) = \cos(x+3x+5x+....+(2n-1)x)$
$+ i\sin(x+3x+5x+...+(2n-1)x)$
$f(x) = \cos n^2x + i\sin n^2x$
$f'(x) = -n^2\sin n^2x + i n^2\cos n^2x$
$\Rightarrow f''x = -n^4\cos n^2x - i n^4\sin n^4x = -n^4 f(x)$

(44) **(A).** $\frac{d}{dx}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 = \frac{d}{dx}\left[x + \frac{1}{x} + 1\right] = 1 - \frac{1}{x^2}$.

(45) **(C).** $x^m y^n = 2(x+y)^{m+n}$

$\Rightarrow m\log x + n\log y = \log 2 + (m+n)\log(x+y)$

Differentiating both sides w.r.t. x,

$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\left[1 + \frac{dy}{dx}\right] \Rightarrow \frac{dy}{dx} = \frac{y}{x}$.

(46) **(C).** Putting $x = \sin A$ and $\sqrt{x} = \sin B$

$y = \sin^{-1}(\sin A\sqrt{1-\sin^2 B} + \sin B\sqrt{1-\sin^2 A})$

$= \sin^{-1}[\sin(A+B)] = A + B = \sin^{-1}x + \sin^{-1}\sqrt{x}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{x-x^2}}$.

(47) **(D).** $\frac{d}{dx}\tan^{-1}\left(\frac{ax-b}{bx+a}\right) = \frac{1}{1+\left(\frac{ax-b}{bx+a}\right)^2}\cdot\frac{d}{dx}\left(\frac{ax-b}{bx+a}\right)$

$= \frac{a^2+b^2}{a^2+b^2+a^2x^2+b^2x^2} = \frac{1}{1+x^2}$.

(48) **(C).** $\frac{d}{dx}[\log_7(\log_7 x)] = \frac{d}{dx}\left(\frac{\log_e(\log_7 x)}{\log_e 7}\right)$

$= \frac{1}{x\log_e x}\cdot\frac{1}{\log_e 7} = \frac{\log_7 e}{x\log_e x}$

(49) **(A).** $\frac{d}{dx}\cos^{-1}\sqrt{\cos x} = \frac{\sin x}{2\sqrt{\cos x}\sqrt{1-\cos x}}$

$= \frac{\sqrt{1-\cos^2 x}}{2\sqrt{\cos x}\sqrt{1-\cos x}} = \frac{1}{2}\sqrt{\frac{1+\cos x}{\cos x}}$.

(50) **(A).** $y = (x\cot^3 x)^{3/2}$

$\therefore \frac{dy}{dx} = \frac{3}{2}(x\cot^3 x)^{1/2}[\cot^3 x + 3x\cot^2 x(-\mathrm{cosec}^2 x)]$

$= \frac{3}{2}(x\cot^3 x)^{1/2}[\cot^3 x - 3x\cot^2 x\,\mathrm{cosec}^2 x]$.

(51) **(A).** Let $y^2 = x\sin x \Rightarrow 2y\frac{dy}{dx} = \sin x + x\cos x$

$\therefore \frac{dy}{dx} = \frac{[\sin x + x\cos x]}{2\sqrt{x\sin x}}$.

(52) **(D).** $\frac{dy}{dx} = \frac{1}{\left(\frac{1+\sin x}{\cos x}\right)}\left[\frac{\cos x\cos x + (1+\sin x)\sin x}{(\cos x)^2}\right]$

$= \left(\frac{\cos x}{1+\sin x}\right)\frac{\cos^2 x + \sin^2 x + \sin x}{(\cos x)^2} = \sec x$.

(53) **(C).** $y = \log x.e^{(\tan x + x^2)}$

$\therefore \frac{dy}{dx} = e^{(\tan x + x^2)}.\frac{1}{x} + \log x.e^{(\tan x + x^2)}(\sec^2 x + 2x)$

$= e^{(\tan x + x^2)}\left[\frac{1}{x} + (\sec^2 x + 2x)\log x\right]$.

