6

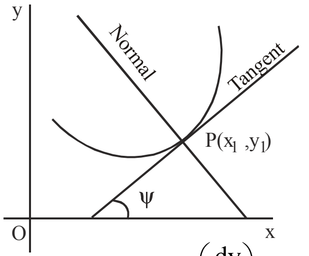
APPLICATION OF DERIVATIVES

TANGENT & NORMAL

SLOPES OF THE TANGENT & THE NORMAL

Slope of the tangent : Let y = f (x) be a continuous curve, and let P (x_1, y_1) be a point on it. Then, $\left(\frac{dy}{dx}\right)_p$ is the slope of the tangent to the curve y = f (x) at point P i.e. ;

$\left(\frac{dy}{dx}\right)_p = \tan\psi =$ slope of the tangent at P, where ψ is the angle which the tangent at P (x_1, y_1) makes with the positive direction of x-axis.



(i) The inclination of tangent with x–axis. = $\tan^{-1}\left(\frac{dy}{dx}\right)$

(ii) Slope of the tangent = $\frac{dy}{dx}$

(iii) If the tangent at $P(x_1,y_1)$ of the curve y = f (x) is parallel to the x–axis (or perpendicular to y-axis) then $\Psi = 0$ i.e. its slope will be zero.

$$m = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0$$

The converse is also true. Hence the tangent at (x_1,y_1) is parallel to x– axis.

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 0$$

(iv) If the tangent at $P(x_1,y_1)$ of the curve y = f (x) is parallel to y– axis (or perpendicular to x-axis) then $\Psi = \pi/2$,and its slope will be infinity i.e.

$$m = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \infty$$

The converse is also true. Hence the tangent at (x_1,y_1) is parallel to y–axis

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \infty$$

(v) If at any point $P(x_1,y_1)$ of the curve y = f (x), the tangent makes equal angles with the axes, then at the point P, $\psi = \pi/4$ or $3\pi/4$, Hence at P, $\tan\Psi = dy/dx = \pm 1$.

The converse of the result is also true. thus at (x_1,y_1) the tangent line makes equal angles with the axes.

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \pm 1$$

Slope of the normal : The normal to a curve at P (x_1, y_1) is a line perpendicular to the tangent at P and passing through P. ∴ Slope of the normal at

$$P = -\frac{1}{\text{slope of tangent at P}} = -\frac{1}{\left(\frac{dy}{dx}\right)_p} = -\left(\frac{dx}{dy}\right)_p$$

Note: If normal makes an angle of θ with positive direction of x-axis then

$$-\frac{dx}{dy} = \tan\theta \quad \text{or} \quad \frac{dy}{dx} = -\cot\theta$$

Example 1 :

The tangent to a given curve is perpendicular to x-axis if dy/dx = ?

Sol. $\frac{dy}{dx} = \infty \Rightarrow \frac{dx}{dy} = 0$

Example 2 :

Find the slope of the tangent to the hyperbola $2x^2 - 3y^2 = 6$ at (3, 2).

Sol. Differentiating the given equation of the curve

$4x - 6y. dy/dx = 0 \;\therefore dy/dx = 2x/3y$

$$\left(\frac{dy}{dx}\right)_{(3,2)} = \frac{2}{3}.\frac{3}{2} = 1$$

Example 3 :

Find the point on the curve $y^2 = x^2 + ax + 25$ touches the axis of x.

Sol. $\frac{dy}{dx} = 0$ as tangent is x - axis

∴ $2x + a = 0$ or $x = -a/2$.

But point lies on the curve.

$$\therefore \; y^2 = x^2 + ax + 25 = \frac{100 - a^2}{4} = 0$$

as it lies on x - axis. ∴ a = ± 10.

Example 4 :

Find the point at the curve $y = 12x - x^3$ where the slope of the tangent is zero.

Sol. $\frac{dy}{dx} = 12 - 3x^2$. Now, slope of the tangent $= 0 \Rightarrow \frac{dy}{dx} = 0$

$\Rightarrow 12 - 3x^2 = 0 \Rightarrow x = 2, -2$.

Then from the equation of the curve $y = 16, -16$

$\Rightarrow$ Required points are $(2, 16)$; $(-2, -16)$

Example 5 :

Find the slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point $t = 2$.

Sol. We have, $\frac{dx}{dt} = 2t + 3$ and $\frac{dy}{dt} = 4t - 2$.

$$\therefore \frac{dy}{dt} = \frac{dy/dt}{dx/dt} = \frac{4t-2}{2t+3}$$

Thus, slope of the tangent to the curve at the point $t = 2$

is $\left.\frac{dy}{dx}\right]_{t=2} = \frac{4(2)-2}{2(2)+3} = \frac{6}{7}$

Example 6 :

Find the tangent of the curve $y = 2x^2 - x + 1$ is parallel to the line $y = 3x + 9$ at the point.

Sol. We have $\frac{dy}{dx} = 4x - 1 = 3$ as tangent is parallel to $y=3x+9$.

$\therefore\ x = 1$ and hence $y = 2$.

Example 7 :

Find the points on the curve $y = x^3 + 5$ at which the tangents are perpendicular to the line $x + 3y = 2$.

Sol. Slope of the given line is $-1/3$ and hence of perpendicular tangent is 3.

$\frac{dy}{dx} = 3x^2 = 3 \quad \Rightarrow x = 1, -1$ and hence $y = 6$ and 4 resp.

EQUATIONS OF TANGENT AND NORMAL

We know that the equation of a line passing through a point (x_1, y_1) and having slope m is $y - y_1 = m(x - x_1)$

Slopes of the tangent and the normal to the curve $y = f(x)$ at a point P (x_1, y_1) are $\left(\frac{dy}{dx}\right)_P$ and $-\frac{1}{\left(\frac{dy}{dx}\right)_P}$ resp.

Therefore the equation of the tangent at P (x_1, y_1) to the curve $y = f(x)$ is $y - y_1 = \left(\frac{dy}{dx}\right)_P (x - x_1)$ ......(1)

Since the normal at P (x_1, y_1) passes through P and has slope $-\frac{1}{\left(\frac{dy}{dx}\right)_P}$, therefore the equation of the normal at P (x_1, y_1) to the curve $y = f(x)$ is

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_P}(x - x_1) \qquad ....(2)$$

Note :

1. If $\left(\frac{dy}{dx}\right)_P = \infty$, then the tangent at (x_1, y_1) is parallel to y-axis and its equation is $x = x_1$.

2. If $\left(\frac{dy}{dx}\right)_P = 0$, then the normal at (x_1, y_1) is parallel to y-axis and its equation is $x = x_1$.

Example 8 :

Find the equation of tangent to the curve $y = \sin x$ at the point $(\pi, 0)$.

Sol. $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x \Rightarrow \left(\frac{dy}{dx}\right)_{(\pi,0)} = -1$

Therefore the equation of tangent at $(\pi, 0)$ is given by

$y - 0 = -1(x - \pi) \Rightarrow x + y = \pi$

Example 9 :

Find the equation of tangent at the point 't' to the curve $x = a\cos^3 t$, $y = a\sin^3 t$.

Sol. $dx/dt = -3a\cos^2 t \sin t$

$dy/dt = 3a\sin^2 t \cos t$

$$\Rightarrow dy/dx = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\frac{\sin t}{\cos t}$$

(slope of tangent at point 't')

Therefore the equation of tangent at the point 't' is written as $y - a\sin^3 t = -\frac{\sin t}{\cos t}(x - a\cos^3 t)$

$$\Rightarrow \frac{y}{\sin t} - a\sin^2 t = -\frac{x}{\cos t} + a\cos^2 t$$

$\Rightarrow x\sec t + y\operatorname{cosec} t = a$

Example 10 :

Find the equation of tangent at those points where the curve $y = x^2 - 3x + 2$ meets x-axis.

Sol. Putting $y = 0$ in the given equation we get $x^2 - 3x + 2 = 0$ which gives $x = 1, 2$. So the given curve meets x-axis at $(1, 0)$ and $(2, 0)$. Now

$$\frac{dy}{dx} = 2x - 3 \Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = -1, \left(\frac{dy}{dx}\right)_{(2,0)} = 1$$

$\therefore$ eqn. of tangent at $(1, 0)$ is

$y = -1(x - 1) \Rightarrow x + y - 1 = 0$

eqn. of tangent at $(2, 0)$ is

$y = 1(x - 2) \Rightarrow x - y - 2 = 0$

Example 11 :

Find the equation of the tangent to the curve $y=\sqrt{9-2x^2}$ at the point where the ordinate and the abscissa are equal.

Sol. Putting y = x in $y=\sqrt{9-2x^2}$, we get

$$x=\sqrt{9-2x^2} \Rightarrow x^2=9-2x^2 \Rightarrow \quad x=\sqrt{3},-\sqrt{3}$$

Since y > 0, therefore , the point is $(\sqrt{3},\sqrt{3})$.

Now, we have, $y^2=9-2x^2$

Differentiating with respect to x, we get

$$2y\frac{dy}{dx}=-4x \Rightarrow \frac{dy}{dx}=-\frac{2x}{y}$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{(\sqrt{3},\sqrt{3})}=-\frac{2\sqrt{3}}{\sqrt{3}}=-2$$

So, the equation of tangent at $(\sqrt{3},\sqrt{3})$ is

$$(y-\sqrt{3})=-2(x-\sqrt{3}) \text{ i.e. } 2x+y-3\sqrt{3}=0$$

Example 12 :

Find the equation of the normal to the curve $y(x-2)(x-3)-x+7=0$ at the point where it cuts the x-axis.

Sol. $\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{y(2x-5)-1}{(x-2)(x-3)}=\frac{1}{20}$ at (7, 0)

∴ Slope of normal is – 20 and its equation is $y-0=-20(x-7)$

Example 13 :

Find the equation of normal to the curve $y=x(2-x)$ at the point (2, 0).

Sol. $y'=2-2x=-2$ at (2, 0)

$$\therefore \quad y-0=\frac{1}{2}(x-2) \quad \therefore \quad x-2y=2$$

LENGTH OF INTERCEPTS MADE ON AXES BY THE TANGENT

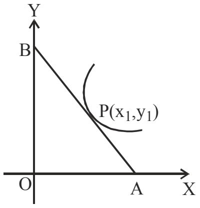
Equation of tangent at any point (x_1,y_1) to the curve

$y=f(x)$ is $\quad y-y_1=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}(x-x_1)$ ....(1)

Equation of x–axis $y=0$ .... (2)

Equation of y–axis $x=0$ .... (3)

Solving (1) and (2) , we get.



$$x=x_1-\left\{\frac{y_1}{(dy/dx)_{(x_1,y_1)}}\right\}$$

$$\therefore \text{ x– intercept : } OA=x_1-\left\{\frac{y_1}{(dy/dx)_{(x_1,y_1)}}\right\}$$

Similarly solving (1) and (3), we get

$$\text{y–intercept, } \quad OB=y_1-x_1\left(\frac{dy}{dx}\right)_{(x_1,y_1)}$$

The length of intercept made by normal on x–axis is $x_1+y_1\left(\frac{dy}{dx}\right)$ and length of intercept on y–axis is $y_1+x_1\left(\frac{dx}{dy}\right)$

Example 14 :

The tangent at any point on the curve $x^4+y^4=a^4$ cuts off intercepts p and q on the coordinate axes then find the value of $p^{-4/3}+q^{-4/3}$.

Sol. $(X-x)f_x+(Y-y)f_y=0$

$(X-x)4x^3+(Y-y)4y^3=0$

or $Xx^3+4Yy^3=x^4+y^4=a^4$ ....(1)

If intercepts are p and q then (p, 0) and (0, q) lie on (1)

$\therefore \quad px^3=a^4,\ qy^3=a^4$

$p^{-4/3}+q^{-4/3}=(a^4)^{-4/3}(x^4+y^4)=a^{-16/3}.\ a^4=a^{-4/3}$

Example 15 :

At a point (a/8, a/8) on the curve $x^{1/3}+y^{1/3}=a^{1/3}$ $(a>0)$ tangent is drawn. If the axes be of length $\sqrt{2}$, then find the value of a.

Sol. Slope of tangent is

$$=-\frac{f_x}{f_y}=\frac{-x^{-2/3}}{y^{-2/3}}=-\left(\frac{y}{x}\right)^{2/3}=-1 \text{ at } (a/8,\ a/8)$$

Tangent is $y-a/8=-1(x-a/8)$ or $x+y=\frac{a}{4}$

Its intercepts on axes are $A=\frac{a}{4}$, $B=\frac{a}{4}$

Portion of tangent intercepted between the axes is $\sqrt{A^2+B^2}=\sqrt{2}$ (given)

$$\therefore \quad \frac{a^2}{16}+\frac{a^2}{16}=2 \quad \text{or } a^2=16 \Rightarrow a=4$$

Example 16 :

The triangle formed by the tangent to the curve $f(x)=x^2+bx-b$ at the point (1, 1) and the coordinate axes, lies in the first quadrant. If its area is 2, then find the value of b.

Sol. $\frac{dy}{dx}=2x+b=2+b$ at (1, 1)

Equation of tangent is $y-1=(2+b)(x-1)$

Its intercepts A and B on the axes are obtained by putting y = 0 and then x = 0

$$\therefore \quad A=1-\frac{1}{2+b}=\frac{b+1}{b+2}$$

$$B=1-(2+b)=-(b+1)$$

$$\Delta = \frac{1}{2}\,AB = 2 \;\therefore AB = 4$$

$$-(b+1)\,(b+1) = 4\,(b+2)$$

or $b^2 + 6b + 9 = 0$ or $(b+3)^2 = 0$

therefore $b = -3$

LENGTH OF PERPENDICULAR FROM ORIGIN TO THE TANGENT

The equation of tangent at point $P(x_1, y_1)$ of the given curve $y - y_1 = \left(\frac{dy}{dx}\right)_p (x - x_1)$

The length of perpendicular from origin (0,0) to the tangent drawn at the point (x_1, y_1) of the curve $y = f(x)$ is

$$p = \left|\frac{y_1 - x_1\left(\frac{dy}{dx}\right)}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right|$$

The length of perpendicular from origin to normal is

$$P' = \left|\frac{x_1 + y_1\left(\frac{dy}{dx}\right)}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right|$$

Example 17 :

Find the distance between the origin and the tangent to the curve $y = e^{2x} + x^2$ drawn at the point $x = 0$.

Sol. Putting $x = 0$ in the given curve, we obtain $y = 1$.

So, the given point is (0, 1).

Now, $y = e^{2x} + x^2 \Rightarrow \frac{dy}{dx} = 2e^{2x} + 2x \Rightarrow \left(\frac{dy}{dx}\right)_{(0,1)} = 2$

The equation of the tangent at (0, 1) is

$y - 1 = 2(x - 0) \Rightarrow 2x - y + 1 = 0$ ..(i)

Required distance = length of the $\perp$ from (0,0) on (i) $= \frac{1}{\sqrt{5}}$

Example 18 :

If p_1 and p_2 be the length of perpendiculars from the origin on the tangent and normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ respectively, then find $4p_1^2 + p_2^2$.

Sol. Take the parametric equation as $x = a\sin^3\theta$, $y = a\cos^3\theta$

Tangent is $x\cos\theta + y\sin\theta = (a/2)\sin 2\theta$.

Normal is $y\cos\theta - x\sin\theta = a\cos 2\theta$

$$\therefore \quad p_1 = \frac{(a/2)\sin 2\theta}{\sqrt{(\cos^2\theta + \sin^2\theta)}} = \frac{a}{2}\sin 2\theta$$

$$P_2 = \frac{a\cos 2\theta}{\sqrt{(\cos^2\theta + \sin^2\theta)}} = a\cos 2\theta$$

$$4p_1^{\,2} + p_2^{\,2} = a^2(\sin^2 2\theta + \cos^2 2\theta) = a^2$$

Example 19 :

Find the distance between the point (1, 1) and the tangent to the curve $y = e^{2x} + x^2$ drawn from the point $x = 0$.

Sol. Putting $x = 0$ in $y = e^{2x} + x^2$ .... (1)

We get $y = 1$

$\therefore$ the given point is P (0, 1)

From (1), $\frac{dy}{dx} = 2e^{2x} = 2x \Rightarrow \left[\frac{dy}{dx}\right]_P = 2$

$\therefore$ equation of tangent at P to (1) is

$y - 1 = 2(x - 0) \Rightarrow 2x - y + 1 = 0$ .... (2)

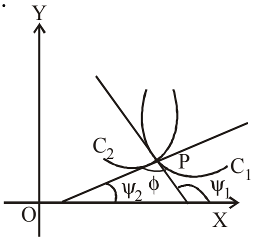
$\therefore$ Required distance = Length of $\perp$ from (1, 1) to (2)

$$= \frac{2 - 1 + 1}{\sqrt{4 + 1}} = \frac{2}{\sqrt{5}}$$

ANGLE OF INTERSECTION OF TWO CURVES

The angle of intersection of two curves is defined to be the angle between the tangents to the two curves at their point of intersection.

Let C_1 and C_2 be two curves having equations $y = f(x)$ and $y = g(x)$ respectively. Let PT_1 and PT_2 be tangents to the curves C_1 and C_2 respectively at their common point of intersection. Then the angle θ between PT_1 and PT_2 is the angle of intersection of C_1 and C_2. Let ψ_1 and ψ_2 be the angles made by PT_1 and PT_2 with the positive direction of x-axis in anticlockwise sense.



Then, $m_1 = \tan\psi_1$ = slope of the tangent to $y = f(x)$ at $P = \left(\frac{dy}{dx}\right)_{C_1}$.

and $m_2 = \tan\psi_2$ = slope of the tangent to $y = g(x)$ at $P = \left(\frac{dy}{dx}\right)_{C_2}$.

From figure it follow that $\phi = \psi_1 - \psi_2$.

$\Rightarrow \tan\phi = \tan(\psi_1 - \psi_2)$

$$\Rightarrow \tan\phi = \frac{\tan\psi_1 - \tan\psi_2}{1 + \tan\psi_1 \tan\psi_2} = \frac{\left(\frac{dy}{dx}\right)_{C_1} - \left(\frac{dy}{dx}\right)_{C_2}}{1 + \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2}}$$

The other angle between the tangents is 180° – ϕ. Generally the smaller of these two angles is taken to be the angle of intersection.

ORTHOGONAL CURVES

If the angle of intersection of two curves is a right angle, the two curve are said to intersect of orthogonal and the curves are called orthogonal curves.

If the curves are orthogonal, then $\phi = \frac{\pi}{2}$.

$$\therefore \quad m_1 m_2 = -1 \Rightarrow \left(\frac{dy}{dx}\right)_{C_1} \cdot \left(\frac{dy}{dx}\right)_{C_2} = -1$$

Example 20 :

Find the angle of intersection between the curves $y = x$ and $y^2 = 4x$ at (4,4).

Sol. Differentiating given equations , we have

$$\left(\frac{dy}{dx}\right)_1 = 1 \text{ and } \left(\frac{dy}{dx}\right)_2 = \frac{2}{y}$$

$$\therefore \text{At } (4,4) \left(\frac{dy}{dx}\right)_1 = 1 \; ; \left(\frac{dy}{dx}\right)_2 = \frac{2}{4} = \frac{1}{2}$$

$$\text{Angle of intersection} = \tan^{-1}\left|\frac{1-\frac{1}{2}}{1+1\left(\frac{1}{2}\right)}\right| = \tan^{-1}(1/3)$$

Example 21 :

Find the angle of intersection of the parabola $y^2 = 4ax$ and $x^2 = 4ay$ at the origin.

Sol. $y^2 = 4ax \Rightarrow 2y\frac{dy}{dx} = 4a$ i.e. $\frac{dy}{dx} = \frac{2a}{y}$

At (0, 0), $\frac{dy}{dx} = \infty = \tan\frac{\pi}{2} = \tan\theta_1$ (say) $\therefore \quad \theta_1 = \frac{\pi}{2}$

Again $x^2 = 4ay \Rightarrow 2x = 4a\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{2a}$

A (0, 0), $\frac{dy}{dx} = 0 = \tan 0 = \tan\theta_2$ (say) $\therefore \quad \theta_2 = 0$

Again between the two curves $= |\theta_1 - \theta_2| = \pi/2$

Example 22 :

Find the angle between the curves y = sinx, y = cos x.

Sol. Clearly $x = \frac{\pi}{4}$ is a common point

$(\because \sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \pi/4 \therefore y = 1/\sqrt{2})$

Now For the first curve $\frac{dy}{dx} = \cos x = \frac{1}{\sqrt{2}}$ where $x = \frac{\pi}{4}$

For the second curve $\frac{dy}{dx} = -\sin x = -\frac{1}{\sqrt{2}}$

$$\therefore \quad \tan\theta = \frac{\frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)}{1 + \frac{1}{\sqrt{2}} \cdot \left(\frac{-1}{\sqrt{2}}\right)} = \frac{\frac{2}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} = \frac{\frac{2}{\sqrt{2}}}{\frac{1}{2}} = 2\sqrt{2}$$

$$\theta = \tan^{-1}(2\sqrt{2})$$

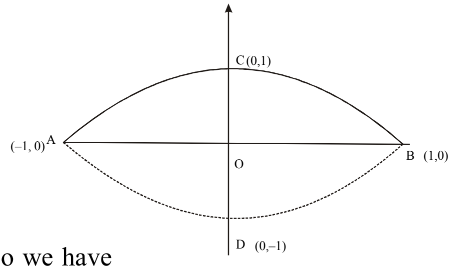
Example 23 :

Find the acute angle between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersection.

Sol. We have $y = |x^2 - 1|$

$$= \begin{cases} x^2 - 1 \text{ if } & x \le -1 \text{ or } x \ge 1 \\ 1 - x^2 \text{ if } & -1 < x < 1 \end{cases}$$

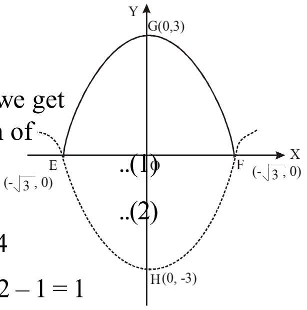
The graph of the curve is



Also we have

$$y = |x^2 - 3| = \begin{cases} x^2 - 3 \text{ if } & x \le -\sqrt{3} \text{or } x \ge \sqrt{3} \\ 3 - x^2 \text{ if } & -\sqrt{3} < x < \sqrt{3} \end{cases}$$

The graph of the curve is



Combining the two curves, we get R is the point of intersection of

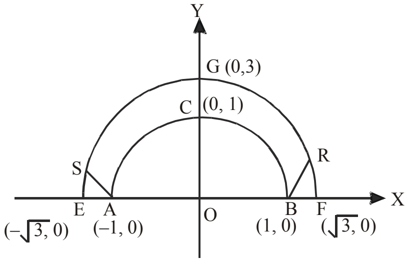
$y = x^2 - 1$ ..(1)

and $y = 3 - x^2$ for $1 < x < \sqrt{3}$ ..(2)

$\therefore \quad x^2 - 1 = 3 - x^2 \Rightarrow 2x^2 = 4$

$\Rightarrow \quad x^2 = 2 \quad \therefore x = \sqrt{2} \quad \therefore y = 2 - 1 = 1$

$\therefore \quad$ R is $(\sqrt{2}, 1)$. Similarly, S is $(-\sqrt{2}, 1)$



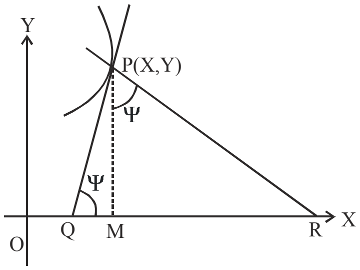
At R, slope of (1) is $2\sqrt{2}$ $\left(\because \frac{dy}{dx} = 2x\right)$

At R, slope of (2) is $-2\sqrt{2}$ $\left(\because \frac{dy}{dx} = -2x\right)$

$$\therefore \quad \tan\theta = \left|\frac{2\sqrt{2} + 2\sqrt{2}}{1 - 8}\right| = \frac{4\sqrt{2}}{7} \quad \therefore \theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$$

LENGTH OF TANGENT, NORMAL, SUB TANGENT & SUB NORMAL

Let tangent and normal to the curve y = f (x) at a point P(x,y) meets the x–axis at point Q and R respectively. Then PQ and PR are called length of tangent and normal respectively, at point P. Also if PM be the perpendicular from p on x– axis, then QM and MR are called length of sub tangent and subnormal respectively at P. So from the diagram at P(x,y)



(i) length of tangent $y \operatorname{cosec} \Psi = y\dfrac{\sqrt{1+(dy/dx)^2}}{(dy/dx)}$

(ii) length of normal = PR = y sec Ψ

(iii) length of sub tangent

$= QM = y \cot \Psi = y / \left(\dfrac{dy}{dx}\right)$

(iv) length of sub normal $= MR = y \tan \Psi = y\left(\dfrac{dy}{dx}\right)$

Example 24 :

Find the length of the tangent at $t = \dfrac{\pi}{4}$ to the curve

$x = a(\cos t + t \sin t)$

$x = a(\sin t - \cos t)\ (a > 0)$

Sol. $y \text{ at } t = \dfrac{\pi}{4} = a\left(\dfrac{1}{\sqrt{2}} - \dfrac{\pi}{4}\dfrac{1}{\sqrt{2}}\right) = \dfrac{a}{\sqrt{2}}\left(1 - \dfrac{\pi}{4}\right)$

$$\left|\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \frac{a\left(\cos\frac{\pi}{4} - \cos\frac{\pi}{4} + \frac{\pi}{4}\sin\frac{\pi}{4}\right)}{a\left(-\sin\frac{\pi}{4} + \frac{\pi}{4}\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)}$$

$$= \frac{\frac{\pi a}{4}\sin\frac{\pi}{4}}{\frac{\pi a}{4}\cos\frac{\pi}{4}} = \tan\frac{\pi}{4} = 1$$

∴ Length of the tangent

$$= \frac{y}{\frac{dy}{dx}}\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{\frac{a}{\sqrt{2}}\left(1-\frac{\pi}{4}\right)}{1}\sqrt{1+1} = a\left(1-\frac{\pi}{4}\right)$$

Example 25 :

If x = a (θ + sinθ), y = a (1 – cosθ), then at θ = π/2, find the length of the normal.

Sol. $\dfrac{dx}{d\theta} = a(1 + \cos\theta)\ ;\ \dfrac{dy}{d\theta} = a\sin\theta$

$$\frac{dy}{dx} = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

$$\text{Length of the normal} = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= a(1-\cos\theta)\sqrt{1+\tan^2\frac{\theta}{2}} = a(1-\cos\theta).\sec\frac{\theta}{2}$$

When θ = π/2

$$\text{Length} = a\left(1-\cos\frac{\pi}{2}\right).\sec\frac{\pi}{4} = a(1-0).\sqrt{2} = \sqrt{2}\,a$$

Example 26 :

The square of the subtangent to the curve $by^2 = (x + a)^2$ is proportional to

(1) $(\text{subnormal})^{3/2}$ (2) Subnormal

(3) $(\text{subnormal})^{1/2}$ (4) None of these

Sol. **(2).** $by^2 = (x + a)^3 \Rightarrow 2by\dfrac{dy}{dx} = 3(x + a)^2$

$$\Rightarrow y\frac{dy}{dx} = \frac{3(x+a)^2}{2b} \text{ i.e. } \frac{dy}{dx} = \frac{3(x+a)^2}{2by}$$

$$\text{Length of the subnormal} = y\frac{dy}{dx} = \frac{3(x+a)^2}{2b}$$

$$\text{Again, subtangent} = \frac{y}{dy/dx} = \frac{y}{\frac{3(x+a)^2}{2by}} = \frac{2by^2}{3(x+a)^2}$$

$\Rightarrow (\text{sub-tangent})^2$

$$= \frac{4b^2y^2}{9(x+a)^4} = \frac{4b^2}{9(x+a)^4}.\frac{(x+a)^6}{b^2} = \frac{4}{9}(x+a)^2$$

$$= \frac{4}{9}.\frac{2b}{3}.\frac{3(x+a)^2}{2b} = \frac{8b}{27}(\text{subnormal})$$

∴ $(\text{sub-tangent})^2$ is proportional to subnormal

Example 27 :

If at any point S of the curve $by^2 = (x + a)^3$, the relation between subnormal SN and subtangent ST be $p(SN) = q(ST)^2$ then find p/q.