(54) **(C).** $\frac{d}{dx}[\cos(\sin x^2)] = -\sin(\sin x^2)\cos x^2.2x$

Putting $x = \sqrt{\frac{\pi}{2}}$, we have

$= -2\sqrt{\frac{\pi}{2}}\sin\left(\sin\frac{\pi}{2}\right)\cos\frac{\pi}{2} = 0\,,\ \left[\because \cos\frac{\pi}{2} = 0\right].$

(55) **(C).** $y = \frac{\log\tan x}{\log\sin x}$

$\Rightarrow \frac{dy}{dx} = \frac{(\log\sin x)\left(\frac{\sec^2 x}{\tan x}\right) - (\log\tan x)(\cot x)}{(\log\sin x)^2}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{\pi/4} = \frac{-4}{\log 2}$ (On simplification).

(56) **(A).** Given $y = (\sin x)^{\tan x}$; $\log y = \tan x\,.\,\log\sin x$

Differentiate with respect to x,

$\frac{1}{y}.\frac{dy}{dx} = \tan x.\cot x + \log\sin x.\sec^2 x$

$\frac{dy}{dx} = (\sin x)^{\tan x}[1 + \log\sin x.\sec^2 x]$.

(57) **(B).** $f(x) = \sin 2x.\cos 2x.\cos 3x + \log_2 2^{x+3}$

$f(x) = \frac{1}{2}\sin 4x\cos 3x + (x+3)\log_2 2$

$f(x) = \frac{1}{4}[\sin 7x + \sin x] + x + 3$

Differentiate w.r.t. x, $f'(x) = \frac{1}{4}[7\cos 7x + \cos x] + 1$

$f'(x) = \frac{7}{4}\cos 7x + \frac{1}{4}\cos x + 1$.

Hence $f'(\pi) = -2 + 1 = -1$.

(58) **(C).** Let $y = \frac{d}{dx}\{\log f(e^x + 2x)\} = \frac{f'(e^x + 2x)(e^x + 2)}{f(e^x + 2x)}$

$\therefore (y)_{x=0} = \frac{1}{f(1)}.f'(1).3 = \frac{2}{3}.3 = 2$

(59) **(B).** $y = (1 + x^2)\tan^{-1}x - x$

$\Rightarrow$

$\frac{dy}{dx} = (1+x^2).\frac{1}{(1+x^2)} + \tan^{-1}x(2x) - 1 = 2x\tan^{-1}x.$

(60) **(C).** Let

$y = \left[\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}\right] = \log e^x + \log\left(\frac{x-2}{x+2}\right)^{3/4}$

$\Rightarrow\ y = x + \frac{3}{4}[\log(x-2) - \log(x+2)]$

$\Rightarrow \frac{dy}{dx} = 1 + \frac{3}{4}\left[\frac{1}{x-2} - \frac{1}{x+2}\right] = 1 + \frac{3}{(x^2-4)}$

$\Rightarrow \frac{dy}{dx} = \frac{x^2-1}{x^2-4}$.

(61) **(C).** $f(x) = \sqrt{ax} + \frac{a^2}{\sqrt{ax}}$, then

$\Rightarrow\ f'(x) = \frac{\sqrt{a}}{2\sqrt{x}} + \frac{a^2}{\sqrt{a}}\left(\frac{-1}{2}x^{-3/2}\right)$

$\Rightarrow\ f'(x) = \frac{\sqrt{a}}{2\sqrt{x}} - \frac{a^2}{2\sqrt{a}}x^{-3/2}$

$\Rightarrow f'(a) = \frac{\sqrt{a}}{2\sqrt{a}} - \frac{a^2}{2\sqrt{a}\,.a^{3/2}} \Rightarrow f'(a) = \frac{1}{2} - \frac{a^2}{2a^2} = 0$.

(62) **(B).** $y = \tan^{-1}(\sec x - \tan x)$

$\frac{dy}{dx} = \frac{1}{1 + (\sec x - \tan x)^2}(\sec x\tan x - \sec^2 x)$

$\frac{dy}{dx} = \frac{\cos^2 x.\sec^2 x(\sin x - 1)}{(1-\sin x)^2 + \cos^2 x}$

$\frac{dy}{dx} = \frac{\sin x - 1}{1 - 2\sin x + \sin^2 x + \cos^2 x} = \frac{\sin x - 1}{2(1-\sin x)} = -\frac{1}{2}$