Sol. $$\frac{dy}{dx} = \frac{3(x+a)^2}{2by}\ ;\ \frac{p}{q} = \frac{\left(y/\frac{dy}{dx}\right)^2}{y\left(\frac{dy}{dx}\right)} = \frac{y}{\left(\frac{dy}{dx}\right)^3} = \frac{8b}{27}$$

POINT OF INFLEXION

If at any point P, the curve is concave on one side and convex on other side with respect to x–axis, then the point P is called the point of inflexion. Thus P is a point of inflexion if at P, $\frac{d^2y}{dx^2} = 0$, but $\frac{d^3y}{dx^3} \neq 0$

Also point p is a inflexion

if $f''(x) = f'''(x) = f^{n-1}(x) = 0$ and $f^n(x) \neq 0$ for odd n.

Example 28 :

Prove that origin for the curve $y = x^3$ is a point of inflexion.

Sol. $\because \quad y = x^3$

$$\therefore \quad \frac{dy}{dx} = 3x^2, \frac{d^2y}{dx^2} = 6x, \frac{d^3y}{dx^3} = 6$$

Clearly at (0,0), $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$

$\therefore$ There is a point of inflexion at (0,0).

LINEAR APPROXIMATION (DIFFERENTIALS)

In many problems we are interested in the change, or approximate change, of values f (x) that correspond to change in x.

If $\Delta x = x - c$ represents a change in x, the corresponding change in f (x) is denoted

$$\Delta f = f(x) - f(c) \qquad \text{..............} (1)$$

If f is differentiable at c, then we know from (1) that

$f(x) \approx f(c) + f'(c)(x - c)$, so

$f(x) - f(c) \approx f'(c)(x - c)$.

Let us rewrite this last statement as

$$\Delta f \approx f'(c) \,.\, \Delta x \qquad \text{..............} (2)$$

Example 29 :

Use differential to approximate $\sqrt{101}$.

Sol. Let $f(x) = \sqrt{x}$; $f(100) = 10$ & $f'(x) = \frac{1}{2\sqrt{x}}$

$$\Delta y = \frac{1}{2\sqrt{100}} = \frac{1}{20} = 0.05 \Rightarrow f(101) = 10 + 0.05 = 10.05$$

TRY IT YOURSELF-1

Q.1 A curve in the plane is defined by the parametric equations $x = e^{2t} + 2e^{-t}$ and $y = e^{2t} + e^t$. An equation for the line tangent to the curve at the point t = ln 2 is –

(A) $5x - 6y = 7$ (B) $5x - 3y = 7$

(C) $10x - 7y = 8$ (D) $3x - 2y = 3$

Q.2 Curve $C_1 : y = e^x \ln x$ and $C_2 : y = \frac{\ln x}{e^x}$ intersect at point P whose absicssa is less than 1. Find equation of normal to curve C_1 at point P.

Q.3 Tangent at point P on the curve $y^2 = x^3$ meets the curve again at point Q. Find $\frac{m_{OP}}{m_{OQ}}$, where O is origin.

Q.4 If θ is the angle b/w $y = x^2$ and $6y = 7 - x^3$ at (a, a). Find θ.

Q.5 Show that at any point on the hyperbola $xy = c^2$, the subtangent varies as the abscissa and the subnormal varies as the cube of the ordinate of the point of contact.

Q.6 Using differentials, find the approximate value of $(82)^{1/4}$ upto 3 places of decimal.

Q.7 If the sum of the squares of the intercepts on the axes cut off by tangent to the curve $x^{1/3} + y^{1/3} = a^{1/3}$, a > 0 at (a/8, a/8) is 2, then a =

(A) 1 (B) 2

(C) 4 (D) 8

Q.8 Tangents are drawn from the origin to the curve y = sin x, then their point of contact lie on the curve –

(A) $\frac{1}{y^2} - \frac{1}{x^2} = 1$ (B) $\frac{1}{x^2} + \frac{1}{y^2} = 1$

(C) $x^2 - y^2 = 1$ (D) $x^2 + y^2 = 1$

Q.9 Find the equations of tangents to the curve $x^2 + y^2 - 2x - 4y + 1 = 0$ which are parallel to the x-axis.

Q.10 Prove that every point of the curve y = bsin(x/a) is a point of inflexion, where it meets the x–axis.

ANSWERS

(1) (C)	**(2)** x = 1/e	**(3)** –2
(4) π/2	**(6)** 3.009	**(7)** (C)
(8) (A)	**(9)** y = 0 and y = 4.	

MONOTONICITY

INTRODUCTION

In this section, we shall study the nature of a function which is governed by the sign of its derivative. If the graph of a function is in upward going direction or in downward coming direction then it is called as monotonic function, and this property of the function is called Monotonicity. If a function is defined in any interval, and if in some part of the interval, graph moves upwards and in the remaining part moves downward then function is not monotonic in that interval.

MONOTONIC FUNCTION

These are of two types

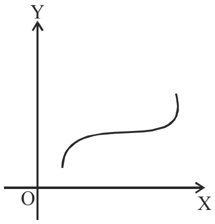
Monotonic Increasing :

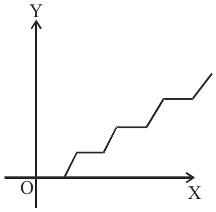
A function f (x) defined in a domain D is said to be monotonic increasing function if the value of (x) does not decrease (increase) by increasing (decreasing) the value of x or

We can say that the value of f (x) should increase (decrease) or remain equal by increasing (Decreasing) the value x.

$$\text{If} \quad \begin{cases} x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \\ \text{or } x_1 < x_2 \Rightarrow f(x_1) \ngtr f(x_2) \end{cases}, \ \forall x_1, x_2 \in D$$

$$\text{or} \quad \begin{cases} x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2) \\ \text{or } x_1 > x_2 \Rightarrow f(x_1) \nless f(x_2) \end{cases}, \ \forall x_1, x_2 \in D$$

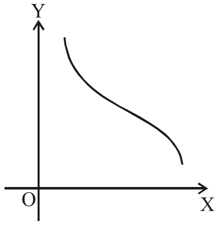


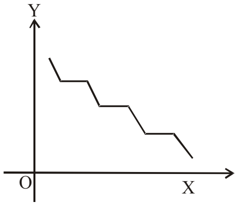


Monotonic Decreasing : A function f (x) defined in a domain D is said to be monotonic decreasing function if the value of (x) does not increase (decrease) by increasing (decreasing) the value of x or We can say that the value of f (x) should increase (decrease) or remain equal by increasing (Decreasing) the value of x.

$$\text{If } \begin{cases} x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2) \\ \text{or } x_1 < x_2 \Rightarrow f(x_1) \nless f(x_2) \end{cases}, \forall x_1, x_2 \in D$$

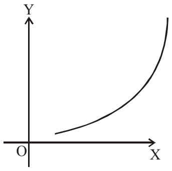
$$\text{or } \begin{cases} x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2) \\ \text{or } x_1 > x_2 \Rightarrow f(x_1) \ngtr f(x_2) \end{cases}, \forall x_1, x_2 \in D$$



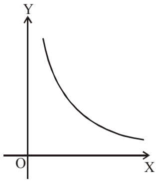


A function is said to be monotonic function in a domain if it is either monotonic increasing or monotonic decreasing in that domain.

Note : If $x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \forall x_1, x_2 \in D$, then f (x) is called strictly increasing in domain D.



Similarly if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \forall x_1, x_2 \in D$ then it is called strictly decreasing in domain D.



For Example

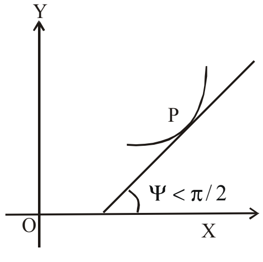
(i) $f(x) = e^x$ is a monotonic increasing function where as g (x)=1/x is monotonic decreasing function.

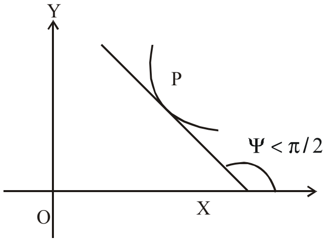
(ii) $f(x) = x^2$ and g(x)=| x | are monotonic increasing for $x > 0$ and monotonic decreasing for $x < 0$. In general they are not monotonic functions.

(iii) sinx, cosx are not monotonic function whereas tanx, cotx are monotonic.

METHOD OF TESTING MONOTONICITY

(i) **At a point :** A function f (x) is said to be monotonic increasing (decreasing) at a point x = a of its domain if it is monotonic increasing (decreasing) in the interval (a–h, a+h) where h is a small positive number. Hence we may observe that f (x) is monotonic increasing at x = a, then at this point tangent to its graph will make an acute angle with the x– axis where as if the function is monotonic decreasing these tangent will make an obtuse angle with x–axis. Consequently f '(a) will be positive or negative according as f (x) is monotonic increasing or decreasing at x = a.





So at x = a, function f (x) is
Monotonic increasing ⇒ $f'(a) > 0$
Monotonic decreasing ⇒ $f'(a) < 0$

Example 30 :

The function f (x)=cosx is decreasing at x = π/3 and increasing at x = 4π/3 since

$f'(\pi/3) = -\sqrt{3}/2 < 0$ and $f'(\pi/3) = +\sqrt{3}/2 > 0$

(ii) **In an interval :** A function f (x) defined in the interval [a,b] will be
Monotonic increasing ⇒ $f'(x) \geq 0$
Monotonic decreasing ⇒ $f'(x) \leq 0$
constant ⇒ $f'(x) = 0 \quad \forall\, x \in (a, b)$
Strictly increasing ⇒ $f'(x) > 0$
Strictly decreasing ⇒ $f'(x) < 0$

Note :

(i) In the above result f ' (x) should not be zero for all value of x otherwise f (x) will be a constant function.

(ii) If in [a,b], f ' (x) < 0, for atleast one value of x and f '(x) > 0 for atleast one value of x then f (x) will not be monotonic in [a,b]. For example,

(1) Function f (x) = sinx is monotonic increasing in [0, π/2] because $f'(x) = \cos x > 0 \forall x \in (0, \pi/2)$

(2) Function $f(x) = e^{-x}$ is monotonically decreasing in [–1, 0], since $f'(x) = -e^{-x} < 0, \forall\, x \in (-1, 0)$

(3) Function f (x) = | x | is not a monotonic functions in the interval [–1, 1] because

$$f(x) = \begin{cases} x; & x > 0 \\ -x; & x < 0 \end{cases}; f'(x) = \begin{cases} 1; & x > 0 \\ -1; & x < 0 \end{cases}$$

(4) $f(x) = \sin^{-1} x + \cos^{-1} x$ is constant function in [–1, 1] because $f(x) = \pi/2 \Rightarrow f'(x) = 0 \Rightarrow x \in (-1, 1)$

EXAMPLES OF MONOTONIC FUNCTION

If a function is monotonic increasing (decreasing) at every point of its domain ,then it is said to be monotonic increasing (decreasing) function.

In the following table we have examples of some monotonic/ not monotonic functions.

Monotonic Increasing	Monotonic Decreasing	Not Monotonic
x^3	$1/x$	x^2
$x\|x\|$	$1-2x$	$\|x\|$
e^x	e^{-x}	$e^x + e^{-x}$
$\log_a x, a > 1$	$\log_a x, a > 1$	$\sin x$
$\tan x$	$\cot x$	$\cos x$
$[x]$		

Example 31:

For all values of x, function $f(x) = 2x^3 + 6x^2 + 7x - 19$ is

(1) Monotonic increasing (2) Monotonic decreasing

(3) Not monotonic (4) None of these

Sol. **(1).** $\because f'(x) = 6x^2 + 12x + 7 = 6(x^2 + 2x) + 7 = 6(x+1)^2 + 1$ which is positive for all value of x. Hence f (x) is monotonic increasing function.

Example 32 :

Function $f(x) = x^3 - 27x + 5$ is monotonic increasing when

(1) $x < -3$ (2) $|x| > 3$

(3) $x \le -3$ (4) $|x| \ge 3$

Sol. **(2).** f (x) will be monotonic increasing if

$$f'(x) > 0 \Rightarrow 3x^2 - 27 > 0 \Rightarrow x^2 > 9 \Rightarrow |x| > 3$$

Example 33 :

The function $f(x) = x - \log(1 + x), x > -1$ is increasing in the interval

(1) $(0, \infty)$ (2) $(-1, 0)$

(3) $(-\infty, 0)$ (4) None of these

Sol. **(1).** We have, $f(x) = x - \log(1 + x), x > -1$

$$\Rightarrow f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} = \frac{x(1+x)}{(1+x)^2}$$

Sign scheme for $\frac{x(1+x)}{(1+x)^2}$

$f'(x) > 0$ if $x < -1$ or $x > 0$

But $x > -1$. $\therefore$ $f'(x) > 0$ if $x > 0$.

Thus, f (x) is increasing in the interval $(0, \infty)$.

Example 34 :

Let $f'(x) < 0$ and $g'(x) > o$ for all real x, then

(1) $f(g(x+1)) > g(g(x+5))$ (2) $f(g(x)) < f(g(f(x+2))$

(3) $g(f(x)) < g(f(x+2))$ (4) $g(f(x)) > g(f(x-2))$

Sol. **(1).** Given, $f'(x) < 0$ and $g'(x) > 0$ therefore g (x) is an increasing function and f ' (x) is a decreasing function

$\therefore$ $x + 1 < x + 5 \Rightarrow g(x+1) < g(x+5)$

$\Rightarrow$ $f(g(x+1)) > f(g(x+5))$

Again $x < x + 1 \Rightarrow g(x+1)$

$\Rightarrow$ $f(g(x)) > f(g(x+1))$

$x < x + 2 \Rightarrow f(x) > f(x+2)$

$\Rightarrow$ $g(f(x)) > g(f(x+1))$

$x > x - 2 \Rightarrow f(x) < f(x-2)$

$\Rightarrow$ $g(f(x)) < g(f(x-2))$

PROPERTIES OF MONOTONIC FUNCTIONS

(i) If f (x) is strictly increasing function on an interval [a, b], then f^{-1} exists and it is also a strictly increasing function.

(ii) If f (x) is strictly increasing function on an interval [a, b] such that it is continuous, then f^{-1} is continuous on [f (a), f(b)].

(iii) If f (x) is continuous on [a, b] such that $f'(c) \ge 0$ ($f'(c) > 0$) for each $c \in (a, b)$, then f (x) is monotonically (strictly) increasing function on [a, b].

(iv) If f (x) is continuous on [a, b] such that $f'(c) \le 0$ ($f'(c) < 0$) for each $c \in (a, b)$, then f (x) is monotonically (strictly) decreasing function on [a, b].

(v) If f (x) and g(x) are monotonically (or strictly) increasing (or decreasing) function on [a, b], then gof (x) is a monotonically (or strictly) increasing function on [a, b].

(vi) If one of the two functions f (x) and g(x) is strictly (or monotonically) increasing and other a strictly (monotonically) decreasing, then gof (x) is strictly (monotonically) decreasing on [a, b].

Example 35 :

Let $f(x) = \cot^{-1}[g(x)]$, where g(x) is an increasing function for $0 < x < \pi$. Then (x) is

(1) increasing in $(0, \pi)$

(2) decreasing in $(0, \pi)$

(3) increasing in $\left(0, \frac{\pi}{2}\right)$ and decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(4) None of these

Sol. **(2).** We have, $f(x) = \cot^{-1}(g(x))$

$$\Rightarrow f'(x) = \frac{-1}{1 + [g(x)]^2} \times g'(x)$$

< 0 for $0 < x < \pi$

$\because$ g(x) is increasing for $0 < x < \pi$, $\therefore g'(x) > 0$]

Thus, f (x) is decreasing in $(0, \pi)$

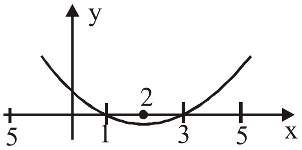
GREATEST AND LEAST VALUE OF FUNCTION

1. **Case I :** If a continuous function y = f (x) is strictly increasing in the closed inveral [a, b] then f (a) is the least value and f (b) is greatest value.
2. **Case II :** If f (x) is decreasing in [a, b] then f (b) is the least and f (a) is the greatest value of f (x) in [a, b].
3. **Case III :** If f (x) is non-monotonic in [a, b] and is continuous then the gratest and least value of f (x) in [a, b] are those f (x) = 0 or f ' (x) does not exist or at the extreme values.

Example 36 :

Find least and greatest value of $f(x) = e^{x^2-4x+3}$ in [–5,5]

Sol. $f(x) = e^{x^2-4x+3}$



For f (x) max → $x^2 - 4x + 3$ be maximum in [–5, 5].

$x^2 - 4x + 3$ will be maximum at x = –5 in the given interval.

i.e., 25 + 20 + 3 = 48

∴ Max $f(x) = e^{48}$ at x = –5

$x^2 - 4x + 3$ will be minimum at x = 2 i.e., 4 – 8 + 3 = –1

∴ Min $f(x) = e^{-1}$ at x = 2.

TRY IT YOURSELF-2

Q.1 Find intervals of monotonicity of $f(x) = \dfrac{x}{\ln x}$.

Q.2 If the function $f(x) = (a + 2)x^3 - 3ax^2 + 9ax - 1$ is always decreasing $\forall\, x \in R$, find a.

Q.3 Find the set of values of x for which $\ln(1 + x) > \dfrac{x}{1+x}$.

Q.4 The function $f(x) = \dfrac{|x-1|}{x^2}$

(A) increases in $(-\infty, 0) \cup (1, 2)$

(B) increases in $(0, 1) \cup (2, \infty)$

(C) decreases in $(0, 1) \cup (2, \infty)$

(D) decreases in $(-\infty, \infty) \cup (1, 2)$

Q.5 Find the intervals in which the given function increases or decreases $f(x) = \dfrac{x^2 + x + 1}{x^2 - x + 1}$.

Q.6 Function $f(x) = \cos x - 2\lambda x$ is monotonic decreasing when

(A) $\lambda > 1/2$ (B) $\lambda < 1/2$

(C) $\lambda < 2$ (D) $\lambda > 2$

ANSWERS

(1) $(0, 1) \cup (1, e)$ **(2)** $a \le -3$

(3) $f(x) > 0\ \forall\, x \in (-1, 0) \cup (0, \infty)$ **(4)** (AC)

(5) Strictly increases in (–1, 1) ; Strictly decreases in $(1, \infty)$

(6) (A)

MAXIMA AND MINIMA

MAXIMUM AND MINIMUM VALUE OF A FUNCTION IN ITS DOMAIN

MAXIMUM : Let f (x) be a function with domain $D \subset R$. Then f (x) is said to attain the maximum value at a point $a \in D$. If $f(x) \le f(a)$ for all $x \in D$.

In such a case, a is called the point of maximum and f (a) is known as the maximum value or the greatest value or the absolute maximum value of f (x).

Consider the function

$f(x) = -(x-1)^2 + 10$ for all $x \in R$

$\because -(x-1)^2 \le 0$ for all $x \in R$

$\therefore\ -(x-1)^2 + 10 \le 10$ for all $x \in R$

$\Rightarrow\ f(x) \le 10$ for all $x \in R$

Thus, 10 is the maximum value of f (x). Clearly f (x) attain this value at x = 1. So x = 1 is the point of maximum or the point of absolute maximum.

MINIMUM: Let f (x) be a function with domain $D \subset R$. Then f (x) is said to attain minimum value at a point $a \in D$ if $f(x) \ge f(a)$ for all $x \in D$.

In such a case the point a is called the point of minimum and f (a) is known as the minimum value or the least value or the absolute minimum value of f (x).

Consider the function

$f(x) = (3x-1)^2 + 5$ for all $x \in R$

$\because\ (3x-1)^2 \ge 0$ for all $x \in R$

$(3x-1)^2 + 5 \ge 5$ for all $x \in R$

Thus, 5 is the minimum value or the least value or the absolute minimum value of f (x) in its domain. Clearly, f (x) attains this value at $x = \dfrac{1}{3}$. So, $x = \dfrac{1}{3}$ is the point of minimum or the point of absolute minimum.

Example 37 :

Find the maximum and the minimum values of the function f (x) = sin (sin x).

Sol. We have f (x) = sinx, $x \in R$ $\because -1 \le \sin x \le 1$ for all $x \in R$

$\therefore \sin(-1) \le \sin(\sin x) \le \sin 1$ for all $x \in R$

[$\because$ sinx is an increasing function on [–1, 1]]

$\Rightarrow -\sin 1 \le f(x) \le \sin 1$ for all $x \in R$

This shows that the maximum value of f (x) is sin 1 and the minimum value is – sin 1.

Example 38 :

Find the maximum and the minimum values of the function $f(x) = 3x^2 + 6x + 8$, $x \in R$

Sol. We have, $f(x) = 3x^2 + 6x + 8$

$= 3(x^2 + 2x + 1) + 5 = 3(x + 1)^2 + 5.$

$\because\ 3(x+1)^2 \ge 0$ for all $x \in R$

$\therefore\ 3(x+1)^2 + 5 \ge 5$ for all $x \in R$

or $f(x) \ge 5$ for all $x \in R$. Thus, 5 is the minimum value of f (x) which it attains at x = – 1.

Since f (x) can be made as large as we please, therefore the maximum value does not exist.

Example 39 :

Find the maximum and the minimum values of the function $f(x) = -|x-1| + 5$ for all $x \in R$

Sol. We have $f(x) = -|x-1| + 5$ for all $x \in R$

$\because\ |x-1| \ge 0$ for all $x \in R$

$\therefore\ -|x-1| \le 0$ for all $x \in R$

$\Rightarrow\ -|x-1| + 5 \le 5$ for all $x \in R \Rightarrow f(x) \le 5$ for all $x \in R$

So, 5 is the maximum value of f (x).

Now, $f(x) = 5 \Rightarrow -|x-1| + 5 = 5 \Rightarrow |x-1| = 0 \Rightarrow x = 1.$

Thus, f (x) attains the maximum value 5 at x = 1.

Since f (x) can be made as small as we please, therefore the minimum value of f (x) does not exist.

Example 40 :

Find the maximum and the minimum values of the function $f(x) = x^3 + 1, x \in R$

Sol. We have $f(x) = x^3 + 1, x \in R$

Here we observe that the values of f (x) increase when the values of x are increased and f (x) can be made as large as we please by giving large values to x. So f (x) does not have the maximum value. Similarly, f (x) can be made as small as we please by giving smaller values to x. So f (x) does not have the minimum value also.

LOCAL MAXIMA & MINIMA

We have talked about the greatest (maximum) and the least (minimum) values of a function in its domain. But there may be points in the domain of a function where the function does not attain the greatest (or the least) value but the values at these points are greatest than or less than the values of the function at the neighbouring points. Such points are known as the points of local minimum or local maximum and we will be mainly discussing about the local maximum and local minimum values of a function.

Local maximum : A function f (x) is said to attain a local maximum at x = a if their exists a neighbourhood $(a - \delta, a + \delta)$ of a such that

$f(x) < f(a)$ for all $x \in (a - \delta, a + \delta), x \neq a$

or $f(x) - f(a) < 0$ for all $x \in (a - \delta, a + \delta), x \neq a$

In such a case f (a) is called the local maximum value of f (a) at x = a.

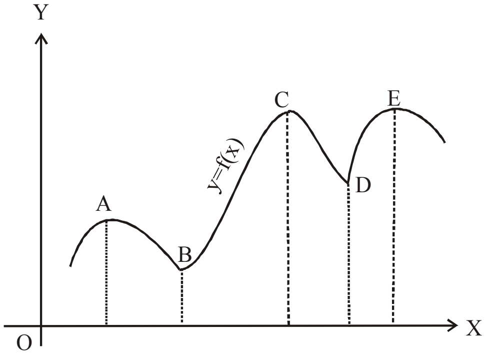
Local minimum : A function f (x) is said to attain a local minimum at x = a if there exists a neighbourhood $(a - \delta, a + \delta)$ of a such that

$f(x) > f(a)$ for all $x \in (a - \delta, a + \delta), x \neq a$

or $f(x) - f(a) > 0$ for all $x \in (a - \delta, a + \delta), x \neq a$

The value of the function at x = a i.e., f (a) is called the local minimum value of f (x) at x = a.

The points at which a function attains either the local maximum values or local minimum values are known as the extreme point or turning points and both local maximum and local minimum values are called the extreme values of f (x). Thus, a function attains an extreme value at x=a if f (a) is either a local maximum value or a local minimum value. Consequently at an extreme point 'a', f (x) – f (a) keeps ths same sign for all values of x in a neighbourhood of a.



In Figure we observe that the x-coordinates of the points A, C, E are points of local maximum and the values at these points i.e., their y-coordinates are the local maximum values of f (x). The x-coordinates of points B and D are points of local minimum and their y-coordinates are the local minimum values of f (x).

Note:

(i) The maximum and minimum points are also known as extreme points.

(ii) A function may have more than one maximum and minimum points.

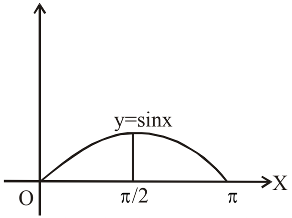
(iii) A maximum value of a function f (x) in an interval [a,b] is not necessarily its greatest value in that interval. Similarly, a minimum value may not be the least value of the function. A minimum value may be greater than some maximum value for a function.

(iv) If a continuous function has only one maximum (minimum) point, then at this point function has its greatest (least) value.

(v) Monotonic functions do not have extreme points.

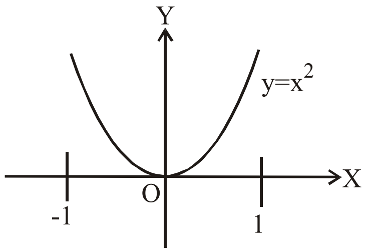
For example,

(1) Function y= sinx, $x \in (0, \pi)$ has a maximum points at $x = \pi/2$ because the value of $\sin = \pi/2$ is greatest in the given interval for sin x.



Clearly function y = sinx is increasing in the interval $(0, \pi/2)$ and decreasing in the interval $(\pi/2, \pi)$ for that reason also it has maxima at $x = \pi/2$. Similarly we can see from the graph of cosx which has a minimum point at $x = \pi$.

(2) $f(x) = x^2, x \in (-1, 1)$ has a minimum point at x = 0 because at x = 0, the value of x^2 is 0, which is less than all the values of foundation at different points of the interval.



Clearly function $y = x^2$ is decreasing in the interval (–1,0) and increasing in the interval (0,1) So it has minima at x= 0.

CONDITIONS FOR MAXIMA & MINIMA

Necessary Condition : We have the following theorem which we state without proof.

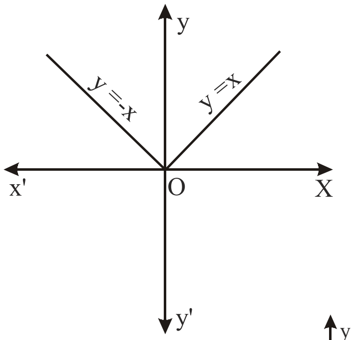
Theorem : A necessary condition for f (a) to be an extreme value of a function f (x) is that $f'(a) = 0$, in case it exists.

Note :

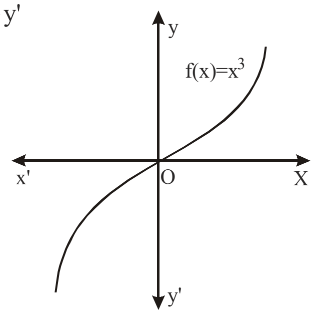
1. This result states that if the derivative exists, it must be zero at the extreme points.

A function may however attain an extreme value at a point without being derivable thereat.

For example, the function f (x) = | x | attains the minimum value at the origin even though it is not derivable at x = 0.



2. This condition is only a necessary condition for the point x = a to be an extreme point.
It is not sufficient i.e., f' (a) = 0 does not necessarily imply that x = a is an extreme point. There are function for which the derivatives vanish at a point but do not have an extreme value thereat . For example, for the function $f(x) = x^3$, f ' (0) = 0 but at x = 0 the function does not attain an extreme value.



3. As discussed in note 2 that all x, for which f ' (x) = 0, do not give us the extreme values. The values of x for which f ' (x) = 0 are called stationary values or critical values of x and the corresponding values of f (x) are called stationary or turning values of f (x).
As we have seen in Remark 2 that f ' (a) = 0 is not the sufficient condition for x = a to be an extreme point. The following theorem provided the sufficient conditions for x = a to be an extreme point. This is known as the first derivative test and is stated without any proof.

Sufficient Condition :
First derivative test for local maxima and minima
Let f (x) be a function differentiable at x = a. Then

(a) x = a is a point of local maximum of f (x) if
(i) f '(a) = 0 and
(ii) f ' (x) changes sign from positive to negative as x passes f ' (x) > 0 at every point in the left neighbourhood (nbd) (a –δ,a) of a and f ' (x) < 0 at every point in the right nbd (a, a + δ) of a.

(b) x = a is a point of local minimum of f (x) if
(i) f ' (a) = 0 and
(ii) f ' (x) changes sign from negative to positive as x passes through a i.e., f ' (x) < 0 at every point in the left nbd (a –δ, a) of a and f ' (x) > 0 at every point in the right nbd (a, a + δ) of a.

(c) If f ' (a) = 0, but f ' (x) does not change sign, that is, f ' (a) has the same sign in the complete nbd of a, then a is neither a point of local maximum nor a point of local minimum.

ALGORITHM FOR DETERMINING EXTREME VALUES OF A FUNCTION BY USING FIRST DERIVATIVE TEST

Step I : Put y = f (x)

Step II : Find $\frac{dy}{dx}$

Step III : Put $\frac{dy}{dx} = 0$ and solve this equation for x. Let c_1, $c_2, c_3, \ldots c_n$ be the roots of this equation. $c_1, c_2, c_3, \ldots, c_n$ are stationary values of x and these are the possible points where the function can attain a local maximum or a local minimum. So we test the function at each one of these points.

Step IV : Consider x = c_1. If $\frac{dy}{dx}$ changes its sign from positive to negative as x increases through c_1, then the function attains a local maximum at x = c_1.

If $\frac{dy}{dx}$ changes its sign from negative to positive as x increases through c_1, then the function attains a local minimum at x = c_1.

If $\frac{dy}{dx}$ does not change sign as x increases through c_1, then x = c_1 is neither a point of local maximum nor a point of local minimum. In this case x = c_1 is a point of inflexion. Similarly we may deal with other values of x.

Example 41 :
Find the local maxima or local minima, if any, of the function
$f(x) = \sin^4 x + \cos^4 x, 0 < x < \pi/2$
using the first derivative test

Sol. We have $y = f(x) = \sin^4 x + \cos^4 x$

$$\therefore \frac{dy}{dx} = 4 \sin^3 x \cos x - 4 \cos^3 x \sin x$$
$$= -4 \cos x \sin x (\cos^2 x - \sin^2 x)$$
$$= -2 \sin 2x \cos 2x = -\sin 4x$$

For a local maximum or a local minimum, we have

$$\frac{dy}{dx} = 0 \Rightarrow -\sin 4x = 0 \Rightarrow -\sin 4x = 0 \Rightarrow \sin 4x = 0$$

$$\Rightarrow 4x = \pi \qquad \left[\because 0 < x < \frac{\pi}{2} \therefore 0 < 4x < 2\pi\right]$$

$$\Rightarrow x = \pi/4$$

In the left nbd of x = π/4, we have

$$x < \frac{\pi}{4} \Rightarrow 4x < \pi \Rightarrow \sin 4x > 0 \Rightarrow -\sin 4x < 0 \Rightarrow \frac{dy}{dx} < 0$$

In the right nbd of x =π/4, we have

$$x > \frac{\pi}{4} \Rightarrow 4x > \pi \Rightarrow \sin 4x < 0 \Rightarrow -4 \sin 4x > 0 \Rightarrow \frac{dy}{dx} > 0$$

Thus, $\frac{dy}{dx}$ changes sign from negative to positive as x increases through $\frac{\pi}{4}$. So $x = \frac{\pi}{4}$ is a point of local minimum.

The local maximum value is

$$f\left(\frac{\pi}{4}\right) = \left(\sin\frac{\pi}{4}\right)^4 + \left(\cos\frac{\pi}{4}\right)^4 = \frac{1}{2}$$

Example 42 :

The $f'(x) = (x-a)^{2n}(x-b)^{2p+1}$ where n and p are positive integers then

(1) x = a is a point of minimum

(2) x = a is a point of maximum

(3) x = a is a point of maximum or minimum

(4)None of these

Sol. **(3).** We have, $f'(x) = (x-a)^{2n}(x-b)^{2p+1}$

$\therefore \quad f'(x) = 0 \Rightarrow x = a, b$

When $x = a - h$, $f'(x) = h^{2n}\ (a + h - b)^{2p+1}$

and when $x = a + h$, $f'(x) = h^{2n}(a + h - b)^{2p+1}$

Thus we see that as x passes through a, f ' (x) does not change sign. Hence, there is neither a maximum nor a minimum at x = a.

Example 43 :

If $f'(x) = (x-a)^{2n}(x-b)^{2m+1}$ where m, n ∈ N, then

(1) x = b is a point of minimum

(2) x = b is a point of maximum

(3) x = b is a point of inflexion

(4) None of these

Sol. **(1).** We have $f'(x) = (x-a)^{2n}(x-b)^{2m+1}$

$\therefore \quad f'(x) = 0 \Rightarrow x = a, b$

Now, for $x = b - h$, $f'(x) = (b - h - a)^{2n}(-h)^{2m+1} < 0$

and for $x = b + h$, $f'(x) = (b + h - a)^{2n} h^{2m+1} > 0$

Thus, as x passes through b, f ' (x) changes sign from negative Hence, x = b is a point of minimum.

HIGHER ORDER DERIVATIVE TEST

(i) The value of the function f (x) at x = a is maximum, if $f'(a) = 0$ and $f''(a) < 0$

(ii) The value of the function f (x) at x = a is minimum if $f'(a) = 0$ and $f''(a) > 0$

(iii) If $f'(a) = 0$, $f''(a) = 0$, $f'''(a) \neq 0$ then x = a is not an extreme point for the function f (x).

(iv) If $f'(a) = 0$, $f''(a) = 0$, $f'''(a) = 0$ then the sign of $f^{(iv)}(a)$ will determine the maximum and minimum value of function i.e. f (x) is maximum, if $f^{(iv)}(a)<0$ and minimum if $f^{(iv)}(a)>0$.

(v) $f'(c) = f''(c) = f'''(c) = ... = f^{n-1}(c) = 0$, and

(vi) $f^n(c)$ exists and is non-zero.

Then, If n is even and $f^n(c) < 0 \Rightarrow x = c$ is a point of local maximum.

If n is even and $f^n(c) > 0 \Rightarrow x = c$ is a point of local minimum

If n is odd ⇒ x = c is neither a point of local maximum nor a point of local minimum.

ALGORITHM FOR DETERMINING EXTREME VALUES OF A FUNCTION

From the above test criteria we obtain the following rule for determining maxima and minima of f (x).

Step I : Find f ' (x)

Step II : Put f ' (x) = 0 and solve this equation for x. Let c_1, c_2,.... c_n be the roots of this equation. c_1, c_2,.... c_n are stationary values of x and these are the possible points where the function can attain a local maximum or a local minimum. So we test function at each one of these point.

Step III : Find f '' (x). Consider $x = c_1$.

If $f''(c_1) < 0$, then $x = c_1$ is a point of local maximum.

If $f''(c_1) > 0$, then $x = c_1$ is a point of local minimum.

If $f''(c_1)=0$, we must find $f'''(x)$ & substitute in it c_1, for x.

If $f'''(c_1) \neq 0$, then $x = c_1$ is neither a point of local maximum nor a point of local minimum and is called the point of inflection.

If $f'''(c_1)=0$, we must find $f^{IV}(x)$ & substitute in it c_1 for x.

If $f^{IV}(c_1) < 0$, then $x = c_1$ is a point of local maximum and if $f^{IV}(c_1) > 0$, then $x = c_1$ is a point of local minimum.

If $f^{IV}(c_1) = 0$, we must find $f^V(x)$, and, so on. Similarly the values of c_2, c_3 ...., may be tested.

Example 44 :

Find the points of maxima and minima for the function $f(x)= x^3 - 9x^2 + 15x - 11$.

Sol. Let $f(x)= x^3 - 9x^2 + 15x - 11$.

then $f'(x) = 3x^2 - 18x + 15 = 3x(x^2 - 6x + 5)$

For maxima and minima $f'(x) = 0 \Rightarrow x^2 - 6x + 15 = 0$

$\Rightarrow \quad (x-1)(x-5) = 0 \Rightarrow x = 1, 5$

Again $f''(1) = -12 < 0 \Rightarrow x = 1$ is a point of maxima

and $f''(5) = 12 > 0 \Rightarrow x = 5$ is a point of minima

Example 45 :

For the curve $y = xe^x$, the point

(1) x = –1 is a point of minimum

(2) x = 0 is a point of minimum

(3) x = – 1 is a point of maximum

(4) x = 0 is a point of maximum

Sol. **(1).** We have, $y = xe^x \Rightarrow dy/dx = e^x + xe^x$

For max. or min. $dy/dx = 0 \Rightarrow e^x(1 + x) = 0 \Rightarrow x = -1$

Now, $\frac{d^2y}{dx^2} = 2e^x + xe^x \Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=-1} = e^{-1}(2-1) > 0$

Hence, x = –1 is a point of local minimum.

Example 46 :

If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$ attains its maximum and minimum at p and q respectively such that $p^2 = q$, then find the value of a.

Sol. Since $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$ attains max. and min. at p and q, respectively. Therefore

$f'(p) = 0$, $f'(q) = 0$, $f''(p) < 0$ and $f''(q) > 0$

Now, $f'(p) = 0$, $f'(q) = 0$

$\Rightarrow \quad 6p^2 - 18ap + 12a^2 = 0$ and $6q^2 - 18aq + 12a^2 = 0$

$\Rightarrow \quad p^2 - 3ap + 2a^2 = 0$ and $q^2 - 3aq + 2a^2 = 0$

$\Rightarrow \quad p = a, 2a, q = a, 2a$ ..(i)

Now, $f''(p) < 0 \Rightarrow 12p - 18a < 0 \Rightarrow p < \frac{3}{2}a$ ...(ii)

and $f''(q) > 0 \Rightarrow 12p - 18a > 0 \Rightarrow q > \frac{3}{2}a$ ...(iii)

From (i), (ii), (iii), we get $p = a, q = 2a$.
Now $p^2 = q \Rightarrow a^2 = 2a \Rightarrow a = 0, 2$
But for $a = 0$, $f(x) = 2x^3 + 1$ which does not attain a max. or min. for any value of x. Hence, $a = 2$

Example 47 :

If $A > 0, B > 0$ and $A + B = \pi/3$, then find the maximum value of tan A tan B.

Sol. We have, $A + B = \pi/3$.

$$\therefore \quad B = \frac{\pi}{3} - A \Rightarrow \tan B = \frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}$$

Let Z = tanA + tan B. Then

$$Z = \tan A + \frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A} = \frac{\sqrt{3}\tan A - \tan^2 A}{1 + \sqrt{3}\tan A}$$

$$\Rightarrow Z = \frac{\sqrt{3}x - x^2}{1 + \sqrt{3}x}, \text{ where } x = \tan A \Rightarrow \frac{dZ}{dx} = \frac{(x + \sqrt{3})(\sqrt{3}x - 1)}{(1 + \sqrt{3}x)}$$

For max Z, $\frac{dZ}{dx} = 0 \Rightarrow x = \frac{1}{\sqrt{3}}, -\sqrt{3}$

$x \neq -\sqrt{3}$ because $A + B = \frac{\pi}{3}$ which implies that $x = \tan A > 0$

it can be easily checked that $\frac{d^2Z}{dx^2} < 0$ for $x = \frac{1}{\sqrt{3}}$

Hence, Z is max. for $x = \frac{1}{\sqrt{3}}$ i.e. $\tan A = \frac{1}{\sqrt{3}}$ or $A = \frac{\pi}{6}$.

For this value of x, $Z = 1/3$

GREATEST & LEAST VALUES OF A FUNCTION IN A GIVEN INTERVAL

If a function f (x) is defined in an interval [a,b], then greatest or least values of this function occurs either at x = a or x = b or at those values of x where f ' (x) = 0.
Remember that a maximum value of the function f (x) in any interval [a,b] is not necessarily its greatest value in that interval.
Thus greatest value of f (x) in interval [a,b]
= Max.[f (a),f (b),f (c)]
Least value of f (x) in interval [a,b]
= Min. [f (a),f (b),f (c)]
Where x = c is a point such that f ' (c) = 0

Example 48 :

Find the greatest value of $x^3 - 12x^2 + 45x$ in the interval [0, 7].

Sol. Let $f(x) = x^3 - 12x^2 + 45x$, then

$f'(x) = 3x^2 - 24x + 45$
$3(x-3) \quad (x-5)$

and $f''(x) = 6x - 24$
Now for maximum and minimum values
$f'(x) = 0 \Rightarrow 3(x-3)(x-5) = 0 \Rightarrow x = 3, 5$
Again $f''(3) = -6 < 0 \Rightarrow$ The function if maximum at x = 3
and $f''(5) = 6 > 0 \Rightarrow$ The function is minimum at x = 5
Now $f(0) = 0, f(3) = 54, f(5) = 50, f(7) = 70$
$\Rightarrow$ The greatest value in [0, 7]
= max. {0, 54 ,50 , 70 } = 70

Example 49 :

If $f(x) = \cos x + \frac{1}{2}\cos 2x - \frac{1}{3}\cos 3x$ then find the difference between the greatest and least values of the function.

Sol. It should be noted that it is not a question of max. or min. but question of greatest and least value of f (x) in a certain interval. The function is periodic with period 2π. Hence the required difference in the difference between greatest and least values in the interval [0, 2π]

$$\frac{dy}{dx} = -(\sin x + \sin 2x - \sin 3x)$$

$$= -\left(2\sin\frac{3x}{2}\cos\frac{x}{2} - 2\sin\frac{3x}{2}\cos\frac{3x}{2}\right)$$

$$= -\ 2\sin\frac{3x}{2}\left(\cos\frac{x}{2} - \cos\frac{3x}{2}\right) = -\ 2\sin\frac{3x}{2}\ 2\sin x.\sin\frac{x}{2}$$

$$\frac{dy}{dx} = 0 \text{ at } x = 0, \frac{2\pi}{3}, \pi, 2\pi \text{ in } [0, 2\pi]$$

Corresponding values of y at the above points are

$$y = \frac{7}{6}, \frac{-13}{12}, \frac{-1}{6}, \frac{7}{6}$$

Hence greatest value is $\frac{7}{6}$ and least is $\frac{-13}{12}$ then

the difference is $\frac{7}{6} - \left(\frac{-13}{12}\right) = \frac{27}{12} = \frac{9}{4}$

Example 50 :

Find the maximum value of $x^3 - 3x$ in the interval [0, 2].

Sol. Let $f(x) = x^3 - 3x$. Then $f'(x) = 3x^2 - 3$. For maximum or minimum $f'(x) = 0 \Rightarrow x = \pm 1$.
But $x = -1 \notin [0, 2]$.
Therefore x = 1 only. It can be easily checked that $f''(x) > 0$ for x = 1. So. f (x) attains a local minimum at x = 1
Now, $f(0) = 0, f(1) = -2$ and $f(2) = 6$
Hence f (x) attains the maximum value at x = 2

PROPERTIES OF MAXIMA & MINIMA

(i) If f (x) is continuous function in its domain, then at least one maxima and one minima must lie between two equal values of x.

(ii) Maxima and minima occur alternately .that is between two maxima there is one minimum and vice-versa.

(iii) If f (x) is a maximum (minimum) at a point x = a, then 1/ f (x), $[f(x) \neq 0]$ will be minimum (maximum) at that point.

(iv) If $f(x) \to \infty$ as $x \to a$ or b and $f'(x) = 0$ only for one value of x (say c) between a and b, then f (c) is necessarily the minimum and the least value.

(v) If $f(x) \to \infty$ as $x \to a$ or b, then f(c) is necessarily the maximum and the greatest values of the function

MAXIMA & MINIMA OF FUNCTION OF TWO VARIABLE:

If a function is defined in terms of two variables and if these are associated with a given relation then by eliminating one variable, we convert function in terms of one variable and then find the maxima and minima by known methods.

Example 51 :

If x + y = 8 then find the maximum value of xy.

Sol. Let Z = xy $\therefore$ $Z = x(8 - x)$ or $Z = 8x - x^2$

$dz/dx = 8 - 2x = 0 \Rightarrow x = 4$

$d^2Z/dx^2 = -2 < 0$

$\Rightarrow$ x = 4 is a maximum point. So maximum value is $Z = 8.4 - 4^2 = 16$.

Example 52 :

Divide 64 into two parts such that the sum of the cubes of two parts is minimum. The two parts are

(1) 44,20 (2) 16, 48
(3) 32, 32 (4) 50, 14

Sol. **(3).** $x + y = 64$

$y = x^3 + y^3 = x^3 + (64 - x)^3$

$$\frac{dy}{dx} = 3\,[x^2 - (64 - x)^2] = 0 = 3\,[(2x - 64)\,64] = 0 \Rightarrow x = 32$$

$$\frac{d^2y}{dx^2} = +\text{ive, Hence min.}$$

Example 53 :

Find the minimum value of px + qy when $xy = r^2$.

Sol. Let Z = px + qy. Then

$$Z = px + \frac{qr^2}{x} \Rightarrow \frac{dZ}{dx} = p - \frac{qr^2}{x^2} \quad [\because xy = r^2]$$

For max or min $\frac{dZ}{dx} = 0 \Rightarrow x = \pm\sqrt{\frac{qr^2}{p}}$

For $x = \sqrt{\frac{qr^2}{p}}$, we have $\frac{d^2z}{dx^2} = \frac{qr^2}{x^3} > 0$

Hence, z is min. for $x = \sqrt{\frac{qr^2}{p}} + \frac{qr^2}{\sqrt{\frac{qr^2}{p}}} = 2r\sqrt{pq}$

SOME STANDARD GEOMETRICAL RESULTS RELATED TO MAXIMA & MINIMA

The following results can easily be established.

(i) The area of rectangle with given perimeter is greatest when it is a square.

(ii) The perimeter of a rectangle with given area is least when it is a square.

(iii) The greatest rectangle inscribed in a given circle is a square.

(iv) The greatest triangle inscribed in a given circle is equilateral.

(v) The semi vertical angle of a cone with given slant height and maximum volume is $\tan^{-1}\sqrt{2}$.

(vi) The height of a cylinder of maximum volume inscribed in a sphere of radius a is $2a/\sqrt{3}$.

Example 54 :

Find the maximum area of a rectangle of perimeter 176cms.

Sol. Let sides of the rectangle be x, y ; then

$2x + 2y = 176$

$\therefore$ Its area $A = xy = x(88 - x) = 88x - x^2$

$$\Rightarrow \frac{dA}{dx} = 88 - 2x, \ \frac{d^2A}{dx^2} = -2 < 0$$

Now, $\frac{dA}{dx} = 0 \Rightarrow x = 44$; Also then $\frac{d^2A}{dx^2} < 0$.

So area will be maximum when x = 44 and maximum area $= 44 \times 44 = 1936$ sq. cms.

Example 55 :

If the sum of length of the hypotenuse and another side of a right angled triangle is given, show that the area of the triangle is max. when the angle between these is

(1) π/12 (2) π/4
(3) π/3 (4) π/2

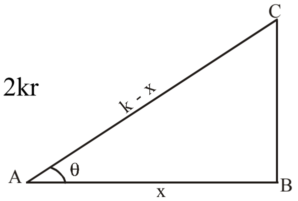
Sol. **(3).** AB + AC = constant = k

If AB = x then AC = k – x

$\therefore$ $BC^2 = (k - x)^2 - x^2 = k^2 - 2kr$

$\therefore$ $\Delta = \frac{1}{2}$ BC. AB

$= \frac{1}{2}\,x\sqrt{(k^2 - 2kx)}$



Let $Z = \Delta^2 = \frac{1}{4}x^2(k^2 - 2kx) = \frac{1}{4}(k^2x^2 - 2kx^3)$

Z will be max. when x = k/3

$$\therefore \cos\theta = \frac{x}{k - x} = \frac{k/3}{k - k/3} = \frac{1}{2} \quad \therefore \quad \theta = \pi/3$$

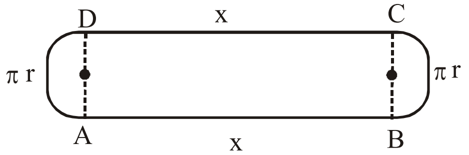
Example 56 :

A running track or 440 it is to be laid out enclosing a football field, the shape of which is a rectangle with a semi -circle at each end. If the area of the rectangular portion is to be maximum then find the length of its sides.

Sol. Perimeter = 440 ft.

$2x + \pi r + \pi r = 440$

or $2x + 2\pi r = 440$ .... (1)



A = Area of rectangular portion = x. 2r

$$A = x\,\frac{(400-2x)}{\pi} = \frac{1}{\pi}(440x - 2x^2)$$

$$\frac{dA}{dx} = \frac{1}{\pi}(440 - 4x) \therefore x = 110$$

$$\frac{d^2A}{dx^2} = -\text{ive} \quad \therefore \text{A is max. when } x = 110.$$

$$\therefore \quad 2r = \frac{440-2x}{\pi} = \frac{440-220}{22/7} = 70$$

RATE OF CHANGE OF VARIABLE

Derivative as a rate measurer : Let $y = f(x)$ be a function of x. Let Δy be the change in y corresponding to a small change Δx in x. Then , $\frac{\Delta y}{\Delta x}$ represents the change in y due to a unit change in x. In other words, $\frac{\Delta y}{\Delta x}$ represents the average rate of change of y w. r. t. x as x changes from x to $x + \Delta x$.

As $\Delta x \to 0$, the limiting value of this average rate of change of y with respect to x in the interval $[x, x + \Delta x]$ becomes the instantaneous rate of change of y w.r.t. x.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \text{Instantaneous rate of change of y w.r.t. x}$$

$$\Rightarrow \frac{dy}{dx} = \text{Rate of change of y. w.r.t. x} \left[\because \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \frac{dy}{dx}\right]$$

The word “instantaneous” is often dropped.

Hence, $\frac{dy}{dx}$ represents the rate of change of y w.r.t. x for a definite value of x.

Note :

1. The value of $\frac{dy}{dx}$ at $x = x_0$ i.e. $\left(\frac{dy}{dx}\right)_{x=x_0}$ represents the rate of change of y with respect to x at $x = x_0$

2. If $x = \phi(t)$ and $y = \psi(t)$, then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, provided that $\frac{dx}{dt} \neq 0$

   Thus, the rate of change of y with respect to x can be calculated by using the rate of change of y and that of x each with respect to t.

3. The term “rate of change” will mean the instantaneous rate of change unless stated otherwise.

Example 57 :

Find the rate of change of the area of a circle with respect to its radius. How fast is the area changing with respect to the radius when the radius is 3cm ?

Sol. Let A be the area of the circle. Then $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$

Thus, the rate of change of the area of the circle w.r.t. its radius r is $2\pi r$.

when r = 3cm, we have $\frac{dA}{dr} = (2\pi \times 3)\ \text{cm}^2/\text{cm} = 6\pi\text{cm}^2/\text{cm}$

Example 58 :

A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x + 3)$. Determine the rate of change of volume with respect to x.

Sol. Let V be the volume of the balloon. Then

$$V = \frac{4\pi}{3}\left\{\frac{3}{4}(2x+3)\right\}^3 = \frac{9\pi}{16}.(2x+3)^3$$

$$\Rightarrow \frac{dV}{dx} = \frac{9\pi}{16}.3(2x+3)^2\,\frac{d}{dx}(2x+3) \Rightarrow \frac{dV}{dx} = \frac{27\pi}{8}(2x+3)^2.$$

Related rates : Generally we come across with the problems in which the rate of change of one of the quantities involved is required corresponding to the given rate of change of another quantity. For example, suppose the rate of change of volume of a spherical balloon is required when the rate of change of its radius is given. In such type of problems we must find a relation connecting such quantities and differentiate this relation w.r.t. time. The procedure is illustrated in the following examples.

Example 59:

The radius of a balloon is increasing at the rate of 10 cm/sec. At what rate is the surface area of the balloon increasing when the radius is 15 cm ?

Sol. Let r be the radius and S be the surface area of the balloon at any time t.

Then, $S = 4\pi r^2$ and $\frac{dr}{dt} = 10$ cm/sec

Now, $S = 4\pi r^2$

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}\ ; \ \frac{dS}{dt} = 80\pi r \left[\because \frac{dr}{dt} = 10cm/sec.\right]$$

$$\Rightarrow \left(\frac{dS}{dt}\right)_{r=15} = 80\pi(15) = 120\pi\, cm^2/sec$$

Example 60 :

The volume of a cube is increasing at a rate of 7 cm^3/sec. How fast is the surface area increasing when the length of an edge is 12 cm ?

Sol. Let x be the length of an edge of the cube, V be the volume and S be the surface area of any time t. Then

$V = x^3$ and $S = 6x^2$

Also, $\frac{dV}{dt} = 7\ cm^3/sec$ [Given]

$$\Rightarrow \frac{d}{dt}(x^3) = 7 \Rightarrow 3x^2\frac{dx}{dt} = 7 \Rightarrow \frac{dx}{dt} = \frac{7}{3x^2}$$

Now, $S = 6x^2 \Rightarrow \frac{dS}{dt} = 12x\frac{dx}{dt}$

$$\Rightarrow \frac{dS}{dt} = 12x \times \frac{7}{3x^2} \qquad \left[\because \frac{dx}{dt} = \frac{7}{3x^2}\right]$$

$$\Rightarrow \frac{dS}{dt} = \frac{28}{x} \Rightarrow \left(\frac{dS}{dt}\right)_{x=12} = \frac{28}{12}\ cm^2/sec. = \frac{7}{3}\ cm^2/sec.$$

Example 61 :

The length x of a rectangle is decreasing at the rate of 2 cm/sec and the width y is increasing at the rate of 2 cm/sec. When x = 12 cm and y = 5 cm, find the rate of change of (i) the perimeter and (ii) the area of the rectangle.

Sol. Let P be the perimeter and A be the area of the rectangle at any time t. Then, $P = 2(x + y)$ and $A = xy$.

It is given that $\frac{dx}{dt} = -2$ cm/sec and $\frac{dy}{dt} = 2$cm/sec.

(i) We have, $P = 2(x + y)$

$$\Rightarrow \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-2+2) = 0\ cm/sec$$

i.e. the perimeter remains constant.

(ii) We have, $A = xy$

$$\Rightarrow \frac{dA}{dt} = \left(\frac{dx}{dt}\right)y + x\left(\frac{dy}{dt}\right)$$

$$\Rightarrow \frac{dA}{dt} = -2\times 5 + 12\times 2\ [\because x = 12\ cm\ and\ y = 5\ cm\ (given)]$$

$$\Rightarrow \frac{dA}{dt} = 14\ cm^2/sec$$

Example 62 :

On the curve $x^3 = 12y$, find the interval of values of x for which the abscissa changes at a faster rate than the ordinate?

Sol. Given $x^3 = 12y$, differentiating w.r.t. y

$$3x^2\frac{dx}{dy} = 12 \quad \therefore \quad \frac{dy}{dx} = \frac{x^2}{4}$$

Now abscissa changes at a faster rate than the ordinate,

then we must have $\left|\frac{dy}{dx}\right| < 1$.

$$\Rightarrow |x^2| < 4, x \neq 0 \Rightarrow -2 < x < 2, x \neq 0 \Rightarrow x \in (-2, 2) - \{0\}$$

IMPORTANT POINTS

* If a quantity y varies with another quantity x, satisfying some rule y= f(x), then $\frac{dy}{dx}$ or (or f '(x)) represents the rate of change of y with respect to x and $\left.\frac{dy}{dx}\right]_{x=x_0}$ (or $f'(x_0)$) represents the rate of change of y with respect to x at $x = x_0$.

* A function f is said to be (a) increasing on an interval (a, b) if $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in (a, b)$. Alternatively, if $f'(x) \geq 0$ for each x in (a, b).
(b) decreasing on (a,b) if $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in (a, b)$. Alternatively, if $f'(x) \leq 0$ for each x in (a, b)

* The equation of the tangent at (x_0, y_0) to the curve $y = f(x)$ is given by $y - y_0 = \left.\frac{dy}{dx}\right]_{(x_0,y_0)}(x - x_0)$

* If $\frac{dy}{dx}$ does not exist at the point (x_0, y_0), then the tangent at this point is parallel to the y-axis and its equation is $x = x_0$.

* Equation of the normal to the curve $y = f(x)$ at a point (x_0,y_0) is given by $y - y_0 = \frac{-1}{\left.\frac{dy}{dx}\right]_{(x_0,y_0)}}(x - x_0)$

* Let $y = f(x)$, Δx be a small increment in x and Δy be the increment in y corresponding to the increment in x, i.e., $\Delta y = f(x + \Delta x) - f(x)$. Then dy given by $dy = f'(x)dx$ or $dy = \left(\frac{dy}{dx}\right)\Delta x$ is a good approximation of Δy when $dx = \Delta x$ is relatively small and we denote it by $dy \approx \Delta y$.

* A point c in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable is called a critical point of f.