(63) **(A).** $f(x) = \frac{2\sin x.\cos x.\cos 2x.\cos 4x.\cos 8x.\cos 16x}{2\sin x}$

$= \frac{\sin 32x}{2^5\sin x}$

$\therefore\ f'(x) = \frac{1}{32}.\frac{32\cos 32x.\sin x - \cos x.\sin 32x}{\sin^2 x}$

$f'\left(\frac{\pi}{4}\right) = \frac{32.\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\times 0}{32\left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{2}$

(64) **(D).** $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} \Rightarrow \frac{dy}{dx} = \frac{1}{t} = \frac{2a}{y}$

$\Rightarrow\ y\frac{dy}{dx} = 2a \Rightarrow y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-(dy/dx)^2}{y} = -\frac{1}{2at^3}$.

(65) **(C).** $\sin y = x\sin(a+y) \Rightarrow x = \frac{\sin y}{\sin(a+y)}$

$$\Rightarrow 1 = \frac{\cos y.\frac{dy}{dx}.\sin(a+y) - \sin y\cos(a+y)\frac{dy}{dx}}{\sin^2(a+y)}$$

$$= \frac{\frac{dy}{dx}.\sin(a+y-y)}{\sin^2(a+y)} \Rightarrow \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}.$$

(66) **(A).** $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Differentiating w.r.t. x of y, we get

$$2ax + 2h\left(y + x\frac{dy}{dx}\right) + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx}(2hx + 2by + 2f) = -(2ax + 2hy + 2g)$$

or $\frac{dy}{dx} = -\frac{(ax+hy+g)}{(hx+by+f)}$.

(67) **(C).** $\frac{dx}{d\theta} = a\cos\theta$ and $\frac{dy}{d\theta} = -b\sin\theta$

$$\Rightarrow \frac{dy}{dx} = \frac{-b}{a}\tan\theta \text{ and } \frac{d^2y}{dx^2} = \frac{-b}{a}\sec^2\theta\frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-b}{a}\sec^2\theta\frac{1}{a\cos\theta} = \frac{-b}{a^2}\sec^3\theta.$$

(68) **(B).** $x = \frac{2t}{1+t^2}$, $y = \frac{1-t^2}{1+t^2}$

Put $t = \tan\theta$

$$x = \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta, \quad y = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2\sin 2\theta}{2\cos 2\theta}$$

$$= -\tan 2\theta = \frac{-2\tan\theta}{1-\tan^2\theta} = \frac{-2t}{1-t^2} = \frac{2t}{t^2-1}$$

(69) **(D).** $y = \cos^{-1}\sqrt{1-t^2} = \sin^{-1} t$

and $x = \sin^{-1}(3t - 4t^3) = 3\sin^{-1} t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\left(\frac{1}{\sqrt{1-t^2}}\right)}{3\left(\frac{1}{\sqrt{1-t^2}}\right)} \Rightarrow \frac{dy}{dx} = \frac{1}{3}$$

(70) $x^y = e^{x-y} \Rightarrow y\log x = x - y \Rightarrow y = \frac{x}{1+\log x}$

$$\Rightarrow \frac{dy}{dx} = \log x(1+\log x)^{-2} = \log x\,[\log ex]^{-2}.$$

(71) **(C).** $y = (1+x)^x$

Taking log on both sides, $\log y = x\log(1+x)$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \log(1+x) + x\frac{1}{(1+x)}$$

Thus $\frac{dy}{dx} = (1+x)^x\left[\frac{x}{1+x} + \log(1+x)\right]$

(72) **(A).** Let $y = x^{\log_e x}$

$$\Rightarrow \log_e y = \log_e x\log_e x = (\log_e x)^2$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 2\log_e x.\frac{1}{x} \quad \therefore \frac{dy}{dx} = 2x^{(\log_e x - 1)}\log_e x.$$

(73) **(A).** $y = x^{\sin x} \Rightarrow \log_e y = \sin x\log_e x$

$$\therefore \frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \cos\log_e x\right]$$

$$= x^{\sin x}\left[\frac{\sin x + x\cos x\log_e x}{x}\right]$$

(74) **(B).** Let $y = (\sin x)^{\log x} \Rightarrow \log_e y = \log_e x\log_e\sin x$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\log_e x} = \left[\frac{1}{x}\log_e\sin x + \cot x\log_e x\right].$$

(75) **(B).** $\frac{d}{dx}\{\tan^{-1}(\sec x + \tan x)\} = \frac{d}{dx}\left\{\tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)\right\}$

$$= \frac{d}{dx}\left\{\tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)\right\} = \frac{d}{dx}\left(\frac{\pi}{4} + \frac{x}{2}\right) = \frac{1}{2}$$

and $\frac{d}{dx}\{\cot^{-1}(\operatorname{cosec} x + \cot x)\}$

$$= \frac{d}{dx}\left\{\cot^{-1}\left(\cot\left(\frac{x}{2}\right)\right)\right\} = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}$$

(76) **(C).** $f(x)\,g(x) = x^3 g(x)$

$3x^2.\,g'(x) = 3x^2 g(x) + x^3 g'(x)$

$3g'(x) = 3g(x) + x\,g'(x)$

$(3-x)\,g'(x) = 3g(x)$

$$\int\frac{g'(x)}{g(x)}dx = \int\frac{3}{3-x}dx + \ln c$$

$\ln g(x) = -3\ln|3-x| + \ln c$

$$\therefore g(x) = \frac{c}{|3-x|^3}\ ;\ g(0) = \frac{c}{27} = \frac{1}{3} \quad \therefore c = 9$$

$\therefore g(x) = \dfrac{9}{|3-x|^3} = g(x)$ decreasing $\therefore g(x) = \dfrac{9}{(x-3)^3}$

(77) **(A).** $f(x-3) \cdot g(x) = (x-3)^3 \cdot g(x) = 9$
$\therefore$ derivative of $f(x-3) \cdot g(x)$ is 0.

(78) $y = f(x) - f(2x)$
$y' = f'(x) - 2f'(2x)$
$y'(1) = f'(1) - 2f'(2) = 5 \quad ....(1)$
and $y'(2) = f'(2) - 2f'(4) = 7 \quad ....(2)$
now let $y = f(x) - f(4x)$
$y' = f'(x) - 4f'(4x)$
$y'(1) = f'(1) - 4f'(4) \quad ....(3)$
substituting the value of $f'(2) = 7 + 2f'(4)$ in (1)
$f'(1) - 2[7 + 2f'(4)] = 5$
$f'(1) - 4f'(4) = 19$

(79) **15.**

$$y^2\left(e^{xy}\left(x\frac{dy}{dx} + y\right)\right) + e^{xy} \cdot 2y\frac{dy}{dx} = 9e^{-3} \cdot 2x$$

put $x = -1$ and $y = 3$

$$9\left(e^{-3}\left(-1\frac{dy}{dx} + 3\right)\right) + e^{-3} \cdot 6\frac{dy}{dx} = -9e^{-3} \cdot 2$$

$$-9\left(\frac{dy}{dx} - 3\right) + 6\frac{dy}{dx} = -18$$

$$3\frac{dy}{dx} = 45 \Rightarrow \frac{dy}{dx} = 15$$

(80) **3.** Put $\cos\phi = \dfrac{2}{\sqrt{13}}$; $\sin\phi = \dfrac{3}{\sqrt{13}}$; $\tan\phi = \dfrac{3}{2}$

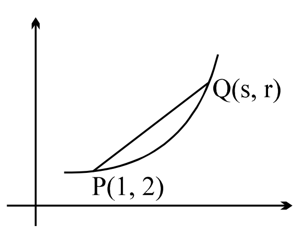
$y = \cos^{-1}\{\cos(x+\phi)\} + \sin^{-1}\{\cos(x-\phi)\}$

$= \cos^{-1}\{\cos(x+\phi)\} + \dfrac{\pi}{2} - \cos^{-1}\{\cos(\phi - x)\}$ (think !)

$= x + \phi + \dfrac{\pi}{2} - \phi + x$

$y = 2x + \dfrac{\pi}{2}$; $z = \sqrt{1+x^2}$. Now compute $\dfrac{dy}{dz}$

(81) **4.** I By definition f '(1) is the limit of the slope of the secant line when $s \to 1$.