* **First derivative test :** Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then :

(a) If $f'(x)$ changes sign from positive to negative as x increases through c, i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of c, and $f'(x) < 0$ at every point sufficiently close to and to the right of c, then c is a point of local maxima.

(b) If $f'(x)$ changes sign from negative to positive as x increases through c, i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of c, and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of local minima.

(c) If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Infact, such a point is called point of inflexion.

* **Second derivative test :** Let f be a function defined on an interval I and $c \in I$. Let f be twice differentiable at c. Then

(a) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$. The values $f(c)$ is local maximum value of f.

(b) $x = c$ is a point of local minima if $f''(c) = 0$ and $f''(c) > 0$. In this case, $f(c)$ is local minimum value of f.

(iii) The test fails if $f''(c) = 0$ and $f''(c) = 0$. In this case, we go back to the first derivative test and find whether c is a point of maxima, minima or a point of inflexion.

* **Rolle's Theorem:** If $f : [a, b] \to R$ is continuous on [a, b] and differentiable on (a, b) such that $f(a) = f(b)$, then there exists some c in (a, b) such that $f'(c) = 0$.

* **Mean value theorem:** If $f : [a, b] \to R$ is continuous on [a, b] and differentiable on (a, b). Then there exists some c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

* **Increasing/Decreasing :**

1. If $f'(x) > 0$ for all x in an interval I then $f(x)$ is increasing on the interval I.
2. If $f'(x) < 0$ for all x in an interval I then $f(x)$ is decreasing on the interval I.
3. If $f'(x) = 0$ for all x in an interval I then $f(x)$ is constant on the interval I.

* **Concave Up/Concave Down :**

1. If $f''(x) > 0$ for all x in an interval I then $f(x)$ is concave up on the interval I.
2. If $f''(x) < 0$ for all x in an interval I then $f(x)$ is concave down on the interval I.

* Length of Sub–tangent = $\left|y\dfrac{dx}{dy}\right|$; Sub–normal = $\left|y\dfrac{dy}{dx}\right|$;

Length of tangent = $\left|y\sqrt{\left\{1+\left(\dfrac{dx}{dy}\right)^2\right\}}\right|$;

Length of normal = $\left|y\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}\right|$

* **Orthogonal trajectory :** Any curve which cuts every member of a given family of curves at right angle, is called an orthogonal trajectory of the family.

* **Some common Parametric coordinates on a curve:**

(a) For $x^{2/3} + y^{2/3} = a^{2/3}$ take parametric coordinate $x = a\cos^3\theta$ & $y = a\sin^3\theta$.

(b) For $\sqrt{x} + \sqrt{y} = \sqrt{a}$ take $x = a\cos^4\theta$ & $y = a\sin^4\theta$.

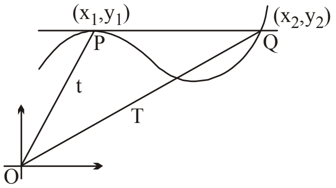
(c) $\dfrac{x^n}{a^n} + \dfrac{y^n}{b^n} = 1$ taken $x = a(\sin\theta)^{2/n}$ & $y = b(\sin\theta)^{2/n}$.

(d) For $c^2(x^2 + y^2) = x^2y^2$ take $x = c\sec\theta$ and $y = c\operatorname{cosec}\theta$.

(e) For $y^2 = x^3$, take $x = t^2$ and $y = t^3$.

* The tangent at P meeting the curve again at Q.

$$\Rightarrow \left.\frac{dy}{dx}\right|_P = \frac{y_2 - y_1}{x_2 - x_1}$$

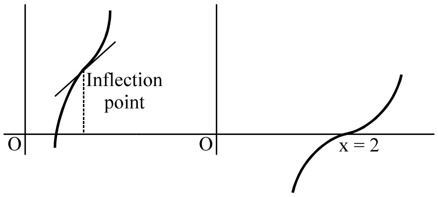


* **Different Graphs of the cubic:**

$y = ax^3 + bx^2 + cx + d$

1. One real & two imaginary roots (always monotonic) $\forall x \in R$

Condition : $f'(x) \ge 0$ or $f'(x) \le 0$ together with either $f'(x) = 0$ has no root (i.e. $D < 0$) or $f'(x) = 0$ has a root $x = \alpha$ then $f(\alpha) = 0$.



(i) either $f'(x) = 0$ has no real root

or (ii) if $f'(x) = 0$ has a root $x = \alpha$ then $f(\alpha) = 0$

e.g. $y = x^3 - 2x^2 + 5x + 4$

$y' = 3x^2 - 4x + 5 \quad (D < 0)$

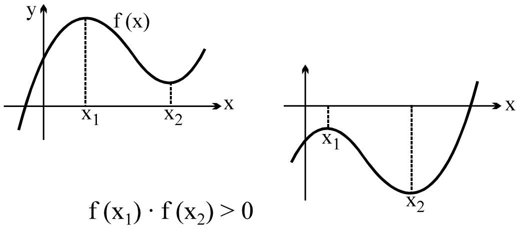
$y = (x - 2)^3$

$y' = 3(x - 2)^2 = 0 \Rightarrow x = 2$, also $f(2) = 0$

gives $x = 2$, $y(2) = 0$

Note: In this case if $f'(x) = 0$ has a root $x = \alpha$ and $f(\alpha) = 0$ this would mean $f(x) = 0$ has repeated roots which is dealt separately.

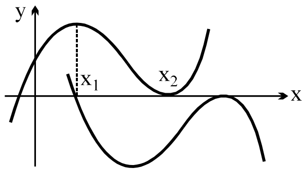
2. Exactly one root and non monotonic



$f(x_1) \cdot f(x_2) > 0$

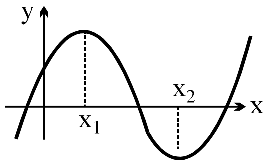
where x_1 & x_2 are the roots of $f'(x) = 0$

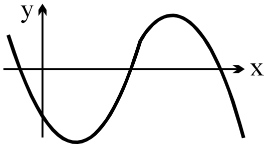
3. Three roots < two concident / One different



$f(x_1)\cdot f(x_2)=0$

4. **All three distinct real roots**

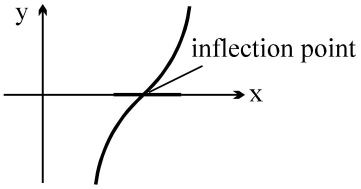




$f(x_1)\cdot f(x_2)<0$

where x_1 & x_2 are the roots of $f'(x)=0$

5. **All three roots concident**



$f'(x)\geq 0$ or $f'(x)\leq 0$ & $f(\alpha)=0$

where α is a root of $f'(x)=0$

e.g. $y=(x-1)^3$

Note :

(i) Graph of every cubic polynomial must have exactly one point of inflection.

(ii) In case (4) if f(a), f(b), f(c) and f(d) alternatively change sign.

ADDITIONAL EXAMPLES

Example 1 :

In the curve $y = c\,e^{x/a}$, then –

(A) subtangent is constant.

(B) subnormal varies as the square of the ordinate.

(C) tangent at (x_1, y_1) on the curve intersects the x-axis at a distance of $(x_1 - a)$ from the origin .

(D) equation of normal at the point where the curve cuts y-axis is $cy + ax = c^2$.

Sol. **(A,B,C,D).** We have, $y = c\,e^{x/a}$

$$\Rightarrow \frac{dy}{dx}=\frac{c}{a}e^{x/a} \Rightarrow \frac{dy}{dx}=\frac{1}{a}y \Rightarrow \frac{y}{dy/dx}=a=\text{const.}$$

$\Rightarrow$ subtangent = const.

Length of the subnormal

$$=y\frac{dy}{dx}=y.\frac{y}{a}=\frac{y^2}{a}\propto \text{(square of the ordinate)}$$

Equation of the tangent at (x_1 , y_1) is

$$y-y_1=\frac{-y_1}{a}(x-x_1)$$

This meets x-axis at a point given by

$$-y=\frac{y_1}{a}(x-x_1)\Rightarrow x=x_1-a$$

The curve meets y-axis at (0, c)

$$\therefore \left(\frac{dy}{dx}\right)_{(0,c)}=c/a$$

So, equation of the normal at (0, c) is

$$y-c=-\frac{1}{c/a}(x-0)\Rightarrow ax+cy=c^2$$

Example 2 :

If the line $\frac{x}{a}+\frac{y}{b}=2$ touches the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ at point (a, b), then find the value of n.

Sol. $$\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$$

$$\Rightarrow \frac{dy}{dx}=-\frac{\frac{n}{a}\left(\frac{x}{a}\right)^{n-1}}{\frac{n}{b}\left(\frac{y}{n}\right)^{n-1}}=-\left(\frac{b}{a}\right)^n\left(\frac{x}{y}\right)^{n-1}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(a,b)}=-\frac{b}{a}$$

So tangent to the curve at (a, b) is

$$y-b=-\frac{b}{a}(x-a)\Rightarrow \frac{x}{a}+\frac{y}{b}=2$$

Example 3 :

Find the angle between the tangent to the curve $y^2 = 2ax$ at the points where x = a/2.

Sol. We have, $y^2 = 2ax$ ..(i)

Put $x=\frac{a}{2}$; $y^2=2a\left(\frac{a}{2}\right)\Rightarrow y=\pm a$

$\therefore$ The points are (a/2, a) and (a/2, –a)

Differentiating (1) with respect to x, we get

$$2y\frac{dy}{dx}=2a\Rightarrow \frac{dy}{dx}=\frac{a}{y}$$

At $\left(\frac{a}{2},a\right)$; $\frac{dy}{dx}=\frac{a}{y}=\frac{a}{a}=1=m_1$ (say)

At $\left(\frac{a}{2},-a\right)$; $\frac{dy}{dx}=\frac{a}{y}=\frac{a}{-a}=-1=m_2$ (say)

Since $m_1m_2 = -1$, the two tangents are at right angles.

Example 4 :

Find the equation of the one of the tangents to the curve $y = \cos(x + y), -2\pi \le x \le 2\pi$ that is parallel to the line $x + 2y = 0$.

Sol. $\frac{dy}{dx} = -\sin(x + y).[1 + dy/dx]$ ...(1)

Since the tangent is parallel to $x + 2y = 0$

therefore $\frac{dy}{dx} = \text{slope} = -\frac{1}{2}$. Putting in (1)

$\sin(x + y) = 1 = \sin(\pi/2)$

$\therefore \cos(x + y) = 0 \quad \therefore y = \cos(x + y) = 0$

$\therefore \sin(x + y) = 1 \Rightarrow \sin x = 1 \qquad \because y = 0$

$\therefore \quad x = \frac{\pi}{2}, -\frac{3\pi}{2}$ as $-2\pi < x < 2\pi$

Hence the points are $[(-3\pi)/2, 0]$ and $[\pi/2, 0]$ where the tangents are parallel to the line $x + 2y = 0$,

The equation of tangents are :

$y - 0 = -\frac{1}{2}(x + 3\pi/2)$ and $y - 0 = -\frac{1}{2}(x - \pi/2)$

or $x + 2y + 3\pi/2 = 0$ and $x + 2y - \pi/2 = 0$

Example 5 :

Find the number of tangents to the curve $y^2 - 2x^3 - 4y + 8 = 0$ that pass through (1, 2).

Sol. Differentiating w.r.t x,

$$2y\frac{dy}{dx} - 6x^2 - 4\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{3x^2}{y-2}$$

$\therefore$ The equation of the tangent at (α, β) is

$$y - \beta = \frac{3\alpha^2}{\beta - 2}(x - \alpha)$$

It passes through (1, 2) if $2 - \beta = \frac{3\alpha^2}{\beta - 2}(1 - \alpha)$

or $(\beta - 2)^2 = 3\alpha^2(\alpha - 1)$

Also, (α, β) satisfies the equation of the curve

$\therefore \quad \beta - 2\alpha^3 - 4\beta + 8 = 0$ or $(\beta - 2)^2 = 2\alpha^3 - 4$

$\therefore \quad (\beta - 2)^2 = 3\alpha^2(\alpha - 1) = 2\alpha^3 - 4$

$\therefore \quad \alpha^3 - 3\alpha^2 + 4 = 0$

or $\quad (\alpha - 2)(\alpha^2 - \alpha - 2) = 0$ or $(\alpha - 2)^2(\alpha + 1) = 0$

When $\alpha = 2$, $(\beta - 2)^2 = 12$ or $\beta = 2 \pm 2\sqrt{3}$

When $\alpha = -1$, $(\beta - 2)^2 = -6$ or β = non real number

$\therefore \quad (\alpha, \beta)$ has two values

Example 6 :

Find the co-ordinates of the point P on the curve $y^2 = 2x^3$, the tangent at which is perpendicular to the line $4x - 3y + 2 = 0$ are given by [Slope of given line is 4/3]

Sol. $\frac{dy}{dx} = \frac{6x^2}{2y} \quad \therefore \frac{6x^2}{2y}.\frac{4}{3} = -1 \; (m_1/m_2 = -1)$

$\therefore y = 4x^2$ or $y^2 = 16x^4 = 2x^3 \quad \therefore y = 0$ or $1/8$

$\therefore y = 0, -\frac{1}{16}$ from $y = -4x^2$. Now $\frac{dy}{dx} = \frac{0}{0}$ at (0, 0) and hence rejected $\therefore$ Required point is $\left(\frac{1}{8}, -\frac{1}{16}\right)$

Example 7 :

Find the angle which the perpendicular from the origin on the tangent makes with the x-axis for the curve whose parametric equations are $x = a\sin^3\theta$, $y = a\cos^3\theta$.

Sol. $\frac{dy}{dx} = \frac{y}{x}$ = slope of the tangent $= -\cot\theta$.

Hence slope of a line through origin and perpendicular to the tangent is $= \tan\theta$ as $m_1m_2 = -1$

Therefore it makes an angle θ with x-axis.

Example 8 :

Find the point of inflexion for the curve $y = (x - a)^n$, where n is odd integer and $n \ge 3$.

Sol. Here $\frac{d^2y}{dx^2} = n(n-1)(x - a)^{n-2}$

Now $\frac{d^2y}{dx^2} = 0 \Rightarrow x = a$

Differentiating equation of the curve n times, we get

$\frac{d^ny}{dx^n} = n! \;\therefore$ at $x = a$, $\frac{d^ny}{dx^n} \ne 0$ and $\frac{d^{n-1}y}{dx^{n-1}} = 0$,

where n is odd. Therefore (a, 0) is the point of inflexion.

Example 9 :

Find the interval of increase of the function $f(x) = x - e^x + \tan(2\pi/7)$.

Sol. We have : $f(x) = x - e^x + \tan\left(\frac{2\pi}{7}\right) \Rightarrow f'(x) = 1 - e^x$

For f (x) to be increasing, we must have

$f'(x) > 0 \Rightarrow 1 - e^x > 0 \Rightarrow e^x < 1$

$\Rightarrow x < 0 \Rightarrow x \in (-\infty, 0)$

Example 10 :

If a function $f(x) = \cos|x| - 2ax + b$ is an increasing function on whole number line, then find the value of a.

Sol. $\because \frac{d}{dx}\cos|x| = -\sin x$, for $x \in R$

$\therefore f'(x) = -\sin x - 2a$

Now f (x) is an increasing function, therefore

$f'(x) > 0 \Rightarrow -\sin x - 2a > 0$

$\Rightarrow a < -\frac{1}{2}\sin x \quad \Rightarrow a \le -\frac{1}{2}$

Example 11 :

If $y = 2x + \text{arc}\cot x + \ell n\left[\sqrt{1+x^2} - x\right]$, then y

(A) increases in $[0, \infty[$

(B) decreases in $[0,\infty[$

(C) neither increases nor decreases in $[0, \infty[$

(D) increases in $]-\infty, 0]$

Sol. **(A, D).** We have

$$y = 2x + \cot^{-1} x + \log\left[\sqrt{1+x^2} - x\right]$$

$$\Rightarrow \frac{dy}{dx} = 2 - \frac{1}{1+x^2} + \frac{1}{\sqrt{1+x^2} - x} \times \left[\frac{x}{\sqrt{1+x^2}} - 1\right]$$

$$= \frac{2x^2+1}{1+x^2} - \frac{1}{\sqrt{1+x^2}} = \frac{(2x^2+1) - \sqrt{1+x^2}}{1+x^2}$$

Now, $\frac{dy}{dx} \geq 0 \Rightarrow (2x^2+1) - \sqrt{1+x^2} \geq 0$

$\Rightarrow \ (2x^2+1)^2 \geq 1 + x^2 \Rightarrow 4x^4 + 3x^2 \geq 0$

Which is true for all real values of x.

$\therefore$ y increases for all real values of x.

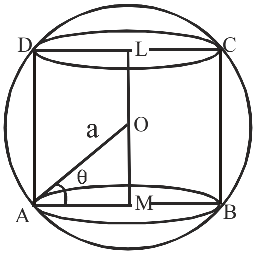
Example 12 :

Find the height of the cylinder of max. volume that can be inscribed in a sphere of radius a.

Sol. If a be the radius and h the height, then from the figure

$r + (h^2/4) = a^2$

$\therefore \ h^2 = 4\,(a^2 - r^2)$



Now $V = \pi r^2 h = \pi\left(a^2 - \frac{1}{4}h^2\right)$

$h = \pi\left(a^2 h - \frac{1}{4}h^3\right)$

$\therefore \ \frac{dV}{dh} = \pi\left(a^2 - \frac{3}{4}h^2\right) = 0$ for max. or min.

This gives $h = (2/\sqrt{3})\,a$

$d^2V/\,dh^2 = -\,6h/4 < 0$

Hence V is max. when $h = 2a/\sqrt{3}$

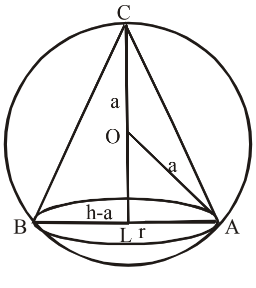
Example 13 :

Find the ratio of the altitude of the cone of greatest volume which can be inscribed in a given sphere to the diameter of the sphere.

Sol. Let h be the height of the cone and r be its radius.

$\therefore h = CL = CO + OL = a + OL$

$\therefore \ OL = h - a$



$r = LA = \sqrt{(OA^2 - OL^2)}$

or $\ r = \sqrt{\{a^2 - (h-a)^2\}}$

$= (2ah - h^2)$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\,(2ah - h^2)\,h = \frac{1}{3}\pi\,(2ah^2 - h^3)$$

$$\frac{dy}{dh} = (\pi/3)\,(4ah - 3h^2) = 0 \quad \therefore\ h = 0 \text{ or } 4a/3$$

h = 0 is rejected $\therefore h = 4a/3 = (2/3)\,(2a)$

h = 2/3 (diameter)

Example 14 :

For the function $f(x) = \int_0^x \frac{\sin t}{t}\,dt$, where $x > 0$,

(A) maximum occurs at $x = n\pi$, n even

(B) minimum occurs at $x = n\pi$, n odd

(C) maximum occurs at $x = n\pi$, n odd

(D) Minimum occurs at $x = n\pi$, n even

Sol. **(C, D).** We have

$$f'(x) = \frac{\sin x}{x} \text{ and } f''(x) = \frac{x\cos x - \sin x}{x^2}$$

For maximum or minimum, $f'(x) = 0$

$$\Rightarrow \frac{\sin x}{x} = 0 \Rightarrow \sin x = 0\ ;\ x \neq 0$$

$\therefore x = n\pi;\ n = 1, 2, 3, \ldots\ldots\ (\because\ x > 0)$

$$\text{At } x = n\pi,\ f''(x) = \frac{n\pi\cos n\pi - \sin n\pi}{(n\pi)^2} = \frac{\cos n\pi}{n\pi} = \frac{(-1)^n}{n\pi}$$

Extreme points are $x = n\pi$, $n = 1, 2, 3.....$ where the maximum occurs at $x = \pi, 3\pi, 5\pi,....$ and the minimum occurs at $x = 2\pi, 4\pi, 6\pi,....$

Example 15 :

If $f(x) = |x| + |x-1| + |x-2|$, then

(A) f (x) has minima at x = 1

(B) f (x) has maxima at x = 0

(C) f (x) has neither maxima nor minima at x = 0

(D) f (x) has neither maxima nor minima at x = 2

Sol. **(A, C, D).** we have, $f(x) = |x| + |x-1| + |x-2|$

$$= \begin{cases} -3x+3 & , \quad x < 0 \\ -x+3 & , \quad 0 \leq x < 1 \\ x+1 & , \quad 1 \leq x < 2 \\ 3x-3 & , \quad x \geq 2 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} -3 & , \quad x < 0 \\ \text{does not exist} & , \quad x = 0 \\ -1 & , \quad 0 < x < 1 \\ \text{does not exist} & , \quad x = 1 \\ 1 & , \quad 1 < x < 2 \\ \text{does not exist} & , \quad x = 2 \\ 3 & , \quad x > 2 \end{cases}$$

Clearly f (x) has minima at x = 1 and neither maxima nor minima at x = 0 and x = 2.

Example 16 :

Find the greatest value of $f(x) = (x+1)^{1/3} - (x-1)^{1/3}$ on $[0, 1]$

Sol. We have, $f(x) = (x+1)^{1/3} - (x-1)^{1/3}$

$$f'(x) = \frac{1}{3}\left[\frac{1}{(x+1)^{1/3}} - \frac{1}{(x-1)^{1/3}}\right] = \frac{(x-1)^{2/3} - (x+1)^{2/3}}{3(x^2-1)^{2/3}}$$

Clearly, $f'(x)$ does not exist at $x = \pm 1$

Now, $f'(x) = 0 \Rightarrow (x-1)^{2/3} = (x+1)^{2/3} \Rightarrow x = 0$

Clearly $f'(x) \neq 0$ for any other value of $x \in [0, 1]$.

The value of $f(x)$ at $x = 0$ is 2.

Hence, the greatest value of $f(x)$ is 2.

Example 17 :

Find the minimum value of $e^{(2x^2-2x-1)\sin^2 x}$.

Sol. Let $y = e^{(2x^2-2x-1)\sin^2 x}$ and $u = (2x^2 - 2x - 1)\sin^2 x$

$$\text{Now } \frac{du}{dx} = (2x^2 - 2x - 1)\, 2\sin x \cos x + (4x-2)\sin^2 x$$

$$= \sin x\,[2\,(2x^2 - 2x)\cos x + (4x-2)\sin x]$$

$$\frac{du}{dx} = 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi$$

$$\frac{d^2u}{dx^2} = \sin x\,\frac{d}{dx}\,[2(2x^2 - 2x - 1)\cos x + (4x-2)\sin x]$$

$$+ \cos x\,[2\cos x\,(2x^2 - 2x - 1) + (4x-2)\sin x]$$

$$\text{At } x = n\pi,\ \frac{d^2u}{dx^2} = 0 + 2\cos^2 n\pi(2n^2\pi^2 - 1) > 0$$

Hence at $x = n\pi$, the value of u and so its corresponding the value of y is minimum and minimum value $= e^0 = 1$

Example 18 :

A wire of length 'a' is cut into two parts which are bent respectively in the form of square and a circle. Find the least value of the sum of the areas so formed.

Sol. Given $4x + 2\pi r = a$

$$A = x^2 + \pi r^2 = \frac{1}{16}(a - 2\pi r)^2 + \pi r^2$$

$$\frac{dA}{dr} = 0 \text{ gives } r = \frac{a}{2(\pi+4)}$$

$$\text{for which } \frac{d^2A}{dr} \text{ is +ive and hence minimum}$$

$$4x = a - 2\pi r = a - \frac{a\pi}{\pi+4} = \frac{4a}{\pi+4}$$

$$\therefore\ x = \frac{a}{\pi+4} \qquad \therefore\ A = x^2 + r^2\pi = \frac{a^2}{4(\pi+4)}$$

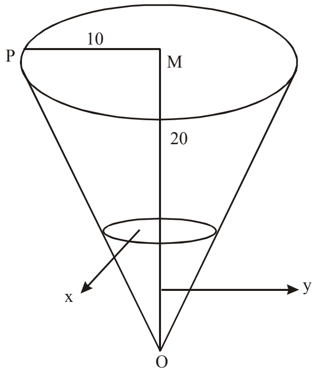
Example 19 :

Water seeps out of a conical filter at the constant rate of 5 c.c./sec. The height of the cone of water in the filter is 15 cm, the height of the filter is 20 cm and radius of the base is 10 cm. Find the rate at which the height of the water decreases.

Sol. Let at any instant, the radius of the base and height of the cone formed by the water in the filter be x and y respectively

$\therefore$ Volume of water in the filter at that time is

$$V = \frac{1}{3}\pi x^2 y \qquad \text{But } \frac{x}{y} = \frac{10}{20} = \frac{1}{2}$$



$$\therefore\ x = \frac{1}{2}y \ \therefore\ V = \frac{1}{3}\pi\frac{1}{4}y^2 . y = \frac{\pi y^3}{12}$$

$$\therefore\ \frac{dv}{dt} = \frac{\pi}{12}3y^2\frac{dy}{dt} = \frac{\pi y^2}{4}\frac{dy}{dt}\ ;\ \frac{dV}{dt} = 5$$

We are to find $\frac{dy}{dt}$, when $y = 15$

$$\therefore\ 5 = \pi\frac{(15)^2}{4}\frac{dy}{dt}$$

$$\therefore\ \frac{dy}{dt} = \frac{5\times 4}{15\times 15}.\frac{1}{\pi} = \frac{4}{45\pi} \text{ cm/sec.}$$

Example 20 :

Show that $\tan^2 x + 6\log\sec x + 2\cos x + 4 \geq 6\sec x$ for $0 \leq x \leq \pi/2$

Sol. Let $f(x) = \tan^2 x + 6\log(\sec x) + 2\cos x + 4 - 6\sec x$

$$f'(x) = 2\tan x\sec^2 x + \frac{6}{\sec x}.\sec x\tan x - 2\sin x - 6\sec x\tan x$$

$$= \frac{2\sin x}{\cos^3 x}\left\{1 + 3\cos^2 x - \cos^3 x - 3\cos x\right\}$$

$$= \frac{2\sin x}{\cos^3 x}(1 - \cos x)^3$$

Now, in $(0, \pi/2)$ $f'(x)$ is positive and hence $f(x)$ increasing. Besides, $f(0) = 0$. Hence, $f(x)$ is positive in $(0, \pi/2)$.

Example 21 :

Find the tangent and normal for $x^{2/3}+y^{2/3}=2$ at (1, 1).

Sol. $x^{2/3}+y^{2/3}=2 \Rightarrow \frac{2}{3}\left(x^{\frac{-1}{3}}+y^{\frac{-1}{3}}y'\right)=0$ or $y'=-\left(\frac{x}{y}\right)^{\frac{-1}{3}}$

At (1, 1) $y'=-1$

Equation of tangent $y-1=-1\,(x-1)\Rightarrow x+y=2$

Equation of normal $y-1=1\,(x-1)\Rightarrow x-y=0$

Example 22 :

Find tangent to $x=a\sin^3 t$ and $y=a\cos^3 t$ at $t=\pi/2$.

Sol. $x=a\sin^3 t;\; y=a\cos^3 t$

$$\frac{dy}{dx}=\frac{-3a\cos^2 t\sin t}{3a\sin^2 t\cos t}=-\cot t$$

At $t=\frac{\pi}{2},\ \frac{dy}{dx}=0$ point is (a, 0)

$\therefore$ Equation of tangent $\Rightarrow\ y=0$

Example 23 :

Find the equation of the normal to the curve $x^2=4y$ which passes through (1, 2).

Sol. $x^2=4y \qquad 2x=4y'$

$$y'=\frac{x_1}{2}\ \&\ \ y_1=\frac{x_1^2}{4}$$

Normal : $y-y_1=\frac{-2}{x_1}(x-x_1)$ or $y-\frac{x_1^2}{4}=\frac{-2}{x_1}(x-x_1)$

It passes through (1, 2)

$$2-\frac{x_1^2}{4}=\frac{-2}{x_1}(1-x_1)=-\frac{2}{x_1}+2$$

$$x_1^3=8\ \Rightarrow x_1=2\ \&\ \ y_1=\frac{x_1^2}{4}=1$$

$\therefore$ Normal is $y-1=\frac{-2}{2}(x-2)=2-x\,;\ x+y=3$

Example 24 :

Tangent at point P on the curve $y^2=x^3$ meets the curve again at point Q. Find $\frac{m_{OP}}{m_{OQ}}$, where O is origin.