Thus $f'(1) = \lim\limits_{s\to 1}\dfrac{s^2 + 2s - 3}{s-1} = \lim\limits_{s\to 1}\dfrac{(s-1)(s+3)}{s-1}$

$= \lim\limits_{s\to 1}(s+3) = 4$

II By substituting x = s into the equation of the secant line, and cancelling by s – 1 again, we get

$y = s^2 + 2s - 1$. This is f (s), and its derivative is $f'(s) = 2s + 2$, so $f'(1) = 4$

(82) **0.**

$$y = \frac{\sin^3 x + \cos^3 x}{\cos x + \sin x} = 1 - \sin x \cos x$$

$$1 - \frac{1}{2}\sin 2x \ ; \ y' = -\cos 2x \Rightarrow y'\left(\frac{\pi}{4}\right) = 0$$

(83) **12.** $y = 1 + x^3 \Rightarrow \dfrac{dy}{dx} = 3x^2 \therefore \dfrac{dx}{dy} = \dfrac{1}{3x^2}$

when $y = 9$ then $x^3 = 8 \Rightarrow x = 2$

hence $\dfrac{dx}{dy} = g'(9) = \dfrac{1}{12}$

(84) **2.** $f'(x) - 2f'(-x) = \cos x$; put $x = \pi/4$

$\therefore f'(\pi/4) - 2f'(\pi/4) = \dfrac{1}{\sqrt{2}} \quad ....(1)$

again put $x = -\pi/4$

$-2f'(\pi/4) + f'(-\pi/4) = \dfrac{1}{\sqrt{2}} \quad ....(2)$

$\Rightarrow f'(\pi/4) = -\dfrac{1}{\sqrt{2}}$

(85) **2.** $\cos(x + 2y)(1 + 2y') = 2[\cos y - x\sin y \cdot y']$
put $x = 0$; $y = \pi$
$1 \cdot (1 + 2y') = 2[-1]$
$1 + 2y' = -2$; $y' = -3/2$

(86) **16.** $(fgh)'(0) = fgh' + ghf' + hfg'$

$$= \frac{(fg)'h + (gh)'f + (hf)'g}{2}$$

$\therefore (fgh)'(0)$

$$= \frac{(fg)'(0)\cdot h(0) + (gh)'(0)\cdot f(0) + (hf)'(0)\cdot g(0)}{2}$$

$$= \frac{6\cdot 3 + 4\cdot 1 + 5\cdot 2}{2} = 16$$

(87) **1.** $u(x) = 7v(x) \Rightarrow u'(x) = 7v'(x)$

and $\dfrac{u(x)}{v(x)} = 7 \Rightarrow \left(\dfrac{u(x)}{v(x)}\right)' = 0 \Rightarrow \dfrac{p+q}{p-q} = \dfrac{7+0}{7-0} = 1$

(88) **6.** $P'(x) = f(x)g'(x) + g(x)f'(x)$
$P'(2) = f(2)g'(2) + g(2)f'(2) = (1)(2) + 4(-1) = -2$

$$Q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

$$Q'(2) = \frac{(4)(-1) - (1)(2)}{16} = -\frac{6}{16} = -\frac{3}{8}$$

$C'(x) = f'(g(x))g'(x)$
$C'(2) = f'(4) \cdot 2 = 3 \times 2 = 6$

(89) We have $g(x) = f\left(\frac{x}{f(x)}\right)$

On differentiating w.r.t. x, we get

$$g'(x) = f'\left(\frac{x}{f(x)}\right) \cdot \left(\frac{f(x) - xf'(x)}{f^2(x)}\right)$$

$$\therefore\ f'(1) = f'\left(\frac{1}{f(1)}\right) \cdot \left(\frac{f(1) - f'(1)}{f^2(1)}\right)$$

As $f(1) = f'(1) \Rightarrow g'(1) = 0$

(90) **8.** Given : $\left(e^{x^2}.g(x)\right)' = e^{x^2}.2x.g'(x)$ .... (1)