Sol. Take P (t^2, t^3) and Q (T^2, T^3)

$$\frac{dy}{dx}=\frac{3x^2}{2y}\ \text{or}\ \left(\frac{dy}{dx}\right)=\frac{3}{2}t$$

Slope line joining P and Q is $=\frac{T^3-t^3}{T^2-t^2}=\frac{T^2+t^2+Tt}{T+t}$

$$\Rightarrow \frac{3}{2}t=\frac{T^2+t^2+Tt}{T+t}\ \text{or}\ 3tT+3t^2=2T^2+2t^2+2Tt$$

$$\Rightarrow T=\frac{-t}{2}\ \Rightarrow \frac{m_{OP}}{m_{OQ}}=-2$$

Example 25 :

Show that for the curve $by^2=(x+a)^3$ the square of the subtangent varies as the subnormal.

Sol. $by^2=(x+a)^3$ or $2byy'=3(x+a)^2$

$$\text{S.T.}=\frac{y}{y'}=\frac{y}{3\,(x+a)^2}\,2by=\frac{2by^2}{3\,(x+a)^2}=\frac{2\,(x+a)}{3}$$

$$\text{S.N.}=yy'=y\frac{3\,(x+a)^2}{2by}=\frac{3\,(x+a)^2}{2b}\ \Rightarrow\ ST^2\propto SN$$

Example 26 :

Show that at any point on the hyperbola $xy=c^2$, the subtangent varies as the abscissa and the subnormal varies as the cube of the ordinate of the point of contact.

Sol. $xy=c^2\ \Rightarrow\ xy'+y=0 \qquad$ or $y'=\frac{-y}{x}$

$$ST=\frac{y}{y'}=-x\,,\ SN=yy'=\frac{-y^2}{x}=\frac{-y^2}{c^2}y=-\frac{-y^3}{c}$$

Example 27 :

Match the following :

Column-I	**Column-II**
(A) If the parabola $y^2=4ax$, $a>0$ cuts the hyperbola $xy=\sqrt{2}$ at right angles, then a =	(P) $4\sqrt{2}$
(B) If the curves $ay+x^2=7$, $a>0$ and $x^3=y$ cut orthogonally at (1, 1), then a =	(Q) $2\sqrt{2}$
(C) If the curves $y^2=4x$ and $xy=a$, $a>0$ cut orthogonally, then a =	(R) 1/2
(D) Curves $2x=y^2$ and $2xy=a$, $a>0$ cut each other at right angles, then a =	(S) 6

Sol. **(A) - R, (B) - S, (C) - P, (D) - Q**

(A) Given curves are, $y^2=4ax$ .....(1)

and $xy=\sqrt{2}$ .....(2)

From (1), $2y\frac{dy}{dx}=4a\ \therefore\ \frac{dy}{dx}=\frac{2a}{y}$ ......(3)

From (2), $y+x\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=\frac{y}{x}$ ....(4)

Putting the value of y from (2) in (1), we get

$$\frac{2}{x^2} = 4ax \Rightarrow x^3 = \frac{1}{2a} \qquad .....(5)$$

For curves (1) and (2) to cut at right angles,

$$\left(\frac{2a}{y}\right)\left(\frac{y}{x}\right) = -1 \Rightarrow 2a = x \ ; 8a^3 = x^3 = \frac{1}{2a} [\text{From } (5)]$$

$$\Rightarrow \ 16a^4 = 1 \ \Rightarrow \ a = 1/2 \qquad [\because a > 0]$$

(B) Given curves are $ay + x^2 = 7$ ....(1)

and $y = x^3$ ....(2)

$$\text{From } (1), \frac{dy}{dx} = \frac{-2x}{a} \qquad ....(3)$$

$$\text{From } (2), \frac{dy}{dx} = 3x^2 \qquad ....(4)$$

For curves (1) & (2) to cut each other orthogonally at (1,1),

$$\left(-\frac{2}{a}\right) \cdot 3 = -1 \ \Rightarrow \ a = 6.$$

(C) Given curves are, $y^2 = 4x$ ....(1)

and $xy = a$ ....(2)

$$\text{From } (1), \ 2y\frac{dy}{dx} = 4 \ \therefore \ \frac{dy}{dx} = \frac{2}{y} \qquad ......(3)$$

$$\text{From } (2), \ 1 \cdot y + x\frac{dy}{dx} = 0$$

$$\therefore \ \frac{dy}{dx} = -\frac{y}{x} \qquad ....(4)$$

Putting the value of y from (2) in (1), we get

$$\frac{a^2}{x^2} = 4x \ \Rightarrow \ a^2 = 4x^3 \qquad ....(5)$$

$$\text{From } (2), \ y = \frac{a}{x} \Rightarrow \frac{dy}{dx} = \frac{-a}{x^2} \quad [\because \text{ from } (2), y = \frac{a}{x}]$$

$$\left(\frac{dy}{dx}\right)_{\text{any curve (1)}} \cdot \left(\frac{dy}{dx}\right)_{\text{for curve (2)}} = \frac{2}{y} \cdot \frac{-a}{x^2} = \frac{-2}{x} \quad ......(6)$$

For curves (1) and (2) to cut each other orthogonally,

$$\frac{-2}{x} = -1 \ \Rightarrow \ x = 2. \qquad [\text{From } (6)]$$

$$\therefore \ \text{From } (5), a = 4\sqrt{2} \qquad [\because a > 1]$$

(D) Given curves are, $y^2 = 2x$ ....(1)

and $xy = a/2$ ....(2)

$$\text{From } (1), \frac{dy}{dx} = \frac{1}{y} \qquad ....(3)$$

$$\text{From } (2), y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x} \qquad ....(4)$$

$$\left(\frac{dy}{dx}\right)_{\text{for curve (1)}} \cdot \left(\frac{dy}{dx}\right)_{\text{for curve (2)}} = \frac{-1}{x} \quad ....(5)$$

Putting the value of y from (2) in (1), we get

$$\frac{a^2}{4x^2} = 2x \Rightarrow 8x^3 = a^2 \qquad ....(6)$$

For the two curves to cut each other at ight angles,

$$-\frac{1}{x} = -1 \ \Rightarrow x = 1$$

$$\therefore \ \text{From } (6), a^2 = 8 \ \Rightarrow \ a = 2\sqrt{2}.$$

Example 28 :

Find intervals of monotonicity of following functions :

(a) $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 7$

(b) $f(x) = \dfrac{2x}{1 + x^2}$

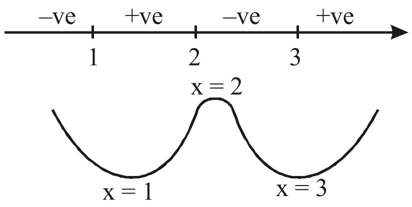
(c) $f(x) = \ln(x^2 - 2x)$

(d) $f(x) = \dfrac{|x-1|}{x^2}$

Sol.(a) We have

$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 7, \ x \in R$

and $f'(x) = 4x^3 - 24x^2 + 44x - 24 = 4(x-1)(x-2)(x-3)$



From the sign scheme for $f'(x)$, we can see that $f(x)$

From the sign scheme for $f'(x)$, we can see that $f(x)$

strictly decreases in $(-\infty, 1)$

strictly increases in (1, 2)

strictly decreases in (2, 3)

strictly increases in $(3, \infty)$.

$$\text{(b) We have } f(x) = \frac{2x}{1 + x^2}, x \in R$$

$$\text{and } f'(x) = \frac{(1 + x^2)\,2 - 2x\,(2x)}{(1 + x^2)^2} = \frac{-2\,(x^2 - 1)}{(1 + x^2)^2}$$

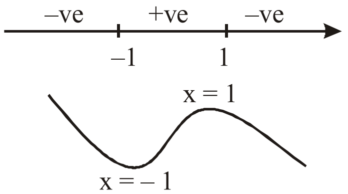
$$= \frac{-2\,(x+1)\,(x-1)}{(1 + x^2)^2}$$

From the sign scheme for $f'(x)$, we can see that $f(x)$

strictly decreases in $(-\infty, 1)$

strictly increases in $(-1, 1)$

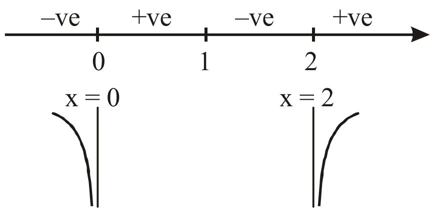
strictly decreases in $(1, \infty)$



(c) We have $f(x) = \ln(x^2 - 2x),\ x \in (-\infty, 0) \cup (2, \infty)$

and $f'(x) = \dfrac{2x-2}{x^2-2x} = \dfrac{2(x-1)}{x(x-2)}$

From the sign scheme for $f'(x)$, we can see that $f(x)$



strictly decreases in $(-\infty, 0) \cup (1, 2)$

strictly increases in $(0, 1) \cup (2, \infty)$.

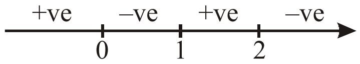
Also, we can see that $f(0^-) = -\infty$ and $f(2^+) = -\infty$.

(d) We have $f(x) = -\dfrac{(x-1)}{x^2},\ x < 1$ and $f(x) = \dfrac{x-1}{x^2}, x \geq 1$

$f'(x) = \dfrac{-2}{x^3} + \dfrac{1}{x^2} = \dfrac{x-2}{x^3}, x < 1$ and $f'(x) = \dfrac{2-x}{x^3}, x > 1$

Now, from the sign scheme for $f'(x)$, we have

$\Rightarrow$ $f(x)$ strictly increases in $(-\infty, 0)$



strictly decreases in (0, 1)

strictly increases in (1, 2)

strictly decreases in $(2, \infty)$.

Example 29 :

If $\phi(x) = f(x) + f(1-x)$ and $f''(x) < 0$ in $(-1, 1)$, then show that $\phi(x)$ strictly increases in (0, 1/2).

Sol. We have $\phi(x) = f(x) + f(1-x)$ and $\phi'(x) = f'(x) - f'(1-x)$ which vanishes at points given by $x = 1 - x$ i.e. $x = 1/2$

$f''(x) < 0 \Rightarrow f'(x)$ is decreasing for $x \in (0, 1/2)$

i.e. $1 - x > x \Rightarrow f'(1-x) < f'(x)$

hence $\phi'(x) > 0\ \forall\ x \in (0, 1/2)$

Hence, $\phi(x)$ strictly increases in (0, 1/2).

Example 30 :

Find the image of interval [–1, 3] under the mapping specified by the function $f(x) = 4x^3 - 12x$.

Sol. $f'(x) = 12x^2 - 12 = 12(x^2 - 1)$

$f'(x) = 0$ at $x = \pm 1$, $f(-1) = 8, f(1) = -8$

$f(3) = 72 \Rightarrow$ greatest value is 72 and least value is –8.

Example 31 :

Find the range of the following functions

$f(x) = \sqrt{x-3} + 2\sqrt{5-x}$.

Sol. We have $f(x) = \sqrt{x-3} + 2\sqrt{5-x}$

whose domain is $x \in [3, 5]$ and its derivative is

$$f'(x) = \frac{1}{2\sqrt{x-3}} - \frac{1}{\sqrt{5-x}} = \frac{\sqrt{5-x} - 2\sqrt{x-3}}{2\sqrt{x-3}\ \sqrt{5-x}}$$

Now, solving

$\sqrt{5-x} > 2\sqrt{x-3}$ i.e. $5 - x > 4(x-3)$ given $x < 17/5$.

Hence, we have

$f'(x) > 0$ $\forall\ x \in (3, 17/5)$ & $f'(x) < 0\ \forall\ x \in (17/5, 5)$

$\Rightarrow$ $f(x)$ strictly increases in (3, 17/5) and strictly decreases in (17/5, 5).

Now, we have

$f(3) = 2\sqrt{2}$, $f(5) = \sqrt{2}$ and

$$f\left(\frac{17}{5}\right) = \sqrt{\frac{17}{5} - 3} + 2\sqrt{5 - \frac{17}{5}} = \sqrt{10}$$

Hence, the range is $y \in [\sqrt{2}, \sqrt{10}]$

Example 32 :

Find the range of the following functions

$$f(x) = \frac{x^4 - x^2 - 2x + 8}{x^4 - x^2 - 2x + 4}$$

Sol. $f(x) = \dfrac{x^4 - x^2 - 2x + 8}{x^4 - x^2 - 2x + 4} = 1 + \dfrac{4}{x^4 - x^2 - 2x + 4}$

$$= 1 + \frac{4}{(x^2-1)^2 + (x-1)^2 + 2}$$

Let $g(x) = (x^2 - 1)^2 + (x - 1)^2 + 2$, whose least value = 2

and greatest value = ∞

Thus, we have for $f(x)$ greatest value $= 1 + \dfrac{4}{2} = 3$

and least value $= 1 + \dfrac{4}{\infty} = 1$.

Also, $f(x)$ is continuous and defined on R. Hence, the range of $f(x)$, is $y \in (1, 3]$.

Example 33 :

Show that $\ln(1+x) > x - \dfrac{x^2}{2}\ \forall\ x \in (0, \infty)$

Sol. Consider the function $f(x) = \ln(1+x) - x + \dfrac{x^2}{2}, x \in (0, \infty)$

Then $f'(x) = \dfrac{1}{1+x} - 1 + x = \dfrac{x^2}{1+x} > 0\ \forall\ x \in (0, \infty)$

$\Rightarrow$ $f(x)$ strictly increases in $(0, \infty)$

$\Rightarrow$ $f(x) > f(0^+) = 0$ i.e. $\ln(1+x) > x - \dfrac{x^2}{2}$

which is the desired result.

Example 34 :

Show that the equation $x^5 - 3x - 1 = 0$ has a unique root in $[1, 2]$.

Sol. Consider the function

$f(x) = x^5 - 3x - 1,\ x \in [1, 2]$

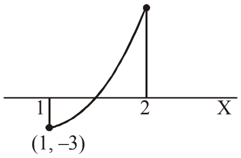
and $f'(x) = 5x^4 - 3 > 0\ \forall\ x \in [1, 2]$

$\Rightarrow$ f(x) striclty increases in [1, 2]

Also, we have

$f(1) = 1 - 3 - 1 = -3$

and $f(2) = 32 - 6 - 1 = 25$



From the shape of the curve shown alongside, we can see that the curve $y = f(x)$ will cut the X-axis exactly once in $[1, 2]$ i.e. f(x) will vanish exactly once in $[1, 2]$

Example 35 :

Prove that $\dfrac{x}{1+x} < \ln(1+x) < x\ \forall\ x > 0$

Sol. Consider the function $f(x) = \ln(1+x) - \dfrac{x}{1+x},\ x > 0.$

Then $f'(x) = \dfrac{x}{1+x} - \dfrac{x}{(1+x)^2} = \dfrac{x}{(1+x)^2} > 0\ \forall\ x > 0$

$\Rightarrow$ f(x) strictly increases in $(0, \infty)$

$\Rightarrow$ $f(x) > f(0^+) = 0$ i.e. $\ln(1+x) > \dfrac{x}{1+x}$ which proves the LHI. Now, consider the function $g(x) = x - \ln(1+x),\ x > 0$

Then $g'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x} > 0\ \forall\ x > 0$

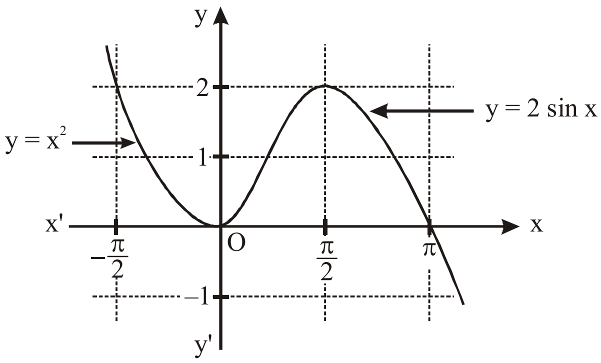
$\Rightarrow g(x)$ strictly increases in $(0, \infty) \Rightarrow g(x) > g(0^+) = 0$

i.e. $x > \ln(1+x)$ which proves the RHI.

Example 36 :

If $f(x) = \begin{cases} x^2, & x \le 0 \\ 2\sin x, & x > 0 \end{cases}$, investigate the function at $x = 0$ for maxima/minima.

Sol. Analyzing the graph of f(x), we get x = 0 is a point of minima.



Example 37 :

The function $y = \dfrac{ax+b}{(x-1)(x-4)}$ has turning point at P (2, 1). Then find the value of a and b.

Sol. $y = \dfrac{ax+b}{(x-1)(x-4)} = \dfrac{ax+b}{x^2 - 5x + 4}$ has turning point at P (2, –1)

$\Rightarrow$ P (2, –1) lies on the curve $\Rightarrow 2a + b = 2$ …(i)

Also $\dfrac{dy}{dx} = 0$ at P(2, –1)

$$\frac{dy}{dx} = \frac{a(x^2 - 5x + 4) - (2x - 5)(ax + b)}{(x^2 - 5x + 4)}$$

At P(2, –1), $\dfrac{dy}{dx} = \dfrac{-2a + 2a + b}{4} = 0$

$\Rightarrow b = 0 \Rightarrow a = 1$ [from equation (i)]

Example 38 :

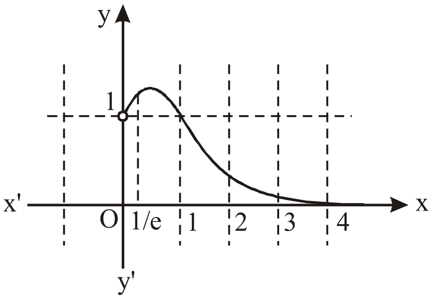
Find the maximum value of $f(x) = \left(\dfrac{1}{x}\right)^x$

Sol. $f(x) = \left(\dfrac{1}{x}\right)^x \Rightarrow f'(x) = \left(\dfrac{1}{x}\right)^x \left(\ln\dfrac{1}{x} - 1\right)$

$f'(x) = 0 \Rightarrow \ln\dfrac{1}{x} = 1 \Rightarrow \dfrac{1}{x} = e \Rightarrow x = \dfrac{1}{e}$

Also for $x < \dfrac{1}{e}$, f'(x) is positive and for $x > \dfrac{1}{e}$, f'(x) is negative. Hence, x = 1/e is point of maxima.

Therefore, the maximum value of function is $e^{1/e}$.



Also $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\right)^x = e^{\lim\limits_{x\to 0} x\ln\left(\frac{1}{x}\right)} = e^{-\lim\limits_{x\to 0} x\ln x} = e^0 = 1$

$\lim\limits_{x\to 0}\left(\dfrac{1}{x}\right)^x = 1.$

Example 39 :

Find the greatest and least values of function

$$f(x)=\begin{cases}-x, & -1\le x<0\\ 2-(x-1)^2, & 0\le x\le 2\end{cases}.$$

Sol. $f(x)=\begin{cases}-x, & -1\le x<0\\ 2-(x-1)^2, & 0\le x\le 2\end{cases}$ and

$$f'(x)=\begin{cases}-1, & -1<x<0\\ -2x(x-1), & 0<x<2\end{cases}$$

Thus, the points at which f(x) may have extreme values, are the critical points $x = 0, 1$ $[f'(1)=0$ and $f'(0)=DNE]$ and the end points $x = -1, 2$

Now, $f(-1)=1, f(1)=2$ and $f(2)=1$.

Since f is discontinuous at x = 0, we also need to find the limiting values of f(x) as $x\to 0$.

We have $f(0^-)\to 0, f(0^+)\to 1$ and $f(0)=1$

the largest and the smallest among the above six values are 2 and 0 respectively. Hence, the greatest value is 2 but the least value does not exist since the function approaches 0 but is never equal to 0.

Example 40 :

Find greatest and least values of

$$f(x)=\frac{a^2}{x}+\frac{b^2}{1-x}, x\in(0,1)\ (a, b>0).$$

Sol. $f(x)=\dfrac{a^2}{x}+\dfrac{b^2}{1-x}$, $x\in(0,1)$

and $f'(x)=\dfrac{-a^2}{x^2}+\dfrac{b^2}{(1-x)^2}$

which exists everywhere in (0, 1) and vanishes at points,

given by $\dfrac{b^2}{(1-x^2)}=\dfrac{a^2}{x^2}$; $a^2(1-x)^2=b^2x^2$

i.e. $a(1-x)=bx$ i.e. $x=\dfrac{a}{a+b}$

To find the greatest and least value, we need to check the values of f(x) at $x=0^+, 1^-, \dfrac{a}{a+b}$.

We have $f(0^+)\to+\infty, f(1^-)\to+\infty$ and $f\left(\dfrac{a}{a+b}\right)=(a+b)^2$

Hence, we have, least value $=(a+b)^2$
and greatest value does not exist.

Example 41 :

Find greatest and least values of

$$f(x)=\frac{(a+x)(b+x)}{(c+x)}, x>-c.$$

Sol. We have $f(x)=\dfrac{(a+x)(b+x)}{(c+x)}, x\in(-c,\infty)$

and $f'(x)=\dfrac{(c+x)(2x+a+b)-[x^2+(a+b)x+ab]}{(c+x)^2}$

$$=\frac{x^2+2cx+ac+bc-ab}{(c+x)^2}, x\in(-c,\infty)$$

which vanishes at points given by
$x^2+2cx+ac+bc-ab=0$

i.e. $x=-c\pm\sqrt{c^2-(ac+bc-ab)}$

$=-c\pm\sqrt{(a-c)(b-c)}$

Thus, the expression for $f'(x)$ can be written as

$$f'(x)=\frac{(x-\alpha)(x-\beta)}{(c+x)^2}$$

choosing $\alpha=-c-\sqrt{(a-c)(b-c)}$ and

$\beta=-c+\sqrt{(a-c)(b-c)}$

The critical point $x=\alpha$ is of no interest since it does lie in the interval $(-c,\infty)$.

Now, we have $f(-c^+)\to\infty, f(\infty)\to\infty$

$$f(\beta)=\frac{\left(a-c+\sqrt{(a-c)(b-c)}\right)\left(b-c+\sqrt{(a-c)(b-c)}\right)}{c-c+\sqrt{(a-c)(b-c)}}$$

$$=\frac{(a-c)(b-c)+(a+b-2c)\sqrt{(a-c)(b-c)}+(a-c)(b-c)}{\sqrt{(a-c)(b-c)}}$$

$=2\sqrt{(a-c)(b-c)}+a+b-2c$

$=a-c+b-c+2\sqrt{(a-c)(b-c)}$

$=\left(\sqrt{(a-c)}+\sqrt{(b-c)}\right)^2$

Hence, we have

Least value $=\left(\sqrt{(a-c)}+\sqrt{(b-c)}\right)^2$ and greatest value does not exist.

Example 42 :

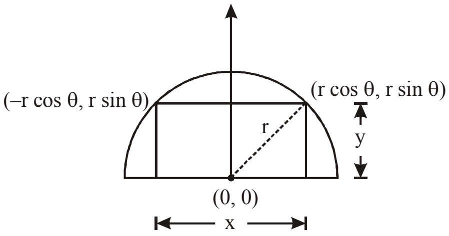
Rectangles are inscribe inside a semi-circle of radius r. Find the rectangle with maximum area.

Sol. Let us choose co-ordinate system with origin as centre of circle. Area, $A=xy$

$\Rightarrow$ $A=2(r\cos\theta)(r\sin\theta), \theta\in(0,\pi/2)$

$\Rightarrow$ $A=r^2\sin 2\theta$

A is maximum when $\sin 2\theta=1 \Rightarrow 2\theta=\pi/2 \Rightarrow \theta=\pi/4$



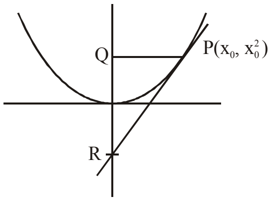
Sides of the rectangle are

$$2r\cos\left(\frac{\pi}{4}\right) = \sqrt{2}\,r \quad \text{and} \quad r\sin\left(\frac{\pi}{4}\right) = \frac{r}{\sqrt{2}}.$$

Example 43 :

The tangent to the parabola $y = x^2$ has been drawn so that the abscissa x_0 of the point of tangency belong to the interval (1, 2). Find x_0 for which the triangle is to be bounded by the tangent, the axis of ordinates, and the straight line $y = x_0^2$ has the greatest area.

Sol. $y = x^2, \quad \frac{dy}{dx} = 2x$



$\Rightarrow$ Equation of the tangent at (x_0, x_0^2) is $y - x_0^2 = 2x_0(x - x_0)$.
It meets y-axis in R $(0, -x_0^2)$. Q is $(0, x_0^2)$

$\Rightarrow$ Z = area of the triangle PQR

$$\frac{1}{2}2x_0^2\,x_0 = x_0^3, 1 \le x_0 \le 2$$

$$\frac{dZ}{dx_0} = 3x_0^2 > 0 \text{ in } 1 \le x_0 \le 2$$

$\Rightarrow$ Z is an increasing function in [1, 2]
Hence, Z, i.e., the area of ΔPQR is greatest at $x_0 = 2$.

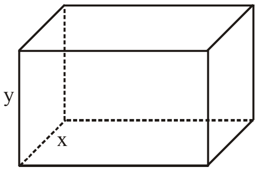
Example 44 :

A sheet of area 40 m^2 is used to make an open tank with square base. Find the dimensions of the base such that volume of this tank is maximum.

Sol. Let the length of base be x m and height be y m
Volume $V = x^2y$
Again x and y are related to the surface area of this tank which is equal to 40 m^2.

$$\Rightarrow x^2 + 4xy = 40 \Rightarrow y = \frac{40 - x^2}{4x}, \qquad x \in (0, \sqrt{40})$$

$$\Rightarrow V(x) = x^2\left(\frac{40 - x^2}{4x}\right) = \frac{40x - x^3}{4}$$



Maximizing volume,

$$V'(x) = \frac{40 - 3x^2}{4} = 0 \Rightarrow x = \sqrt{\frac{40}{3}} \text{ m}$$

$$\text{and } V''(x) = \frac{-3x}{2} \Rightarrow V''\left(\sqrt{\frac{40}{3}}\right) < 0$$

$$\Rightarrow \text{ volume is maximum at } x = \sqrt{\frac{40}{3}} \text{ m.}$$

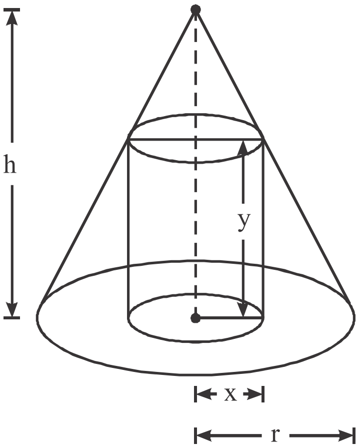
Example 45 :

If a right-circular cylinder is inscribed in a given cone. Find the dimensions of the cylinder such that its volume is maximum.

Sol. Let x be the radius of cylinder and y be its height
Volume $V = \pi x^2 y$
x, y can be related by using similar triangles

$$\frac{y}{r - x} = \frac{h}{r} \Rightarrow y = \frac{h}{r}(r - x)$$

$$\Rightarrow V(x) = \pi x^2 \frac{h}{r}(r - x), x \in (0, r)$$



$$\Rightarrow V(x) = \frac{\pi h}{r}(rx^2 - x^3) \Rightarrow V'(x) = \frac{\pi h}{r}x(2r - 3x)$$

$V'(x) = 0 \Rightarrow x = 2r/3$

$$\text{Also} \quad V''(x) = \frac{\pi h}{r}(2r - 6x) \quad \Rightarrow V''\left(\frac{2r}{3}\right) < 0$$

This volume is maximum when, $x = \frac{2r}{3}$ and $y = \frac{h}{3}$.

Example 46 :

Find the value of a if $x^3 - 3x + a = 0$ has three real distinct roots.

Sol. Let $f(x) = x^3 - 3x + a$

Let $f'(x) = 0 \Rightarrow 3x^2 - 3 = 0 \Rightarrow x = \pm 1$

For three distinct roots, $f(1)\,f(-1) < 0$

$\Rightarrow (1 - 3 + a)(-1 + 3 + a) < 0$

$\Rightarrow (a + 2)(a - 2) < 0$

$\Rightarrow -2 < a < 2$

Example 47 :

Prove that three exist exactly two non-similar isosceles triangle ABC such that $\tan A + \tan B + \tan C = 100$.

Sol. Let $A = B$, then $2A + C = 180^\circ$ and $2\tan A + \tan C = 100$

Now $2A + C = 180^\circ \Rightarrow \tan 2A = -\tan C$ …(i)

Also $2\tan A + \tan C = 100$

$\Rightarrow 2\tan A - 100 = -\tan C$ …(ii)

From (i) and (ii), $2\tan A - 100 = \dfrac{2\tan A}{1 - \tan^2 A}$

Let $\tan A = x$, then $\dfrac{2x}{1 - x^2} = 2x - 100$

$\Rightarrow x^3 - 50x^2 + 50 = 0$

Let $f(x) = x^3 - 50x^2 + 50$. Then $f'(x) = 3x^2 - 100x$. Thus $f'(x) = 0$ has roots $0, \dfrac{100}{3}$. Also $f(0)\, f\left(\dfrac{100}{3}\right) < 0$. Thus $f(x) = 0$ has exactly three distinct real roots. Therefore, tan A and hence A has three distinct values but one of them will be obtuse angle. Hence, there exist exactly two non similar isosceles triangles.

Example 48 :

Find the set of value of m for the cubic

$$x^3 - \frac{3}{2}x^2 + \frac{5}{2} = \log_{1/4}(m) \text{ has 3 distinct solutions.}$$

Sol. Consider $y = x^3 - \dfrac{3}{2}x^2 + \dfrac{5}{2}$

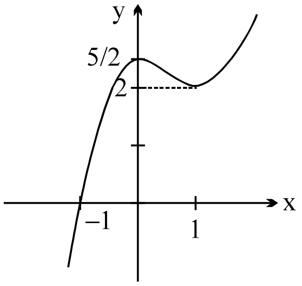
$$\frac{dy}{dx} = 3x^2 - 3x = 3x(x - 1) = 0 \Rightarrow x = 0 \text{ or } 1$$

$$\frac{d^2y}{dx^2} = 6x - 3\ ;\ \left.\frac{d^2y}{dx^2}\right|_{x=0} = -3 \text{ i.e. } < 0$$

$\Rightarrow$ maximum at $x = 0$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = \text{is } 3 \text{ i.e.} > 0 \Rightarrow \text{mimimum}$$

Hence the graph of the cubic is now for 3 distinct roots



$2 < \log_{1/4}(m) < 5/2$

$2 < -\log_4(m) < 5/2$

$-5/2 < \log_4(m) < -2$

$1/32 < m < 1/16$

QUESTION BANK

CHAPTER 6 : APPLICATION OF DERIVATIVES

EXERCISE - 1 [LEVEL-1]

PART - 1 - TANGENT AND NORMAL

Q.1 P is the point of contact of the tangent from the origin to the curve $y = \log_e x$. The length of the perpendicular drawn from the origin to the normal at P is
(A) 1/2e (B) 1/e
(C) $2\sqrt{e^2+1}$ (D) $\sqrt{e^2+1}$

Q.2 For the curve $4x^5 = 5y^4$, the ratio of the cube of the subtangent at a point on the curve to the square of the subnormal at the same point is –
(A) $x\,(4/5)^4$ (B) $y\,(5/4)^4$
(C) $(4/5)^4$ (D) $(5/4)^4$

Q.3 The angle between $y^2 = 4x$ and $x^2 + y^2 = 12$ at a point of their intersection is –
(A) $\tan^{-1}(1/2)$ (B) $\tan^{-1} 2\sqrt{2}$
(C) $\tan^{-1} 2$ (D) $\tan^{-1}\sqrt{2}$

Q.4 Length of the subtangent at (x_1, y_1) on $x^n y^m = a^{m+n}$, $m, n > 0$, is –
(A) $\frac{n}{m}|x_1|$ (B) $\frac{n}{m}x_1$
(C) $\frac{m}{n}|x_1|$ (D) $\frac{n}{m}|y_1|$

Q.5 The length of the sub-tangent, ordinate and the sub-normal are in –
(A) AGP (B) A.P.
(C) H.P (D) G.P.

Q.6 If $\sin^{-1} a$ is the acute angle between the curves $x^2 + y^2 = 4x$ and $x^2 + y^2 = 8$ at (2, 2), then a =
(A) 1 (B) 0
(C) $1/\sqrt{2}$ (D) $\sqrt{3}/2$

Q.7 If the length of the sub-tangent at any point to the curve $xy^n = a$ is proportional to the abscissa, then 'n' is –
(A) any non-zero real number (B) 2
(C) –2 (D) 1

Q.8 Slope of Normal to the curve $y = x^2 - \frac{1}{x^2}$ at (–1, 0) is –
(A) 1/4 (B) –1/4
(C) 4 (D) – 4

Q.9 If $y = 4x - 5$ is tangent to the curve $y^2 = px^3 + q$ at (2, 3), then
(A) p = 2, q = –7 (B) p = –2, q = 7
(C) p = –2, q = –7 (D) p = 2, q = 7

Q.10 If $x + 4y = 14$ is a normal to the curve $y^2 = \alpha x^3 - \beta$ at (2, 3), then the value of $\alpha + \beta$ is –
(A) 3 (B) 7
(C) 2 (D) 9

Q.11 The length of the subtangent at any point of the curve $x^m y^n = a^{m+n}$ is proportional to –
(A) ordinate (B) abscissa
(C) $(\text{ordinate})^n$ (D) $(\text{abscissa})^n$

Q.12 The distance between the origin and the normal to the curve $y = e^{2x} + x^2$ at the point whose abscissa is 0, is –
(A) $1/\sqrt{5}$ (B) $2/\sqrt{5}$
(C) $3/\sqrt{5}$ (D) $2/\sqrt{3}$

Q.13 If the line $ax + by + c = 0$ is a normal to the rectangular hyperbola $xy = 1$, then –
(A) $a \ge 0, b \ge 0$ (B) $a < 0, b < 0$ or $a > 0, b > 0$
(C) $a > 0, b < 0$ (D) $a < 0, b > 0$

Q.14 The length of subtangent to the curve $x^2y^2 = a^4$ at the point (–a, a) is
(A) 3a (B) 2a
(C) a (D) 4a

Q.15 The line $2x + \sqrt{6}y = 2$ is a tangent to the curve $x^2 - 2y^2 = 4$. The point of contact is
(A) $(4, -\sqrt{6})$ (B) $(7, -2\sqrt{6})$
(C) (2, 3) (D) $(\sqrt{6}, 1)$

Q.16 The point of the curve $y^2 = 2(x - 3)$ at which the normal is parallel to the line $y - 2x + 1 = 0$ is
(A) (5, 2) (B) (–1/2, –2)
(C) (5, –2) (D) (3/2, 2)

Q.17 The equation of tangent to the curve $y = 2\cos x$ at $x = \pi/4$ is
(A) $y - \sqrt{2} = 2\sqrt{2}\left(x - \frac{\pi}{4}\right)$ (B) $y + \sqrt{2} = \sqrt{2}\left(x + \frac{\pi}{4}\right)$
(C) $y - \sqrt{2} = -\sqrt{2}\left(x - \frac{\pi}{4}\right)$ (D) $y - \sqrt{2} = \sqrt{2}\left(x - \frac{\pi}{4}\right)$

Q.18 The point on the curve $y^2 = x$, where tangent makes 45° angle with x-axis, is –
(A) $\left(\frac{1}{2}, \frac{1}{4}\right)$ (B) $\left(\frac{1}{4}, \frac{1}{2}\right)$
(C) (4, 2) (D) (1, 1)

PART - 2 - MONOTONICITY

Q.19 The set of real values of x for which $f(x) = \frac{x}{\log x}$ is increasing, is –
(A) $\{x : x \ge e\}$ (B) empty
(C) $\{x : x < e\}$ (D) {1}

Q.20 The function $f(x) = \frac{x}{3} + \frac{3}{x}$ decreases in the interval
(A) (–3, 3) (B) $(-\infty, 3)$
(C) $(3, \infty)$ (D) (–9, 9)

Q.21 If $f(x)=x^3-6x^2+9x+3$ be a decreasing function, then x lies in

(A) $(-\infty,-1)\cap(3,\infty)$
(B) $(1,3)$
(C) $(3,\infty)$
(D) None of these

Q.22 The function $f(x)=\tan^{-1}(\sin x+\cos x)$, $x>0$ is always an increasing function on the interval

(A) $(0,\pi)$
(B) $(0,\pi/2)$
(C) $(0,\pi/4)$
(D) $(0,3\pi/4)$

Q.23 $2x^3+18x^2-96x+45=0$ is an increasing function when

(A) $x\le -8, x\ge 2$
(B) $x<-2, x\ge 8$
(C) $x\le -2, x\ge 8$
(D) $0\le x\le -2$

Q.24 $f(x)=(x-2)^5(x+1)^4$ is decreasing in interval (–1, 1/A). Find the value of A.

(A) 3
(B) 7
(C) 2
(D) 9

Q.25 Function $f(x)=\dfrac{a\sin x+b\cos x}{c\sin x+d\cos x}$ is monotonic decreasing if

(A) $ad-bc<0$
(B) $ad-bc>0$
(C) $ab-cd<0$
(D) $ab-cd>0$

Q.26 If function $f(x)=kx^3-9x^2+9x+3$ is monotonic increasing in every interval then

(A) $k<3$
(B) $k\le 3$
(C) $k>3$
(D) $k\ge 3$

Q.27 The function $f(x)=x^{1/x}$ is increasing in the interval

(A) (e,∞)
(B) $(-\infty,e)$
(C) $(-e,e)$
(D) None of these

Q.28 $f(x)=x^3+ax^2+bx+5\sin^2 x$ is an monotonically increasing function in the set of real numbers if a and b satisfy the condition –

(A) $a^2-3b-15<0$
(B) $a^2-3b-15>0$
(C) $a^2-3b+15<0$
(D) $a>0, b>0$

Q.29 The function $\sin x-bx+c$ will be increasing in the interval $(-\infty,\infty)$, if

(A) $b\le 1$
(B) $b\le 0$
(C) $b<-1$
(D) $b\ge 0$

PART - 3 - MAXIMA AND MINIMA

Q.30 A wire of length 20cm is bent in the form of a sector of a circle. The maximum area that can be enclosed by the wire is –

(A) 20 sq. cm
(B) 25 sq. cm
(C) 10 sq. cm
(D) 30 sq. cm

Q.31 The sum of two positive numbers is given. If the sum of their cubes is minimum, then –

(A) one is thrice the other
(B) they are equal
(C) one is twice the other
(D) they are unequal

Q.32 The perimeter of a sector is a constant. If its area is to be maximum, then the sectorial angle is

(A) 2^c
(B) $\pi^c/6$
(C) $\pi^c/4$
(D) 4^c

Q.33 The maximum value of xe^{-x} is

(A) $-1/e$
(B) e
(C) $1/e$
(D) $-e$

Q.34 The maximum area of a rectangle that can be inscribed in a circle of radius 2 units is –

(A) 8π sq. units
(B) 4 sq. units
(C) 5 sq. units
(D) 8 sq. units

Q.35 On the interval [0, 1] the function $x^{25}(1-x)^{75}$ takes its maximum value at the point.

(A) 0
(B) 1/4
(C) 1/2
(D) 1/3

Q.36 The function $f(x)=\dfrac{x^2-3x+2}{x^2+2x-3}$ is equal to

(A) min. at $x=-3$, max. at $x=1$
(B) max. at $x=-3$
(C) Increasing in its domain
(D) Decreasing in its domain

Q.37 The value of a in order that $f(x)=\sin x-\cos x-ax+b$ decreases for all real values is given by –

(A) $a\ge\sqrt{2}$
(B) $a<\sqrt{2}$
(C) $a\ge 1$
(D) $a<1$

Q.38 The function $f(x)=1+x(\sin x)[\cos x]$, $0<x\le\pi/2$ (where [.] is G.I.F.)

(A) is continuous on $(0,\pi/2)$
(B) is strictly increasing in $(0,\pi/2)$
(C) is strictly decreasing in $(0,\pi/2)$
(D) has global maximum value 2

Q.39 Let $f(x)=\begin{cases}|x|, 0<|x|\le 2\\ 1\ ,\ x=0\end{cases}$, then at $x=0$ f has

(A) A local maximum
(B) No local maximum
(C) A local minimum
(D) No extremum

Q.40 The total revenue in Rupees received from the sale of x units of a product is given by $R(x)=3x^2+36x+5$. The marginal revenue, when $x=15$ is

(A) 116
(B) 96
(C) 90
(D) 126

Q.41 The sum of two numbers is fixed. Then its multiplication is maximum, when

(A) Each number is half of the sum
(B) Each number is 1/3 and 2/3 respectively of the sum
(C) Each number is 1/4 and 3/4 respectively of the sum
(D) None of these

Q.42 The minimum value of the expression $7-20x+11x^2$ is

(A) $\dfrac{177}{11}$
(B) $-\dfrac{177}{11}$
(C) $-\dfrac{23}{11}$
(D) $\dfrac{23}{11}$

Q.43 Let $f(n)=20n-n^2$ (n = 1, 2, 3, ...), then –

(A) $f(n)\to\infty$ as $n\to\infty$
(B) f (n) has no maximum
(C) the maximum value of f (n) is greater than 200
(D) The maximum value of f (n) is 100

PART - 4 - RATE OF CHANGE OF VARIABLE

Q.44 A sphere increases its volume at the rate of π cc/s. The rate at which its surface area increases when the radius is 1 cm is –

(A) $\frac{\pi}{2}$ sq. cm/s (B) $\frac{3\pi}{2}$ sq. cm/s

(C) π sq. cm/s (D) 2π sq. cm/s

Q.45 If a ball is thrown vertically upwards and the height s reached in time t is given by $s = 22t - 11t^2$, then the total distance traveled by the ball is –

(A) 22 units (B) 44 units

(C) 33 units (D) 11 units

Q.46 A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/sec. At that instant, when the radius of circular wave is 8 cm, how fast is the enclosed area increasing?

(A) 6π cm^2/s (B) 8π cm^2/s

(C) (8/3) cm^2/s (D) 80π cm^2/s

Q.47 A balloon which always remains sphereical is being inflated by pumping in 10 cube centimeters of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15 cms.

(A) $\frac{1}{90\pi}$ cm / sec (B) $\frac{1}{9\pi}$ cm / sec

(C) $\frac{1}{30\pi}$ cm / sec (D) $\frac{1}{\pi}$ cm / sec

PART - 5 - MISCELLANEOUS

Q.48 The equation $\sin x + x \cos x = 0$ has at least one root in –

(A) $(-\pi/2, 0)$ (B) $(0, \pi)$

(C) $(\pi, 3\pi/2)$ (D) $(0, \pi/2)$

Q.49 Function $f(x) = \cos x - 2\lambda x$ is monotonic decreasing when

(A) $\lambda > 1/2$ (B) $\lambda < 1/2$

(C) $\lambda < 2$ (D) $\lambda > 2$

Q.50 If $a^2 x^4 + b^2 y^4 = c^4$, then the maximum value of xy is

(A) $\frac{c}{\sqrt{ab}}$ (B) $\frac{c^2}{2\sqrt{ab}}$

(C) $\frac{c}{2\sqrt{ab}}$ (D) $\frac{c^2}{2ab}$

Q.51 The curve represented parametrically by the equations $x = 2 \ln \cot t + 1$ & $y = \tan t + \cot t$

(A) tangent and normal intersect at the point (2, 1)

(B) normal at $t = \pi/4$ is parallel to y – axis

(C) tangent at $t = \pi/4$ is parallel to the line $y = x$

(D) tangent at $t = \pi/4$ is parallel to x – axis

Q.52 The maximum value of $(x-p)^2+(x-q)^2+(x-r)^2$ will be at x equal to-

(A) $\frac{p+q+r}{3}$ (B) $3\sqrt{qpr}$

(C) qpr (D) $p^2 + q^2 + r^2$

Q.53 The value of θ, $\theta \in [0, \pi/2]$ for which the sum of intercepts on co-ordinate axes by tangent at point$(3\sqrt{3} \cos \theta, \sin \theta)$ of ellipse $\frac{x^2}{27} + y^2 = 1$ is minimum, is :

(A) $\pi/6$ (B) $\pi/4$

(C) $\pi/3$ (D) $\pi/2$

Q.54 Let the function f (x) be defined as follows :

$$f(x) = \begin{cases} x^3 + x^2 - 10x, & -1 \le x < 0 \\ \cos x & , 0 \le x < \frac{\pi}{2} \\ 1 + \sin x & , \frac{\pi}{2} \le x \le \pi \end{cases}$$

. Then f (x) has –

(A) a local minimum at $x = \pi/2$

(B) a local maximum at $x = \pi/2$

(C) absolute minimum at $x = -1$

(D) absolute maximum at $x = \pi$

Q.55 If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ and $x^2 - y^2 = c^2$ cut each other at right angles, then –

(A) $a^2 + b^2 = 2c^2$ (B) $b^2 - a^2 = 2c^2$

(C) $a^2 - b^2 = 2c^2$ (D) $a^2 b^2 = 2c^2$

Q.56 The greatest area of the rectangular plot which can be laid out within a triangle of base 36ft. & altitude 12ft. equals (Assume that one side of the rectangle lies on the base of the triangle)

(A) 90 (B) 108

(C) 72 (D) 126

Q.57 $f(x) = \begin{cases} \sin \frac{\pi x}{2}, & 0 \le x < 1 \\ 3 - 2x, & x \ge 1 \end{cases}$, then

(A) f (x) has a local minimum at x = 1

(B) f (x) has a local maximum at x = 1

(C) f (x) does not have any local maximum or minimum at x = 1

(D) f (x) has a global minimum at x = 1

Q.58 For the curves, $x^3 + 2 = 3xy^2$ and $y^3 + 2 = 3x^2y$ which of the following are true?

(i) They are orthogonal.

(ii) They are symmetric with respect to the axes of coordinates.

(iii) They are reflections of each other with respect to y = x.

(A) (i) Only (B) (ii) and (iii) Only

(C) (i) and (iii) Only (D) (i), (ii) and (iii)

Q.59 A curve y = f (x) passes through the point (4, 3) and the normal to the curve at the point happens to be a tangent to the circle $x^2 + y^2 = 25$. The value of f ' (4) is

(A) –3/4 (B) 3/4

(C) 4/3 (D) – 4/3

Q.60 Let $f : R \to R$ be a function such that $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)\ \forall\ x \in R$. Then $f(x) = 0$ has –

(A) Three real and distinct roots

(B) Three real roots of which two are equal

(C) Two imaginary roots

(D) Three real and coincident roots

Q.61 The interval in which $f(x) = 2 \sin x + \tan x - 3x$ increases is

(A) $(-\pi/2, 0)$ (B) $(0, \pi/2)$

(C) $(-\pi/2, \pi/2)$ (D) $(\pi/2, 3\pi/2)$

Q.62 Let the function $g : R \to (-\pi/2, \pi/2)$ be given by $g(t) = \pi/2 - 2\cot^{-1}(3^{-t})$. Then g is –

(A) even and is strictly increasing in $(-\infty, \infty)$

(B) odd and is strictly decreasing in $(-\infty, \infty)$

(C) even and is strictly decreasing in $(-\infty, \infty)$

(D) odd and is strictly increasing in $(-\infty, \infty)$

Q.63 Let $f(x) = \cot^{-1}[g(x)]$, where $g(x)$ is an increasing function for $0 < x < \pi$. Then $f(x)$ is

(A) increasing in $(0, \pi)$

(B) decreasing in $(0, \pi)$

(C) increasing in $(0, \pi/2)$ and decreasing in $(\pi/2, \pi)$

(D) None of these

Q.64 Suppose x_1 & x_2 are the point of maximum and the point of minimum respectively of the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$ respectively, then for the equality $x_1^2 = x_2$ to be true the value of 'a' must be

(A) 0 (B) 2

(C) 1 (D) 1/4

Q.65 $\{a_1, a_2, \ldots\ldots, a_4, \ldots\ldots\}$ is a progression where

$a_n = \dfrac{n^2}{n^3 + 200}$. The largest term of this progression is :

(A) a_6 (B) a_7

(C) a_8 (D) none

Q.66 The largest possible value of the expression

$y = \sqrt{x-2} + 2\sqrt{3-x}$ is –

(A) 3 (B) $\sqrt{5}$

(C) 2 (D) 17

Q.67 If the normal to the curve $y = f(x)$ at the point $(3, 4)$ makes an angle $3\pi/4$ with the positive x-axis, then $f'(3) =$

(A) –1 (B) –3/4

(C) 4/3 (D) 1

Q.68 The function $f(x) = \begin{vmatrix} x-1 & x+1 & 2x+1 \\ x+1 & x+3 & 2x+3 \\ 2x+1 & 2x-1 & 4x+1 \end{vmatrix}$ has –

(A) one point of maximum and one point of minimum

(B) one point of maximum only

(C) one point of minimum only

(D) none of the above

Q.69 The normal at 2, 6 to the curve $x = 1 + t$, $y = 2 + 4t$ has the intercepts on the axes given by

(A) 50, 25/4 (B) 50, 25/2

(C) 48, 25 (D) None of these

Q.70 The function $f(x) = \cot^{-1} x + x$ increases in the interval

(A) $(1, \infty)$ (B) $(-1, \infty)$

(C) $(-\infty, \infty)$ (D) $(0, \infty)$

Q.71 At a point $(a/8, a/8)$ on the curve $x^{1/3} + y^{1/3} = a^{1/3}$ $(a > 0)$ tangent is drawn. If the axes be of length $\sqrt{2}$, then find the value of a.

(A) 1 (B) 2

(C) 4 (D) 8

Q.72 If the line $ax + by + c = 0$ is normal to the curve $xy + 5 = 0$, then

(A) $a > 0, b > 0$ (B) $b > 0, a < 0$

(C) $b < 0, a > 0$ (D) none of these

Q.73 If $f(x) = x^3 + ax^2 + bx - 5\cos^2 x$ is an increasing function for all real values of x, then a and b satisfy the condition

(A) $a^2 - 3b - 15 < 0$ (B) $a^2 - 3b - 15 > 0$

(C) $a^2 - 3b + 15 < 0$ (D) $a^2 - 3b + 15 > 0$

Q.74 $f(x) = 2x^4 - 5x^2 + 7$ attains-

(A) A maximum of $\dfrac{87}{16}$ at $x = \dfrac{\sqrt{5}}{2}$

(B) A minimum of $\dfrac{87}{16}$ at $x = -\dfrac{\sqrt{5}}{2}$

(C) A maximum of $\dfrac{31}{8}$ at $x = \dfrac{\sqrt{5}}{2}$

(D) A minimum of $\dfrac{31}{8}$ at $x = -\dfrac{\sqrt{5}}{2}$

Q.75 If $a < b < c < d$ and $x \in R$ then the least value of the function, $f(x) = |x-a| + |x-b| + |x-c| + |x-d|$ is

(A) $a + c - b - d$ (B) $a + b + c + d$

(C) $c + d - a - b$ (D) $a + b - c - d$

Q.76 Maximum value of $[\sin x] + [\cos x]$ is (where [.] represents greatest integer function)

(A) 0 (B) 1

(C) 2 (D) 3

Q.77 The point on the curve $y^2 = x^2 + ax + 25$ touches the axis of x are –

(A) ± 5 (B) ± 10

(C) ± 15 (D) none of these

Q.78 If a function $f(x) = \cos|x| - 2ax + b$ is an increasing function on whole number line, then the value of a is

(A) b (B) b/2

(C) $a \le -1/2$ (D) $a > -3/2$

Q.79 Statement 1 : The cubic equation $x^3 + 2x^2 + x + 5 = 0$ has three real roots.

Statement 2 : The cubic equation $x^3 + 2x^2 + x + 5 = 0$ has only one real root.

Statement 3 : The cubic equation $x^3 + 2x^2 + x + 5 = 0$ has only real root α, such that $[\alpha] = -3$.

Statement 4 : The cubic equation $x^3 + 2x^2 + x + 5 = 0$ has three real roots α, β, γ, such that $[\alpha] = -3$, $[\beta] = -2$, $[\gamma] = -1$,(where [.] denotes the greatest integer function)

(A) TFFT (B) FTTF

(C) TFFF (D) TFTF

Q.80 If a > b > 0, then maximum value of $\frac{ab\,(a^2-b^2)\sin x\cos x}{a^2\sin^2 x+b^2\cos^2 x}$, where $x\in\left(0,\frac{\pi}{2}\right)$ is –

(A) a^2-b^2

(B) $\frac{a^2-b^2}{2}$

(C) $\frac{a^2+b^2}{2}$

(D) None of these

Q.81 A truck is to be driven 300 km. on a highway at a constant speed of x kmph. Speed rules of the highway required that $30 \le x \le 60$. The fuel costs Rs. 10 per litre and is consumed at the rate of $2+\frac{x^2}{600}$ litres per hour. The wages of the driver are Rs. 200 per hour. The most economical speed to drive the truck, in kmph, is –

(A) 30

(B) 60

(C) $30\sqrt{3.3}$

(D) $20\sqrt{3.3}$

Q.82 The radius of a right circular cylinder increases at the rate of 0.1 cm/min, and the height decreases at the rate of 0.2 cm/min. The rate of change of the volume of the cylinder, in cm^3/min, when the radius is 2 cm and the height is 3 cm is

(A) -2π

(B) $-8\pi/5$

(C) $-3\pi/5$

(D) $2\pi/5$

Q.83 Let $x_1=(\tan\theta)^{\cot\theta}, x_2=(\cot\theta)^{\cot\theta}, x_3=(\tan\theta)^{\tan\theta}$ and $x_4=(\cot\theta)^{\tan\theta}$ where $0<\theta<\pi/4$, then

(A) $x_1<x_2<x_3<x_4$

(B) $x_1<x_3<x_4<x_2$

(C) $x_1<x_4<x_3<x_2$

(D) $x_1<x_2<x_4<x_3$

Q.84 Maximum value of $x^2 \ln(1/x)$ is –

(A) 2e

(B) e

(C) 1/e

(D) 1/2e

Q.85 Let f (x) be defined as $f(x)=\begin{cases}\tan^{-1}\alpha-5x^2, & 0<x<1\\ -6x & , x\ge 1\end{cases}$;

f (x) can have a maximum at x = 1 if value of α is

(A) 0

(B) – 1

(C) – 2

(D) – tan 1

Q.86 The curve $y - e^{xy} + x = 0$ has a vertical tangent at –

(A) (1, 1)

(B) (0, 1)

(C) (1, 0)

(D) no point

Q.87 Length of the tangent at t = π/4 to the curve $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$ $(a > 0)$ is

(A) $a\left(1-\frac{\pi}{4}\right)$

(B) $a\left(\frac{\pi}{4}-1\right)$

(C) $a(\pi-4)$

(D) None of these

Q.88 If t, n, t', n' are the lengths of tangent, normal subtangent and subnormal at a point P (x, y) on any curve y = f (x) then

(A) $t^2 + n^2 = t'n'$

(B) $\frac{1}{t^2}+\frac{1}{n^2}=\frac{1}{t'n'}$

(C) t'n' = tn

(D) nt' = n't

Q.89 If a variable tangent to the curve $x^2y = c^3$ makes intercepts a, b on x and y axis respectively, then the value of a^2b is

(A) 27 c^3

(B) (4/27) c^3

(C) (27/4) c^3

(D) (4/9) c^3

Q.90 The slope of normal at the point with abscissa x = – 2 of the graph of the function $f(x) = |x^2 - |x||$ is –

(A) 1/3

(B) –1 / 3

(C) 1/6

(D) – 1/6

Q.91 Difference between the greatest and the least values of the function $f(x) = x(\ln x - 2)$ on $[1, e^2]$ is

(A) 2

(B) e

(C) e^2

(D) 1

Q.92 The true set of real values of x for which the function, $f(x) = x\ln x - x + 1$ is positive is –

(A) $(1, \infty)$

(B) $(1/e, \infty)$

(C) $[e, \infty]$

(D) $(0, 1) \cup (1, \infty)$

Q.93 For which values of 'a' will the function $f(x)=x^4+ax^3+\frac{3x^2}{2}+1$ will be concave upward along the entire real line

(A) $a\in[0,\infty)$

(B) $a\in(-2,2)$

(C) $a\in[-2,2]$

(D) $a\in(0,\infty)$

Q.94 If slope of $y=\frac{ax}{b-x}$ at (1, 1) be 2, then b =

(A) 0

(B) 2

(C) 1

(D) None of these

Q.95 If x and y are real numbers satisfying the relation $x^2 + y^2 - 6x + 8y + 24 = 0$ then minimum value of $f(x) = \log_2(x^2 + y^2)$ is –

(A) 1

(B) 2

(C) 3

(D) 4

Q.96 The curve $y=\frac{2x}{1+x^2}$ has –

(A) exactly three points of inflection separated by a point of maximum and a point of minimum.

(B) exactly two points of inflection with a point of maximum lying between them.

(C) exactly two points of inflection with a point of minimum lying between them.

(D) exactly three points of inflection separated by two points of maximum.