But actually

$$\left(e^{x^2}.g(x)\right)' = e^{x^2}.g'(x) + g(x).e^{x^2}.2x \quad .... (2)$$

From eq. (1) and (2),

$$e^{x^2}.g'(x) + g(x).e^{x^2}.2x = e^{x^2}.2x.g'(x)$$

$$g'(x).(2x-1) = g(x).2x \quad \therefore \frac{g'(x)}{g(x)} = \frac{2x}{2x-1}$$

Integrating,

$$\ln(g(x)) = \int \frac{(2x-1)+1}{2x-1}dx = x + \frac{1}{2}\ln(2x-1) + C$$

$$\therefore\ g(x) = c.e^x.e^{\ln\sqrt{2x-1}}$$

$$g(x) = c\sqrt{2x-1}.e^x \ ;\ g(1) = e \Rightarrow c = 1$$

hence $g(x) = \sqrt{2x-1}.e^x$

$\therefore\ g(x) = 3e^5 \Rightarrow k = 3$ and $c = 5 \Rightarrow k + c = 8$

(91) **2.** $g(f(x)) = x$

$\Rightarrow g'(f(x))f'(x) = 1$ ....(1)

If $f(x) = 1 \Rightarrow x = 0, f(0) = 1$

Substitute $x = 0$ in (1), we get

$$g'(1) = \frac{1}{f'(0)} \Rightarrow g'(1) = 2$$

$$f'(x) = 3x^2 + \frac{1}{2}e^{x/2} \Rightarrow f'(0) = \frac{1}{2}$$

(92) **9.** $\frac{dy}{dx} = -\sin(3\cos^{-1}x)\cdot\frac{-3}{\sqrt{1-x^2}}$

$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = 9(1-\cos^2(3\cos^{-1}x)) = 9(1-y^2)$$

$$\Rightarrow (1-x^2)^2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\cdot(-2x) = -18y\cdot\frac{dy}{dx}$$

$$\Rightarrow (x^2-1)\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = 9y$$

EXERCISE-2

(1) **(C).** $y = \log_y x \Rightarrow y = \frac{\log_e x}{\log_e y} \Rightarrow y\log_e y = \log_e x$

Differentiating both side w.r.t. x

$$y.\frac{1}{y}\frac{dy}{dx} + \log_e y.\frac{dy}{dx} = \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx}(1+\log_e y) = \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{1}{x(1+\log_e y)}$$

(2) **(D).** $x = 3\cos\theta - 2\cos^3\theta$

$\frac{dx}{d\theta} = 3(-\sin\theta) - 2.3\cos^2\theta.(-\sin\theta)$

$= -3\sin\theta + 6\cos^2\theta\sin\theta$

$= 3\sin\theta(-1 + 2\cos^2\theta) = 3\sin\theta\cos 2\theta$ .... (1)

and $y = 3\sin\theta - 2\sin^3\theta$

$\frac{dy}{d\theta} = 3\cos\theta - 2.3\sin^2\theta.\cos\theta$

$= 3\cos\theta[1 - 2\sin^2\theta] = 3\cos\theta\cos 2\theta$ .... (2)

$$\because \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos\theta\cos 2\theta}{3\sin\theta\cos 2\theta} = \cot\theta$$

(3) **(B).** $y = (x + \sqrt{1+x^2})^n$ .... (1)

$$y_1 = n(x+\sqrt{1+x^2})^{n-1}\left(1 + \frac{2x}{2\sqrt{1+x^2}}\right)$$

$$= n(x+\sqrt{1+x^2})^{n-1}\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow y_1 = n\frac{(x+\sqrt{1+x^2})^n}{\sqrt{1+x^2}}$$

$$\Rightarrow y_1\sqrt{1+x^2} = n(x+\sqrt{1+x^2})^n$$

$$\Rightarrow y_1\sqrt{1+x^2} = ny \quad \text{[From (1)]}$$

Squaring both side $y_1^2(1+x^2) = n^2y^2$

again differntiating w.r.t. x

$$\Rightarrow y_1^2(2x) + (1+x^2)2y_1y_2 = n^2 2yy_1$$

$$\Rightarrow y_1x + (1+x^2)y_2 = n^2y$$

(4) **(D).** $f(x+y) = f(x).f(y) \Rightarrow f(x+5) = f(x)f(5)$

$\Rightarrow f(x+5) = 2f(x)\ \{\because f(5) = 2\}$

$\Rightarrow f'(x+5) = 2f'(x)$

Put $x = 0 \Rightarrow f'(5) = 2f'(0)$ $(\because f'(0) = 3)$

$= 2 \times 3 = 6$

Alter : $\because f(x+y) = f(x).f(y)$

$\therefore$ function is in form of $e^{\lambda x}$

$\therefore f(x) = e^{\lambda x} \Rightarrow f(5) = e^{5\lambda} = 2$ .... (1) (given)

$f'(x) = \lambda e^{\lambda x}$

$f'(0) = \lambda e^0 = \lambda \Rightarrow \lambda = 3 \ (\because f'(0) = 3)$ .... (2)

Now $f'(5) = \lambda e^{5\lambda}$

From (1) and (2) $= 3 \times 2 = 6$

(5) **(D).** $f(x) = x^n \Rightarrow f(1) = 1^n = 1$

$f(x) = nx^{n-1} \Rightarrow f'(1) = n \cdot 1^{n-1} = n$

$f''(x) = n(n-1)x^{n-2} \Rightarrow f''(1) = n(n-1)$

$f'''(x) = n(n-1)(n-2)x^{n-3} \Rightarrow f'''(1) = n(n-1)(n-2)$

Similarly,

$f^n(x) = n(n-1)(n-2) .... 1 = n!$

$$\therefore f(1) - \frac{f'(1)}{1!} + \frac{f''(r)}{2!} - \frac{f'''(r)}{3!} + ... + \frac{(-1)^n f^n(1)}{n!}$$

$$= 1 - \frac{n}{1!} + \frac{n(n-1)}{2!} - \frac{n(n-1)(n-2)}{3!} + ... + (-1)^n \frac{n!}{n!}$$

$$= {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + ... + (-1)^n\, {}^nC_n = 0$$

(6) **(B).** $\because$ f(x) is polynomial of second degree

let $f(x) = px^2 + qx + r$

$\therefore f'(x) = 2px + q$ .... (1)

$\therefore f(1) = f(-1)$

$\Rightarrow p + q + r = p - q + r \Rightarrow 2q = 0 \Rightarrow q = 0$

Put this value in (1) we get $f'(x) = 2px$

Now $f'(a) = 2ap$

$f'(b) = 2bp$

$f'(c) = 2cp$ $\quad\because$ a, b, c are in A.P.

$\therefore$ 2Aa, 2Ab, 2Ac, will also be in A.P.

(7) **(C).** $x = e^{y + e^{y + ... \text{to } \infty}}$

$\Rightarrow x = e^{y+x} \Rightarrow \log x = y + x \Rightarrow y = \log x - x$

differentiating w.r.t. x $\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x} - 1 = \dfrac{1-x}{x}$

(8) **(D).** $x^m . y^n = (x+y)^{m+n}$

Taking log both side

$\log x^m . y^n = \log (x+y)^{m+n}$

$\Rightarrow m \log x + n \log y = (m+n) \log (x+y)$

Differntiating both side w.r.t. x

$$\Rightarrow \frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = (m+n) . \frac{1}{x+y}\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{n}{y} - \frac{m+n}{x+y}\right) = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\Rightarrow \frac{dy}{dx}\left[\frac{nx + ny - my - ny}{y(x+y)}\right] = \frac{mx + nx - mx - my}{x(x+y)}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y}\right) = \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

(9) **(A).** At $x = 1 \Rightarrow \cot y = 0$

$$y' = \frac{2(\cot y - 1)}{2\,\mathrm{cosec}^2 y} = \frac{-2}{2} = -1$$

(10) $$g'(x) = 2(f(2f(x)+2))\left(\frac{d}{dx} f(2f(x)+2)\right)$$

$= 2f(2f(x)+2)\, f'(2f(x)+2) . (2f'(x))$

$\Rightarrow g'(0) = 2f(2f(0)+2) . f'(2f(0)+2) . 2(f'(0) = 4f(0) f'(0) = 4(-1)(1) = -4$

(11) **(D).** $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{dx/dy}\right) = \frac{d}{dy}\left(\frac{1}{dx/dy}\right) . \frac{dy}{dx}$$

$$= -\frac{1}{\left(\frac{dx}{dy}\right)^2} . \frac{\frac{d^2x}{dy^2}}{\frac{dx}{dy}} = \frac{-\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}$$