Q.97 If $f(x) = 2x^3 - 3(a+1)x^2 + 6ax - 12$ has maximum at x_1 and minimum at x_2 and if $2x_1 = x_2$ then value of 'a' is –

(A) 1

(B) 1/2

(C) –1

(D) 3

Q.98 Tangent of acute angle between the curves $y = |x^2 - 1|$ and $y=\sqrt{7-x^2}$ at their points of intersection is –

(A) $\frac{5\sqrt{3}}{2}$

(B) $\frac{3\sqrt{5}}{2}$

(C) $\frac{5\sqrt{3}}{4}$

(D) $\frac{3\sqrt{5}}{4}$

Q.99 The difference between greatest and least value of $f(x) = 2\sin x + \sin 2x, \quad x \in \left[0, \frac{3\pi}{2}\right]$ is –

(A) $\frac{3\sqrt{3}}{2}$ (B) $\frac{3\sqrt{3}}{2} - 2$

(C) $\frac{3\sqrt{3}}{2} + 2$ (D) None of these

Q.100 The number of tangents to the curve $x^{3/2} + y^{3/2} = 2a^{3/2}$, $a > 0$ which are equally inclined to the axes, is –

(A) 2 (B) 1

(C) 0 (D) 4

Q.101 Coffee is draining from a conical filter, height and diameter both 15 cms into a cylinderical coffee pot diameter 15 cm. The rate at which coffee drains from the filter into the pot is 100 cu cm /min. The rate in cms/min at which the level in the pot is rising at the instant when the coffee in the pot is 10 cm, is

(A) $\frac{9}{16\pi}$ (B) $\frac{25}{9\pi}$

(C) $\frac{5}{3\pi}$ (D) $\frac{16}{9\pi}$

EXERCISE - 2 [LEVEL-2]

Q.1 The tangent to the curve $xy = 25$ at any point on it cuts the coordinate axes at A B, then the area of the triangle OAB is

(A) 100 sq. units (B) 50 sq. units

(C) 25 sq. units (D) 75 sq. units

Q.2 The tangent to the curve $y = x^3 + 1$ at (1, 2) makes an angle θ with y-axis, then the value of $\tan\theta$ is

(A) –1/3 (B) 3

(C) –3 (D) 1/3

Q.3 The two curves $x^3 - 3y^2 + 2 = 0$ and $3x^2y - y^3 = 2$

(A) touch each other (B) cut at right angle

(C) cut at angle $\pi/3$ (D) cut at angle $\pi/4$

Q.4 The greatest value of the function

$$f(x) = \tan^{-1} x - \frac{1}{2}\log x \ \text{ in } \left[\frac{1}{\sqrt{3}}, \sqrt{3}\right] \text{ is –}$$

(A) $\frac{\pi}{6} + \frac{1}{4}\log 3$ (B) $\frac{\pi}{6} - \frac{1}{4}\log 3$

(C) $\frac{\pi}{3} - \frac{1}{4}\log 3$ (D) $\frac{\pi}{3} + \frac{1}{2}\log 3$

Q.5 Let $f(x) = x + \tan^3 x$, g (x) is inverse function of f (x), find $\sqrt{343g'\left(\frac{\pi}{4}+1\right)}$.

(A) 3 (B) 7

(C) 2 (D) 9

Q.6 The point(s) on the curve $y^3 + 3x^2 = 12y$ where the tangent is vertical (parallel to y-axis), is (are)

(A) $\left(\pm\frac{4}{\sqrt{3}}, -2\right)$ (B) $\left(\pm\frac{\sqrt{11}}{3}, 1\right)$

(C) (0, 0) (D) $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

Q.7 All the values of λ for which the curve

$y = \frac{x^4}{4} - \frac{3x^2}{2} + \lambda x - 3$ has three tangents parallel to the axis of x lie in the interval (–k,k) then find the integral value of λ.

(A) 3 (B) 7

(C) 2 (D) 9

Q.8 If the normal at the point " t_1" on the curve $xy = c^2$ meets the curve again at " t_2", then

(A) $t_1^3 t_2 = 1$ (B) $t_1^3 t_2 = -1$

(C) $t_1 t_2^3 = -1$ (D) $t_1 t_2^3 = 1$

Q.9 The curve $y^2 = 2x$ and $2xy = k$ cut at right angles if

(A) $k^2 = 8$ (B) $k^2 = 4$

(C) $k^2 = 2$ (D) None of these

Q.10 If $f(x) = xe^{x(1-x)}$, then f (x) is

(A) increasing in [–1/2, 1] (B) decreasing in R

(C) increasing in R (D) decreasing in [–1/2, 1]

Q.11 Function $f(x) = \tan^{-1}(\sin x + \cos x)$ is monotonic increasing when

(A) $x < 0$ (B) $x > 0$

(C) $0 < x < \pi/2$ (D) $0 < x < \pi/4$

Q.12 $f(x) = 2x^2 - \log|x|\ (x \neq 0)$ is monotonic increasing in the interval

(A) $(1/2, \infty)$ (B) $(-\infty, -1/2) \cup (1/2, \infty)$

(C) $(-\infty, -1/2) \cup (0, 1/2)$ (D) $(-1/2, 0) \cup (1/2, \infty)$

Q.13 The function $f(x) = 2\log(x-2) - x^2 + 4x + 1$ increases on the interval

(A) (1, 2) (B) (2, 3)

(C) (5/2, 3) (D) Both (B) and (C)

Q.14 If the relation between sub-normal SN and sub-tangent ST at any point S on the curve; $by^2 = (x + a)^3$ is $p\,(SN) = q(ST)^2$, then the value of p/q –

(A) 8a/27 (B) 27/8b

(C) 8b/27 (D) 8/27

Q.15 A ladder 10 meters long rests with one end against a vertical wall, the other end on the floor, the lower end moves away from the wall at the rate of 2 meter/minute. The rate at which the upper end falls when its base is 6 meters away from the wall, is –

(A) – 3 meters/min. (B) –2/3 meters/min.

(C) – 3/2 meters/min. (D) None of these

Q.16 The function $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3$, $(0 \le x \le \pi)$ is –

(A) Increasing in $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$

(B) Increasing in $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{2\pi}{3}, \pi\right)$

(C) Decreasing in $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$

(D) all of above

Q.17 The interval in which the function $2x^3 + 15$ increases less rapidly than the function $9x^2 - 12x$, is –

(A) $(-\infty, 1)$ (B) $(1, 2)$
(C) $(2, \infty)$ (D) None of these

Q.18 AB is a diameter of a circle and C is any point the circumference of the circle, then –

(A) area of Δ ABC is maximum when it is an isosceles
(B) area of Δ ABC is minimum when it is isosceles
(C) the perimeter of Δ ABC is minimum when it is isosceles
(D) the perimeter of Δ ABC is maximum when it is isosceles

Q.19 The interval in which $f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is decreasing

(A) $(-\infty, \infty)$ (B) $(-\infty, 0)$
(C) $(0, \infty)$ (D) $(1, \infty)$

Q.20 The maximum value of $x^{1/x}$ is –

(A) $(1/e)^e$ (B) $e^{1/e}$
(C) e (D) 1/e

Q.21 The altitude of a cone is 20cm. and its semi-vertical angle is 30°. If the semi-vertical angle is increasing at the rate of 2° per second, then the radius of the base is increasing at the rate of –

(A) 30 cm/sec (B) 160/3 cm/sec
(C) 10 cm/sec (D) 160 cm/sec.

Q.22 If $y = a\log|x| + bx^2 + x$ has its extremum values at $x = -1$ and $x = 2$, then

(A) $a = 2, b = -1$ (B) $a = 2, b = -1/2$
(C) $a = -2, b = 1/2$ (D) None of these

Q.23 The function $\frac{(e^{2x} - 1)}{(e^{2x} + 1)}$ is

(A) Increasing (B) Odd
(C) Even (D) Both (A) and (B)

Q.24 The values of 'a' for which the function $(a+2)x^3 - 3ax^2 + 9ax - 1$ decreases monotonically throughout for all real x, are

(A) $a < -2$ (B) $a > -2$
(C) $-3 < a < 0$ (D) $-\infty < a \le -3$

Q.25 Find the coordinates of a point of the parabola $y = x^2 + 7x + 2$ which is closest to the straight line $y = 3x - 3$.

(A) (–2, –8) (B) (–3, –7)
(C) (–1, –6) (D) (–5, –9)

Q.26 If $z = y + f(v)$, where $v = \left(\frac{x}{y}\right)$ then $v\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$ is

(A) –1 (B) 1
(C) 0 (D) 2

Q.27 Co-ordinates of a point on the curve $y = x \log x$ at which the normal is parallel to the line $2x - 2y = 3$ are

(A) (0,0) (B) (e, e)
(C) $(e^2, 2e^2)$ (D) $(e^{-2}, -2e^{-2})$

Q.28 The equation of the tangent to curve $y = be^{-x/a}$ at the point where it crosses y-axis is

(A) $ax + by = 1$ (B) $ax - by = 1$
(C) $\frac{x}{a} - \frac{y}{b} = 1$ (D) $\frac{x}{a} + \frac{y}{b} = 1$

Q.29 If $f(x) = 3x^2 + 15x + 5$, then the approximate value of $f(3.02)$ is –

(A) 47.66 (B) 57.66
(C) 67.66 (D) 77.66

Q.30 The curve given by $x + y = e^{xy}$ has a tangent parallel to the y-axis at the point

(A) (0, 1) (B) (1, 0)
(C) (1, 1) (D) (–1, –1)

Q.31 Find the minimum value of the function

$$\frac{40}{3x^4 + 8x^3 - 18x^2 + 60}.$$

(A) 1 (B) 1/4
(C) 1/2 (D) 2/3

Q.32 The largest term in the sequence $a_n = \frac{n^2}{n^3 + 200}$ is given by

(A) 529/49 (B) 8/29
(C) 49/543 (D) None of these

Q.33 What are the minimum and maximum values of the function $x^5 - 5x^4 + 5x^3 - 10$

(A) –37, –9
(B) 10, 0
(C) It has 2 min. and 1 max. values
(D) It has 2 max. and 1 min. values

Q.34 The maximum value of $f(x) = \frac{x}{4 + x + x^2}$ on [–1, 1] is

(A) –1/4 (B) –1/3
(C) 1/6 (D) 1/5

Q.35 If P = (1, 1), Q = (3, 2) and R is a point on x-axis then the value of PR + RQ will be minimum at

(A) (5/3, 0) (B) (1/3, 0)
(C) (3, 0) (D) (1, 0)

Q.36 Function $f(x) = x^4 - \frac{x^3}{3}$ is

(A) Increasing for x > 1/4 and decreasing for x < 1/4
(B) Increasing for every value of x
(C) Decreasing for every value of x
(D) None of these

Q.37 Find the minimum value of 64 secx + 27 cosecx , $0 < x < \pi/2$

(A) 137 (B) 125

(C) 25 (D) 75

Q.38 In its domain, $f(x) = \dfrac{\sin^{-1} x}{\cot^{-1} x}$ is –

(A) a increasing function

(B) a strictly increasing function

(C) a decreasing function

(D) a strictly decreasing function

Q.39 Number of point of inflexion on curve y = g (x) such that

$$g'(x) = \frac{x^2 - 2x - 1}{(x-1)^2}$$

(A) 0 (B) 1

(C) 2 (D) >2

Q.40 The curve $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at point P (–2, 0) and cuts y-axis at a point Q where its gradient is 3. Find a, b, c.

(A) 0, 0, 3 (B) $\frac{1}{2}, \frac{1}{4}, 3$

(C) $\frac{1}{4}, \frac{1}{2}, 3$ (D) $-\frac{1}{2}, \frac{-3}{4}, 3$

Q.41 Tangent to the curve $x = a\sqrt{\cos 2\theta \cos\theta}$,

$y = a\sqrt{\cos\theta}\sin\theta$ at the point corresponding to $\theta = \dfrac{\pi}{6}$ is–

(A) parallel to the x-axis (B) parallel to y-axis

(C) parallel to y = x (D) None of these

Q.42 If curves $y^2 = 6x$ and $9x^2 + by^2 = 16$, intersect orthogonally then b =

(A) 4 (B) 2

(C) 9/2 (D) 2/9

Q.43 Find the number of critical points of $f(x) = \dfrac{|x-1|}{x^2}$.

(A) 4 (B) 2

(C) 1 (D) 3

Q.44 If the function $f(x) = \dfrac{cx + d}{(x-1)(x-4)}$ has a turning point at the point (2, –1) then –

(A) c = 2, d = 0 (B) c = 1, d = 0

(C) c = 1, d = – 1 (D) c = 1, d = 1

Q.45 Find the value of n for which the area of the triangle formed by the axes of coordinates and any tangent to the curve $x^n y = a^n$ is constant.

(A) 4 (B) 2

(C) 1 (D) 3

Q.46 If equation of normal at point $(m^2, -m^3)$ on the curve $x^3 - y^2 = 0$ is $y = mx - 2m^3$, then m^2 equals –

(A) 2/9 (B) –2/9

(C) 2/3 (D) – 2/3

Q.47 If tangent at any point of the curve $y = x^3 + \lambda x^2 + x + 5$ makes acute angle with x-axis, then –

(A) $0 < \lambda < 3$ (B) $-\sqrt{3} < \lambda < \sqrt{3}$

(C) $|\lambda| < 1$ (D) $\lambda \in (0, 1)$

Q.48 Cosine of the angle of intersection of curves $y = 3^{x-1} \log x$ and $y = x^x - 1$ is –

(A) 0 (B) 1

(C) 1/2 (D) 1/3

Q.49 If m be the slope of a tangent to the curve $e^y = 1 + x^2$ then–

(A) $|m| > 1$ (B) $m < 1$

(C) $|m| < 1$ (D) $|m| \le 1$

Q.50 Let $f(x) = e^x \cos x$ and slope of the curve y = f (x) is maximum at x = a then a equals –

(A) 0 (B) π/2

(C) 3π/2 (D) None of these

Q.51 The point of the curve $y = x^2$ that it closest to (4, –1/2) is

(A) (1, 1) (B) (2, 4)

(C) (2/3, 4/9) (D) (4/3, 16/9)

Q.52 If $0 \le x \le 1$ and $f(x) = \begin{vmatrix} x & 1 & 1 \\ -1 & x & 1 \\ 1 & -1 & x \end{vmatrix}$ then –

(A) f (x) has local maximum at x = 2/3

(B) f (x) has local minimum at x = 1/3

(C) least value of f (x) is 2

(D) least value of f (x) is 4

Q.53 Angle between two curves y = f (x) and y = g (x) is the angle between tangents to these curves at the common point of intersection.

Given curves are $y = |x^2 - 1|$ and $y = |x^2 - 3|$.

Choose the correct options –

(A) The common point of intersection is

$(\sqrt{2}, 1)$ & $(-\sqrt{2}, 1)$

(B) The acute angle between the curves at their point of intersection is $\tan^{-1}\left(\frac{3\sqrt{2}}{7}\right)$ & $\pi - \tan^{-1}\left(\frac{3\sqrt{2}}{7}\right)$

(C) only (A) is correct

(D) Both (A) and (B) are correct

Q.54 If $f''(x) > 0 \ \forall \ x \in R$, $f'(3) = 0$ and $g(x) = f(\tan^2 x - 2\tan x + 4)$, $0 < x < \pi/2$, then g (x) is increasing in –

(A) $\left(0, \frac{\pi}{4}\right)$ (B) $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ (C) $\left(0, \frac{\pi}{3}\right)$ (D) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Q.55 Let $a \in (0, 4/27)$ be such that $r_1 = a(1 + 2r_1 + 3r_1^2 + ...\infty)$, $r_2 = a(1 + 2r_2 + 3r_2^2 + ...\infty)$, $r_1 < r_2$ then –

(A) $r_1 \in \left(0, \frac{1}{3}\right), r_2 \in \left(\frac{1}{3}, 1\right)$ (B) $r_1 \in \left(-\frac{1}{3}, 0\right), r_2 \in \left(0, \frac{1}{3}\right)$

(C) $r_1, r_2 \in \left(0, \frac{1}{3}\right)$ (D) $r_1, r_2 \in \left(-\frac{1}{3}, 0\right)$

Directions : Assertion-Reason type questions.

Each questions contain Statement -1 (Assertion) and Statement -2 (Reason). Each question has 4 choices (A), (B), (C) and (D) out of which ONLY ONE is correct.

(A) Statement-1 is True, Statement-2 is True,Statement-2 is a correct explanation for Statement -1

(B) Statement-1 is True, Statement-2 is True;Statement-2 is NOT a correct explanation for Statement - 1

(C) Statement - 1 is True, Statement- 2 is False

(D) Statement -1 is False, Statement -2 is True

Q.56 **Statement 1 :** If $f(x) = (x - 3)^3$, then f (x) has neither maximum nor minimum at $x = 3$

Statement 2 : $f'(x) = 0, f''(x) = 0$ at $x = 3$.

Q.57 **Statement 1 :** If $f(x) = \max. \{x^2 - 2x + 2, |x - 1|\}$ the greatest value of f (x) on the interval [0, 3] is 5.

Statement 2 : Greatest value of $f(C) = \max. \{5, 2\} = 5$.

Q.58 **Statement 1 :** Sum of left hand derivative and right hand derivative of $f(x) = |x^2 - 5x + 6|$ at $x = 2$ is equal to zero.

Statement 2 : Sum of left hand derivative and right hand derivative of $f(x) = |(x - a)(x - b)|$ at $x = a$ $(a < b)$ is equal to zero, (where $a, b \in R$)

Q.59 **Statement 1 :** $f(x) = x + \cos x$ is strictly increasing.

Statement 2 : If f (x) is strictly increasing, then f' (x) may vanish at some infinite number of points.

Q.60 Let $u = \sqrt{C+1} - \sqrt{C}$, $v = \sqrt{C} - \sqrt{C-1}$ and let $f(x) = \ln(1 + x) \; \forall \; x \in (-1, \infty)$

Statement 1 : $f(u) > f(v) \; \forall \; C > 1$

Statement 2 : f (x) is increasing hence for $u > v$, $f(u) > f(v)$.

Q.61 **Statement -1 :** In a triangle ABC if sides a, b are constants and the base angles A and B vary, then

$$\frac{dA}{\sqrt{a^2 - b^2 \sin^2 A}} = \frac{dB}{\sqrt{b^2 - a^2 \sin^2 B}}$$

Statement -2 : In a triangle ABC, b sin A = a sin B.

Q.62 **Statement 1 :** The tangent at x = 1 to the curve $y = x^3 - x^2 - x + 2$ again meets the curve at $x = -2$.

Statement 2 : When a equation of a tangent solved with the curve, repeated roots are obtained at point of tangency.

Q.63 **Statement 1 :** Tangent drawn at the point (0, 1) to the curve $y = x^3 - 3x + 1$ meets the curve thrice at one point only.

Statement 2 : Tangent drawn at the point (1, –1) to the curve $y = x^3 - 3x + 1$ meets the curve at 1 point only.

Q.64 **Statement 1 :** Let $f : [0, \infty) \to [0, \infty)$ and $g : [0, \infty) \to [0, \infty)$ be non-increasing and non-decreasing functions respectively and $h(x) = g(f(x))$. If f and g are differentiable for all points in their respective domains and h (0) = 0 then h (x) is constant function.

Statement 2 : $g(x) \in [0, \infty) \Rightarrow h(x) \geq 0$ and $h'(x) \leq 0$

Q.65 Let $f : R \to R$ is differentiable and strictly increasing function throughout its domain.

Statement 1 : If | f (x) | is also strictly increasing function, then f (x) = 0 has no real roots.

Statement 2 : At ∞ or – ∞, f (x) may approach to 0, but cannot be equal to zero.

Passage (Q.66-Q.68)

a (t) is a function of t such that da/dt = 2 for all values of t and a = 0 when t = 0. Further $y = m(t)x + c(t)$ is tangent to the curve $y = x^2 - 2ax + a^2 + a$ at the point whose abscissa is 0. Then –

Q.66 If the rate of change of distance of vertex of $y = x^2 - 2ax + a^2 + a$ from the origin with respect to t is k, then k =

(A) 2 (B) $2\sqrt{2}$

(C) $\sqrt{2}$ (D) $4\sqrt{2}$

Q.67 If the rate of change of c (t) with respect to t, when t = k is ℓ, then –

(A) $16\sqrt{2} - 2$ (B) $8\sqrt{2} + 2$

(C) $10\sqrt{2} + 2$ (D) $16\sqrt{2} + 2$

Q.68 The rate of change of m (t), with respect to t, at $t = \ell$ is –

(A) –2 (B) 2

(C) – 4 (D) 4

Q.69 Assume that f is continuous on [a, b], a > 0 and differentiable on an open interval (a, b). If $\frac{f(a)}{a} = \frac{f(b)}{b}$, then there exist $x_0 \in (a, b)$ such that

(A) $x_0 f'(x_0) = f(x_0)$ (B) $f'(x_0) + x_0 f(x_0) = 0$

(C) $x_0 f'(x_0) + f(x_0) = 0$ (D) $f'(x_0) = x_0^2 f(x_0)$

Q.70 Number of positive integral values of 'a' for which the curve $y = a^x$ intersects the line y = x is –

(A) 0 (B) 1

(C) 2 (D) More than 2

Q.71 Point 'A' lies on the curve $y = e^{-x^2}$ and has the coordinate (x, e^{-x^2}) where x > 0. Point B has the coordinates (x, 0). If 'O' is the origin then the maximum area of the triangle AOB is

(A) $\frac{1}{\sqrt{2e}}$ (B) $\frac{1}{\sqrt{4e}}$ (C) $\frac{1}{\sqrt{e}}$ (D) $\frac{1}{\sqrt{8e}}$

Q.72 The angle at which the curve $y = Ke^{Kx}$ intersects the y-axis is :

(A) $\tan^{-1} k^2$ (B) $\cot^{-1}(k^2)$

(C) $\sec^{-1}(\sqrt{1 + k^4})$ (D) none

Q.73 The minimum value of the polynomial $x(x+1)(x+2)(x+3)$ is :

(A) 0 (B) 9/16

(C) – 1 (D) – 3/2

Q.74 A curve y = f(x) passes through the point P(1, 1). The normal to the curve at P is a (y – 1) + (x – 1) = 0. If the slope of the tangent at any point on the curve is proportional to the ordinate of the point, then the equation of the curve is-

(A) $y = e^{a(x-1)}$ (B) $y = e^{a(1-x)}$

(C) $y = e^{a/2(x-1)}$ (D) $e^{a/2(x+1)}$

Q.75 The minimum value of $\dfrac{\tan\left(x+\dfrac{\pi}{6}\right)}{\tan x}$ is :

(A) 0 (B) 1/2
(C) 1 (D) 3

Q.76 The values of x for which

$1 + x\ln\,(x+\sqrt{x^2+1}) \ge \sqrt{1+x^2}$ are-

(A) $x \le 0$ (B) $0 \le x \le 1$
(C) $x \ge 0$ (D) None of these

Q.77 Let $f(x) = \dfrac{\tan^n x}{\sum\limits_{r=0}^{2n} \tan^r x}$, $n \in N$, where $x \in [0, \pi/2)$

(A) f (x) is bounded and it takes both of it's bounds and the range of f (x) contains exactly one integral point.

(B) f (x) is bounded and it takes both of it's bounds and the range of f (x) contains more than one integral point.

(C) f (x) is bounded but minimum and maximum does not exists.

(D) f (x) is not bounded as the upper bound does not exist.

Q.78 Let C be the curve $y = x^3$ (where x takes all real values). The tangent at A meets the curve again at B. If the gradient at B is K times the gradient at A then K is equal to

(A) 4 (B) 2
(C) –2 (D) 1/4

Q.79 Which of the following statement is true for the function

$$f(x) = \begin{cases} \sqrt{x} & x \ge 1 \\ x^3 & 0 \le x \le 1 \\ \dfrac{x^3}{3} - 4x & x < 0 \end{cases}$$

(A) It is monotonic increasing $\forall\ x \in R$.

(B) $f'(x)$ fails to exist for 3 distinct real values of x.

(C) $f'(x)$ changes its sign twice as x varies from $(-\infty, \infty)$.

(D) function attains its extreme values at x_1 & x_2, such that $x_1, x_2 > 0$.

NOTE : The answer to each question is a NUMERICAL VALUE.

Q.80 If $f(x) = 7e^{\sin^2 x} - e^{\cos^2 x} + 2$, then find the value of $\sqrt{7f_{min} + f_{max}}$.

Q.81 Let F (x) be a cubic polynomial defined by

$F(x) = \dfrac{x^3}{3} + (a-3)\,x^2 + x - 13$. Find the sum of all possible integral value(s) of 'a' for which F (x) has negative point of local minimum in the interval [1, 100].

Q.82 Let $F(x) = \begin{cases} -2x + \log_{1/2}(k^2 - 6k + 8), & -2 \le x < -1 \\ x^3 + 3x^2 + 4x + 1 & , -1 \le x \le 3 \end{cases}$

Find the sum of all possible positive integer(s) in the range of k such that F (x) has the smallest value at x = –1.

Q.83 Let $P(x) = x^{10} + a_1x^9 + a_2x^8 + ... + a_{10}$ be a polynomial with real coefficients. Suppose P (0) = –1, P (1) = 2, P (2) = –1.
Let R be the number of real zero's of P (x) then R ≥ A. Find the value of A.

Q.84 If 1200 sq. cm of material is available to make a box with a square base and an open top. Find the largest possible volume of the box (in cubic cm).

Q.85 The lower corner of a leaf in a book is folded over so as to just reach the inner edge of the page. The fraction of width folded over if the area of the folded part is minimum is A/3. Find the value of A.

Q.86 A rectangle with one side lying along the x-axis is to be inscribed in the closed region of the xy plane bounded by the lines y = 0, y = 3x, and y = 30 – 2x. The largest area of such a rectangle is 135/A. Find the value of A.

Q.87 A closed vessel tapers to a point both at its top E and its bottom F and is fixed with EF vertical when the depth of the liquid in it is x cm, the volume of the liquid in it is, $x^2(15 - x)$ cu. cm. The length EF is (in cm.)

Q.88 A horse runs along a circle with a speed of 20 km/hr. A lantern is at the centre of the circle . A fence is along the tangent to the circle at the point at which the horse starts. The speed with which the shadow of the horse move along the fence at the moment when it covers 1/8 of the circle in km/hr is –

Q.89 The radius of a right circular cylinder increases at a constant rate. Its altitude is a linear function of the radius and increases three times as fast as radius. When the radius is 1cm the altitude is 6 cm. When the radius is 6cm, the volume is increasing at the rate of 1Cu cm/sec. When the radius is 36cm, the volume is increasing at a rate of n cu. cm/sec. The value of 'n' is equal to:

Q.90 The number of values of x where the function

$f(x) = \cos x + \cos\,(\sqrt{2}x)$ attains its maximum is

Q.91 Tangents are drawn from P(6, 8) to the circle $x^2 + y^2 = r^2$ Find the radius of the circle such that the area of the triangle formed by tangents and chord of contact is maximum.

Q.92 If y is a function of x and log (x + y) – 2xy = 0, then the value of y' (0) is equal to

Q.93 If f(x) is a twice differentiable function such that f (a) = 0, f (b) = 2, f (c) = 1, f (d) = 2, f (e) = 0, where a < b < c < d< e, then the minimum number of zeroes of $g(x) = (f'(x))^2 + f''(x) \,.\, f(x)$ in the interval [a, e] is _____.