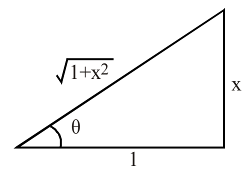
(12) **(A).** $y = \sec(\tan^{-1} x)$

Let $\tan^{-1} x = \theta$

$x = \tan\theta$

$y = \sec\theta$

$y = \sqrt{1+x^2}$



$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+x^2}} . 2x \ ; \text{At } x = 1, \ \frac{dy}{dx} = \frac{1}{\sqrt{2}}$$

(13) **(D).** $f(g(x)) = x \ ; \quad f'(g(x))\, g'(x) = 1$

$g'(x) = 1 + \{g(x)\}^5$

(14) **(C).** $x \in \left(0, \frac{1}{4}\right) \ ; \ 3x^{3/2} \in \left(0, \frac{3}{8}\right) \ ; \ \theta \in \left(0, \tan^{-1}\left(\frac{3}{8}\right)\right)$

$$y = \tan^{-1}\left(\frac{6x\sqrt{x}}{1 - 9x^3}\right) = \tan^{-1}\left(\frac{2x^{3/2}}{1 - (3x^{3/2})^2}\right)$$

Let $\tan^{-1}(3x^{3/2}) = \theta$

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1 - \tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta) = 2\theta = 2\tan^{-1}(3x^{3/2})$$

$$y' = \frac{2}{1 + 9x^3} \times 3 \cdot \frac{3}{2} x^{1/2} = \frac{9}{1 + 9x^3}\sqrt{x}$$

(15) **(D).** $\dfrac{dx}{dt} = 3\sec^2 t \ ; \ \dfrac{dy}{dt} = 3\sec t \tan t$

$$\frac{dy}{dx} = \frac{\tan t}{\sec t} = \sin t \ ; \ \frac{d^2y}{dx^2} = \cos t \frac{dt}{dx}$$

$$= \frac{\cos t}{3\sec^2 t} = \frac{\cos^3 t}{3} = \frac{1}{3 \cdot 2\sqrt{2}} = \frac{1}{6\sqrt{2}}$$

(16) **(B).** $y = f(f(f(x))) + (f(x))^2$

$$\frac{dy}{dx} = f'(f(f(x)))\,f'(f(x))\,f'(x) + 2f(x)\,f'(x)$$

$$= f'(1)\,f'(1)\,f'(1) + 2f(1)\,f'(1)$$

$$= 3 \times 5 \times 3 + 2 \times 1 \times 3 = 27 + 6 = 33$$

(17) **(B).** $k \cdot x^{k-1} + k \cdot y^{k-1}\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = -\left(\frac{x}{y}\right)^{k-1} \;;\; \frac{dy}{dx} + \left(\frac{x}{y}\right)^{k-1} = 0$$

$$k - 1 = -\frac{1}{3} \;;\; k = 1 - \frac{1}{3} = \frac{2}{3}$$

(18) **(A).** $y = \sqrt{\frac{2\cos^2\alpha}{\sin\alpha\cos\alpha} + \frac{1}{\sin^2\alpha}}$

$$= \sqrt{2\cot\alpha + \operatorname{cosec}^2\alpha} = |1 + \cot\alpha| = -1 - \cot\alpha$$

$$\frac{dy}{d\alpha} = \operatorname{cosec}^2\alpha$$

$\Rightarrow \left(\frac{dy}{d\alpha}\right)$ at $\alpha = \frac{5\pi}{6}$ will be 4.

(19) **(D).** $x = 2\sin\theta - \sin 2\theta$

$$\Rightarrow \frac{dx}{d\theta} = 2\cos\theta - 2\cos 2\theta = 4\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)$$

$y = 2\cos\theta - \cos 2\theta$

$$\frac{dy}{d\theta} = -2\sin\theta + 2\sin 2\theta = 4\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right)$$

$$\frac{dy}{dx} = \cot\left(\frac{3\theta}{2}\right) \Rightarrow \frac{d^2y}{dx^2} = \frac{-\frac{3}{2}\operatorname{cosec}^2\left(\frac{3\theta}{2}\right)}{4\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\theta=\pi} = \frac{3}{8}$$

(20) **(C).** $f\,g(x) = x$

$f'(g(x))\,g'(x) = 1$

Put $x = a$

$\Rightarrow f'(b)\,g'(a) = 1 \;;\; f'(b) = 1/5$