Q.94 The total number of local maxima and local minima of the function $f(x)=\begin{cases}(2+x)^3, & -3<x\le -1\\ x^{2/3} & , -1<x<2\end{cases}$

Q.95 The maximum value of the function $f(x)=2x^3-15x^2+36x-48$ on the set $A=\{x\,|\,x^2+20\le 9x\}$ is

Q.96 Let f be a function defined on R (the set of all real numbers) such that $f'(x)=2010\,(x-2009)\,(x-2010)^2\,(x-2011)^3\,(x-2012)^4$, for all $x\in R$. If g is a function defined on R with values in the interval $(0,\infty)$ such that $f(x)=\ln(g(x))$, for all $x\in R$, then the number of points in R at which g has a local maximum is :

Q.97 The number of distinct real roots of $x^4-4x^3+12x^2+x-1=0$ is

Q.98 Let $f: IR\to IR$ be defined as $f(x)=|x|+|x^2-1|$. The total number of points at which *f* attains either a local maximum or a local minimum is

Q.99 Let p (x) be a real polynomial of least degree which has a local maximum at x = 1 and a local minimum at x = 3. If p (1) = 6 and p (3) = 2, then p' (0) is –

Q.100 The number of points in $(-\infty,\infty)$, for which $x^2-x\sin x-\cos x=0$, is –

EXERCISE - 3 [PREVIOUS YEARS JEE MAIN QUESTIONS]

Q.1 If the function $f(x) = 2x^3 - 9ax^2 + 12a^2 x + 1$, where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a equals **[AIEEE 2003]**
(A) 1/2 (B) 3
(C) 1 (D) 2

Q.2 The real number x when added to its inverse gives the minimum value of the sum at x equal to- **[AIEEE 2003]**
(A) – 2 (B) 2
(C) 1 (D) – 1

Q.3 If $u = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} + \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$ then the difference between the maximum and minimum values of u^2 is given by – **[AIEEE 2004]**
(A) $2(a^2 + b^2)$ (B) $2\sqrt{a^2 + b^2}$
(C) $(a + b)^2$ (D) $(a - b)^2$

Q.4 A function y = f(x) has a second order derivative $f''(x) = 6(x - 1)$. If its graph passes through the point (2,1) and at that point the tanget to the graph is $y = 3x - 5$, then the function, is- **[AIEEE 2004]**
(A) $(x-1)^2$ (B) $(x-1)^3$
(C) $(x+1)^3$ (D) $(x+1)^2$

Q.5 The normal to the curve $x = a(1+ \cos \theta)$, $y = a \sin \theta$ at 'θ' always passes through the fixed point- **[AIEEE 2004]**
(A) (a, 0) (B) (0, a)
(C) (0,0) (D) (a, a)

Q.6 If $2a + 3b + 6c = 0$, then at least one root of the equation $ax^2 + bx + c = 0$ lies in the interval- **[AIEEE 2004]**
(A) (0,1) (B) (1,2)
(C) (2,3) (D) (1,3)

Q.7 If the equation $a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x = 0$; $a_1 \neq 0$, $n \geq 2$, has a positive root $x = \alpha$, then the equation $na_n x^{n-1} + (n - 1) a_{n-1} x^{n-2} + ... + a_1 = 0$ has a positive root, which is - **[AIEEE-2005]**
(A) greater than α (B) smaller than α
(C) greater than or equal to α (D) equal to α

Q.8 The normal to the curve $x = a (\cos \theta + \theta \sin \theta)$, $y = a (\sin \theta - \theta \cos \theta)$ at any point 'θ' is such that -
(A) it passes through the origin **[AIEEE-2005]**
(B) it makes anlge $\frac{\pi}{2} + \theta$ with the x-axis
(C) it passes through $\left(a\frac{\pi}{2}, -a\right)$
(D) it is at a constant distance from the origin.

Q.9 A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of $50 cm^3/min$. When the thickness of ice is 5 cm, then the rate of which the thickness of ice decreases, is - **[AIEEE-2005]**
(A) $\frac{1}{36\pi}$ cm/min. (B) $\frac{1}{18\pi}$ cm/min.
(C) $\frac{1}{54\pi}$ cm/min. (D) $\frac{5}{6\pi}$ cm/min.

Q.10 A function is matched below against an interval whre it is supposed to be increasing. Which of the follownig pairs is incorrectly matched ? **[AIEEE-2005]**

Interval	function
(A) $(-\infty, \infty)$	$x^3 + 6x^2 + 6$
(B) $[2, \infty)$	$3x^2 - 2x + 1$
(C) $(-\infty, -4]$	$x^3 - 3x^2 + 3x + 3$
(D) $\left(-\infty, \frac{1}{3}\right]$	$2x^3 - 3x^2 - 12x + 6$

Q.11 Let f be differentiable for all x. If $f(1) = -2$ and $f'(x) \geq 2$ for $x \in [1, 6]$, then - **[AIEEE-2005]**
(A) $f(6) \geq 8$ (B) $f(6) < 8$
(C) $f(6) < 5$ (D) $f(6) = 5$

Q.12 Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) is – **[AIEEE 2006]**
(A) π/2 (B) π/6
(C) π/4 (D) π/3

Q.13 The function $f(x) = \frac{x}{2} + \frac{2}{x}$ has a local minimum at –
(A) $x = -2$ (B) $x = 0$ **[AIEEE 2006]**
(C) $x = 1$ (D) $x = 2$

Q.14 A triangular park is enclosed on two sides by a fence and on the third side by a straight river bank. The two sides having fence are of same length x. The maximum area enclosed by the park is – **[AIEEE 2006]**
(A) $\sqrt{\frac{x^3}{8}}$ (B) $\frac{1}{2}x^2$
(C) πx^2 (D) $\frac{3}{2}x^2$

Q.15 A value of C for which the conclusion of Mean Value Theorem holds for the function $f(x) = \log_e x$ on the interval [1, 3] is- **[AIEEE 2007]**
(A) $2\log_3 e$ (B) $(1/2) \log_e 3$
(C) $\log_3 e$ (D) $\log_e 3$

Q.16 The function $f(x) = \tan^{-1} (\sin x + \cos x)$ is an increasing function in- **[AIEEE 2007]**
(A) (π/4, π/2) (B) (–π/2, π/4)
(C) (0, π/2) (D) (–π/2, π/2)

Q.17 If p and q are positive real numbers such that $p^2 + q^2 = 1$, then the maximum value of (p + q) is- **[AIEEE 2007]**
(A) 2 (B) 1/2
(C) $1/\sqrt{2}$ (D) $\sqrt{2}$

Q.18 Suppose the cubic $x^3 - px + q$ has three distinct real roots where $p > 0$ and $q > 0$. Then which one of the following holds ? **[AIEEE 2008]**

(A) The cubic has minima at $-\sqrt{\frac{p}{3}}$ and maxima at $\sqrt{\frac{p}{3}}$

(B) The cubic has manima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

(C) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

(D) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $-\sqrt{\frac{p}{3}}$

Q.19 Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval $[-1,1]$ - **[AIEEE 2009]**

(A) P (–1) is the minimum and P(1) is the maximum of P
(B) P(–1) is not minimum but P(1) is the maximum of P
(C) P(–1) is the minimum but P(1) is not the maximum of P
(D) Neither P(–1) is the minimum nor P(1) is the maximum of P

Q.20 The shortest distance between the line $y - x = 1$ and the curve $x = y^2$ is - **[AIEEE 2009]**

(A) $\frac{3\sqrt{2}}{8}$ (B) $\frac{2\sqrt{3}}{8}$
(C) $\frac{3\sqrt{2}}{5}$ (D) $\frac{\sqrt{3}}{4}$

Q.21 The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x-axis, is – **[AIEEE 2010]**

(A) y = 1 (B) y = 2
(C) y = 3 (D) y = 0

Q.22 Let $f : R \to R$ be defined by $f(x) = \begin{cases} k - 2x, & \text{if } x \le -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$.

If f has a local minimum at x = –1, then a possible value of k is – **[AIEEE 2010]**

(A) 0 (B) –1/2
(C) –1 (D) 1

Q.23 For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t}\,\sin t\, dt$. Then f has :

(A) local maximum at π and 2π. **[AIEEE 2011]**
(B) local minimum at π and 2π
(C) local minimum at π and local maximum at 2π (D) local maximum at π and local minimum at 2π

Q.24 The shortest distance between line $y - x = 1$ and curve $x = y^2$ is – **[AIEEE 2011]**

(A) $\frac{\sqrt{3}}{4}$ (B) $\frac{3\sqrt{2}}{8}$ (C) $\frac{8}{3\sqrt{2}}$ (D) $\frac{4}{\sqrt{3}}$

Q.25 A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is : **[AIEEE 2012]**

(A) 9/7 (B) 7/9
(C) 2/9 (D) 9/2

Q.26 Let $a, b \in R$ be such that the function f given by $f(x) = \ln|x| + bx^2 + ax,\ x \ne 0$ has extreme values at $x = -1$ and $x = 2$. **[AIEEE 2012]**

Statement-1 : f has local maximum at x = – 1 and at x = 2.

Statement-2 : a = 1/2 and b = –1/4

(A) Statement-1 is false, Statement-2 is true.
(B) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1.
(C) Statement-1 is true, statement-2 is true; statement-2 is not a correct explanation for Statement-1.
(D) Statement-1 is true, statement-2 is false.

Q.27 The real number k for which the equation, $2x^3 + 3x + k = 0$ has two distinct real roots in [0, 1] **[JEE MAIN 2013]**

(A) lies between 1 and 2 (B) lies between 2 and 3
(C) lies between –1 and 0 (D) does not exist.

Q.28 The intercepts on x-axis made by tangents to the curve, $y = \int_0^x |t|\, dt,\ x \in R$, which are parallel to the line y = 2x, are equal to – **[JEE MAIN 2013]**

(A) ± 1 (B) ± 2
(C) ± 3 (D) ± 4

Q.29 If f and g are differentiable functions in [0, 1] satisfying $f(0) = 2 = g(1), g(0) = 0$ and $f(1) = 6$, then for some $c \in]0, 1[$ **[JEE MAIN 2014]**

(A) $2f'(c) = g'(c)$ (B) $2f'(c) = 3g'(c)$
(C) $f'(c) = g'(c)$ (D) $f'(c) = 2g'(c)$

Q.30 If x = –1 and x = 2 are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$, then – **[JEE MAIN 2014]**

(A) α = –6, β = 1/2 (B) α = –6, β = –1/2
(C) α = 2, β = –1/2 (D) α = 2, β = 1/2

Q.31 Let f (x) be a polynomial of degree four having extreme values at x = 1 and x = 2. If $\lim_{x \to 0}\left[1 + \frac{f(x)}{x^2}\right] = 3$, then f(2)=

(A) – 4 (B) 0 **[JEE MAIN 2015]**
(C) 4 (D) –8

Q.32 The normal to the curve, $x^2 + 2xy - 3y^2 = 0$ at (1, 1) **[JEE MAIN 2015]**

(A) Meets the curve again in the second quadrant
(B) Meets the curve again in the third quadrant:
(C) Meets the curve again in the fourth quadrant
(D) Does not meet the curve again

Q.33 A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then: **[JEE MAIN 2016]**

(A) $(4-\pi)x = \pi r$ (B) $x = 2r$
(C) $2x = r$ (D) $2x = (\pi + 4) r$

Q.34 Consider $f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right), x \in \left(0, \frac{\pi}{2}\right)$.

A normal to y = f (x) at x = π/6 also passes through the point **[JEE MAIN 2016]**

(A) (0, 2π/3) (B) (π/6, 0)
(C) (π/4, 0) (D) (0, 0)

Q.35 The normal to the curve y (x – 2) (x – 3) = x + 6 at the point where the curve intersects the y-axis, passes through the point : **[JEE MAIN 2017]**

(A) (1/2, –1/3) (B) (1/2, 1/3)
(C) (–1/2, –1/2) (D) (1/2, 1/2)

Q.36 Twenty meters of wire is available for fencing off a flower-bed in the form of a circular sector. Then the maximum area (in sq. m) of the flower-bed, is : **[JEE MAIN 2017]**

(A) 25 (B) 30
(C) 12.5 (D) 10

Q.37 If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is: **[JEE MAIN 2018]**

(A) 4 (B) 9/2
(C) 6 (D) 7/2

Q.38 Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$,

$x \in R - \{-1, 0, 1\}$. If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of h (x) is: **[JEE MAIN 2018]**

(A) $-2\sqrt{2}$ (B) $2\sqrt{2}$
(C) 3 (D) –3

Q.39 The maximum volume (in cu. m) of the right circular cone having slant height 3m is : **[JEE MAIN 2019 (Jan)]**

(A) $3\sqrt{3}\,\pi$ (B) 6π (C) $2\sqrt{3}\,\pi$ (D) $(4/3)\,\pi$

Q.40 If θ denotes the acute angle between the curves, $y = 10 - x^2$ and $y = 2 + x^2$ at a point of their intersection, then | tan θ | is equal to : **[JEE MAIN 2019 (Jan)]**

(A) 4/9 (B) 7/17 (C) 8/17 (D) 8/15

Q.41 The shortest distance between the line y = x and the curve $y^2 = x - 2$ is : **[JEE MAIN 2019 (April)]**

(A) $\frac{7}{4\sqrt{2}}$ (B) $\frac{7}{8}$
(C) $\frac{11}{4\sqrt{2}}$ (D) 2

Q.42 If S_1 and S_2 are respectively the sets of local minimum and local maximum points of the function, $f(x) = 9x^4 + 12x^3 - 36x^2 + 25, x \in R$, then : **[JEE MAIN 2019 (April)]**

(A) $S_1 = \{-2, 1\}; S_2 = \{0\}$ (B) $S_1 = \{-2, 0\}; S_2 = \{1\}$
(C) $S_1 = \{-2\}; S_2 = \{0, 1\}$ (D) $S_1 = \{-1\}; S_2 = \{0, 2\}$

Q.43 Let f : [0, 2] → R be a twice differentiable function such that f"(x) > 0, for all x ∈ (0, 2). If f(x) = f(x) + f(2 – x), then f is :**[JEE MAIN 2019 (April)]**

(A) decreasing on (0, 2)
(B) decreasing on (0, 1) and increasing on (1, 2)
(C) increasing on (0, 2)
(D) increasing on (0, 1) and decreasing on (1, 2)

Q.44 The height of a right circular cylinder of maximum volume inscribed in a sphere of radius 3 is **[JEE MAIN 2019 (April)]**

(A) $2\sqrt{3}$ (B) $\sqrt{3}$
(C) $\sqrt{6}$ (D) $\frac{2}{3}\sqrt{3}$

Q.45 Let S be the set of all values of x for which the tangent to the curve $y = f(x) = x^3 - x^2 - 2x$ at (x, y) is parallel to the line segment joining the points (1, f (1)) and (–1, f (–1)), then S is equal to : **[JEE MAIN 2019 (April)]**

(A) {–1/3, –1} (B) {1/3, –1}
(C) {–1/3, 1} (D) {1/3, 1}

Q.46 If the tangent to the curve, $y = x^3 + ax - b$ at the point (1, –5) is perpendicular to the line, –x + y + 4 = 0, then which one of the following points lies on the curve ? **[JEE MAIN 2019 (April)]**

(A) (–2, 2) (B) (2, –2)
(C) (2, –1) (D) (–2, 1)

Q.47 Let f(x) is a five degree polynomial which has critical points x = ±1 and $\lim_{x \to 0}\left(2 + \frac{f(x)}{x^3}\right) = 4$ then which one is **incorrect**. **[JEE MAIN 2020 (Jan)]**

(A) f (x) has minima at x = 1 & maxima at x = –1.
(B) f (1) – 4f (–1) = 4.
(C) f (x) is maxima at x = 1 and minima at x = –1.
(D) f (x) is odd.

Q.48 Let $f(x) = x^3 - 4x^2 + 8x + 11$, if LMVT is applicable on f(x) in [0, 1], value of c is : **[JEE MAIN 2020 (Jan)]**

(A) $\frac{4-\sqrt{7}}{3}$ (B) $\frac{4-\sqrt{5}}{3}$
(C) $\frac{4+\sqrt{7}}{3}$ (D) $\frac{4+\sqrt{5}}{3}$

Q.49 For $f(x) = \ln\left(\frac{x^2+\alpha}{7x}\right)$. Rolle's theorem is applicable on [3, 4], the value of f "(c) is equal to **[JEE MAIN 2020 (JAN)]**

(A) 1 / 12 (B) –1 / 12
(C) 1 / 6 (D) –1 / 6

Q.50 Let $f(x) = x\cos^{-1}(\sin(-|x|)),\ x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ then **[JEE MAIN 2020 (JAN)]**

(A) $f'(0) = -\pi/2$

(B) $f'(x)$ is not defined at $x = 0$

(C) $f'(x)$ is increasing in $(-\pi/2, 0)$ and $f'(x)$ is decreasing in $(0, \pi/2)$.

(D) $f'(x)$ is decreasing in $(-\pi/2, 0)$ and $f'(x)$ is increasing in $(0, \pi/2)$.

Q.51 If normal at P on the curve $y^2 - 3x^2 + y + 10 = 0$ passes through the point $(0, 3/2)$ then slope of tangent at P is n. The value of $|n|$ is equal to **[JEE MAIN 2020 (JAN)]**

Q.52 Let $f(x)$ be a polynomial of degree 3 such that $f(-1) = 10$, $f(1) = -6$, $f(x)$ has a critical point at $x = -1$ and $f'(x)$ has a critical point at $x = 1$. Then $f(x)$ has a local minima at $x =$ _______. **[JEE MAIN 2020 (JAN)]**

Q.53 A spherical iron ball of 10 cm radius is coated with a layer of ice of uniform thickness the melts at a rate of 50cm^3/min. When the thickness of ice is 5cm, then the rate (in cm/min.) at which of the thickness of ice decreases, is : **[JEE MAIN 2020 (JAN)]**

(A) $1/36\pi$ (B) $5/6\pi$

(C) $1/18\pi$ (D) $1/54\pi$

Q.54 Let a function $f : [0, 5] \to \mathbf{R}$ be continuous, $f(1) = 3$ & F be defined as: $F(x) = \int_1^x t^2 g(t)\,dt$, where $g(t) = \int_1^t f(u)\,du$.

Then for the function F, the point $x = 1$ is : **[JEE MAIN 2020 (JAN)]**

(A) a point of local minima. (B) not a critical point.

(C) a point of inflection. (D) a point of local maxima.

ANSWER KEY

EXERCISE - 1

Q	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
A	D	C	B	C	D	C	A	A	A	D	B	B	B	C	A	C	C	B	A	A	B	C	A	A	A	D	B	C	C	B
Q	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**	**41**	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**	**51**	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**	**60**
A	B	A	C	D	B	C	A	A	A	D	A	C	D	D	A	D	A	B	A	D	D	A	A	B	C	B	B	C	B	A
Q	**61**	**62**	**63**	**64**	**65**	**66**	**67**	**68**	**69**	**70**	**71**	**72**	**73**	**74**	**75**	**76**	**77**	**78**	**79**	**80**	**81**	**82**	**83**	**84**	**85**	**86**	**87**	**88**	**89**	**90**
A	C	B	B	B	B	B	D	D	A	C	C	A	A	D	C	B	B	C	B	B	B	D	B	D	D	C	A	B	C	A
Q	**91**	**92**	**93**	**94**	**95**	**96**	**97**	**98**	**99**	**100**	**101**																			
A	B	D	C	B	D	A	B	C	C	B	D																			

EXERCISE - 2

Q	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**
A	B	A	B	C	B	D	C	B	A	A	D	D	D	C	C	A	B	A	B	B	B	B	D	D	A
Q	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**	**41**	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**
A	B	D	D	D	B	D	C	A	C	A	A	B	B	A	D	A	C	D	B	C	C	B	B	D	A
Q	**51**	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**	**60**	**61**	**62**	**63**	**64**	**65**	**66**	**67**	**68**	**69**	**70**	**71**	**72**	**73**	**74**	**75**
A	A	C	C	D	A	B	B	A	B	D	A	D	C	A	A	B	D	C	A	B	D	B	C	A	D
Q	**76**	**77**	**78**	**79**	**80**	**81**	**82**	**83**	**84**	**85**	**86**	**87**	**88**	**89**	**90**	**91**	**92**	**93**	**94**	**95**	**96**	**97**	**98**	**99**	**100**
A	C	A	A	C	8	5040	12	4	4000	2	2	10	40	33	1	5	1	6	2	7	1	2	5	9	2

EXERCISE - 3

Q	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
A	D	C	D	B	A	A	B	BD	B	D	A	A	D	B	A	B	D	D	B	A
Q	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
A	C	C	D	B	C	B	D	A	D	C	B	C	B	A	D	A	B	B	C	D
Q	**41**	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**	**51**	**52**	**53**	**54**						
A	A	A	B	A	C	B	A	A	A	D	4	3	C	A						

CHAPTER-6:
APPLICATION OF DERIVATIVES

SOLUTIONS TO TRY IT YOURSELF
TRY IT YOURSELF-1

(1) **(C).** $x = e^{2t} + 2e^{-t}$ and $y = e^{2t} + e^{t}$

At $t = \ln 2$; $x = 4 + 1 = 5,\ y = 4 + 2 = 6$

$$\frac{dy}{dx} = \frac{2e^{2t} + e^{t}}{2e^{2t} - 2e^{-t}} = \frac{8+2}{8-1} = \frac{10}{7}$$

$\Rightarrow$ Equation of tangent is $y - 6 = \frac{10}{7}(x - 5)$

$7y - 42 = 10x - 50$ or $10x - 7y = 8$

(2) For point of intersection

$$e^{x} \ln x = \frac{\ln x}{e^{x}} \Rightarrow \ln x = 0 \text{ or } e^{2}x^{2} = 1$$

$\Rightarrow$ $x = 1$ or $x = \pm 1/e$ but $0 < x < 1$

Point P is $(1/e, -1)$

For curve C_1, $\frac{dy}{dx} = e(1 + \ln x)$

$\Rightarrow$ Slope of tangent at point P is equal to $e\left(1 + \ln\frac{1}{e}\right) = 0$

$\Rightarrow$ Equaton of normal is $x = 1/e$.

(3) Take P (t^2, t^3) and Q (T^2, T^3)

$$\frac{dy}{dx} = \frac{3x^2}{2y} \text{ or } \left(\frac{dy}{dx}\right) = \frac{3}{2}t$$

Slope line joining P and Q is

$$\frac{T^3 - t^3}{T^2 - t^2} = \frac{T^2 + t^2 + Tt}{T + t} \Rightarrow \frac{3}{2}t = \frac{T^2 + t^2 + Tt}{T + t}$$

or $3tT + 3t^2 = 2T^2 + 2t^2 + 2Tt \Rightarrow T = -t/2$

$$\Rightarrow \frac{m_{OP}}{m_{OQ}} = -2$$

(4) $y = x^2$ and $6y = 7 - x^3$ Point (a, a) is $(1, 1)$.

$y'_1 = 2x$ and $y'_2 = -x^2/2$

$y'_1 \times y'_2 = -1 \quad \therefore\ \theta = \pi/2$

(5) $xy = c^2 \Rightarrow xy' + y = 0$ or $y' = \frac{-y}{x}$

$$ST = \frac{y}{y'} = -x,\ SN = yy' = \frac{-y^2}{x} = \frac{-y^2}{c^2}y = \frac{-y^3}{c}$$

(6) Let $f(x) = x^{1/4}$ $\quad \therefore\ f'(x) = \frac{1}{4}(x)^{-3/4} = \frac{1}{4x^{3/4}}$

Also, $f(x + \delta x) = (x + \delta x)^{1/4}$

Now, $f(x + \delta x) = f(x) + \delta x\, f'(x)$ (approximately)

$$\Rightarrow (x + \delta x)^{1/4} = x^{1/4} + \delta x \cdot \frac{1}{4x^{3/4}}$$

We have to find $(82)^{1/4}$ and we know the value of $(81)^{1/4}$ which is equal to 3.

Putting at $x = 81,\ x + \delta x = 82$, so that $dx = 1$

we get $(82)^{1/4} = (81)^{1/4} + 1 \cdot \frac{1}{4(81)^{3/4}}$

$$= 3 + \frac{1}{4 \times 3^3} = 3 + \frac{1}{108} = 3.009$$

(7) **(C).** Given, $x^{1/3} + y^{1/3} = a^{1/3},\ a > 0$

$$\therefore\ \frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}\frac{dy}{dx} = 0 \qquad \therefore\ \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{2/3}$$

At P $(a/8, a/8)$, $dy/dx = -1$

Equation of tangent at P is

$$y - \frac{a}{8} = -1\left(x - \frac{a}{8}\right) \text{ or } x + y = \frac{a}{4}$$

It intercepts on the axes are $a/4,\ a/4$.

Given, $\frac{a^2}{16} + \frac{a^2}{16} = 2 \Rightarrow a^2 = 16 \Rightarrow a = 4\ (\because\ a > 0)$

(8) **(A).** Given curve is $y = \sin x$ ...... (1)

Let the tangent to curve (1) at P (α, β) pass through $(0, 0)$.

Equation of tangent at (α, β) is

$y - \beta = \cos\alpha\,(x - \alpha)$ ...... (2)

Since (2) passes through $(0, 0)$

$\therefore\ -\beta = -\alpha\cos\alpha$ or $\cos\alpha = \frac{\beta}{\alpha}$ ...... (3)

Also, (α, β) lies on (1), $\therefore\ \sin\alpha = \beta$ ...... (4)

From eq. (3) and (4), $1 = \frac{\beta^2}{\alpha^2} + \beta^2$

$$\text{or } \alpha^2 - \beta^2 = \alpha^2\beta^2 \text{ or } \frac{\alpha^2 - \beta^2}{\alpha^2\beta^2} = 1 \text{ or } \frac{1}{\beta^2} - \frac{1}{\alpha^2} = 1$$

$\therefore$ (α, β) lies on curve $\frac{1}{y^2} - \frac{1}{x^2} = 1$.

(9) Equation of the curve is $x^2 + y^2 - 2x - 4y + 1 = 0$.

$$\therefore\ 2x + 2y\frac{dy}{dx} - 2 - \frac{4dy}{dx} = 0$$

$$\Rightarrow (2y - 4)\frac{dy}{dx} = 2 - 2x \quad \therefore\ \frac{dy}{dx} = \frac{2 - 2x}{2y - 4} = \frac{1 - x}{y - 2}$$

Since tangents are parallel to the x-axis, slope of each of the tangents = 0.

$$\therefore\ \frac{1 - x}{y - 2} = 0 \Rightarrow 1 - x = 0 \Rightarrow x = 1$$

At $x = 1$, $1^2 + y^2 - 2(1) - 4y + 1 = 0$

$\Rightarrow$ $y^2 - 4y = 0 \Rightarrow y(y - 4) = 0 \Rightarrow y = 0$ or $y = 4$

$\therefore$ The points are (1, 0) and (1, 4).

$\therefore$ The equation of tangent through (1, 0) and parallel to the x-axis is $y = 0$.

and the equation of tangent through (1, 4) and parallel to the x-axis is $y = 4$

$\therefore$ The equations of tangents are $y = 0$ and $y = 4$.

(10) For the points of intersection of curve and x–axis, $y = 0$

$\therefore$ $\sin x / a = 0 \Rightarrow x = a.n\pi$

Here $\frac{dy}{dx} = \frac{b}{a}\cos\frac{x}{a}$; $\frac{d^2y}{dx^2} = -\frac{b}{a^2}\sin\frac{x}{a}$

$$\frac{d^3y}{dx^3} = -\frac{b}{a^3}\cos\frac{x}{a}$$

Now at $x = an\pi$, we have

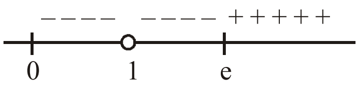
$$\frac{d^2y}{dx^2} = -\frac{b}{a^2}\sin n\pi = 0 \text{ and } \frac{d^3y}{dx^3} = -\frac{b}{a^3}\cos n\pi \neq 0$$

Therefore $x = an\pi$ is a point of inflexion of the curve. This is the point where the curve meets the x–axis.

TRY IT YOURSELF-2

(1) $f'(x) = \frac{\ln x - 1}{(\ln x)^2}$

Sign of f' (x)



Strictly increasing in (e, ∞) and strictly decreasing in $(0, 1) \cup (1, e)$

(2) $f'(x) = 3(a + 2)x^2 - 6ax + 9a \leq 0 \;\forall\; x \in R$

$\Rightarrow$ $3(a + 2) < 0$ and $36a^2 - 4 . 3(a + 2) . 9a \leq 0$

$\Rightarrow$ $a < -2$ and $a^2 - 3a(a + 2) \leq 0$

$\Rightarrow$ $a < -2$ .... (1) and $a^2 + 3a \geq 0$

$\Rightarrow$ $a \in (-\infty, -3] \cup [0, \infty)$

$\Rightarrow$ $a \leq -3$

(3) $f(x) = \ln(1 + x) - \frac{x}{1 + x} = \ln(1 + x) + \frac{1}{1 + x} - 1$

Domain : $x > -1$

$$f'(x) = \frac{1}{1 + x} - \frac{1}{(1 + x)^2} = \frac{x}{(1 + x)^2}$$

$f'(x) \geq 0 \;\forall\; x \geq 0 \Rightarrow f(x) \uparrow$

& $f'(x) \leq 0 \;\forall\; x \leq 0 \Rightarrow f(x) \downarrow$

$f'(0) = 0$

$\therefore$ $f(x) > f(0) \;\forall\; x \in D_f - \{0\}$

$\therefore$ $f(x) > 0 \;\forall\; x \in (-1, 0) \cup (0, \infty)$

(4) **(AC).** $f(x) = \frac{|x - 1|}{x^2} = \begin{cases} \frac{x - 1}{x^2}, & x \geq 1 \\ \frac{1 - x}{x^2}, & x < 1, x \neq 0 \end{cases}$

$$\therefore \; f'(x) = \begin{cases} \frac{x - 2}{x^3}, & x < 1,\; x \neq 0 \\ \text{does not exist}, & x = 1 \\ \frac{2 - x}{x^3}, & x > 1 \end{cases}$$

Clearly, $f'(x) > 0$ for $x < 0$ or $1 < x < 2$

and $f'(x) < 0$ for $0 < x < 1$ or $x > 2$

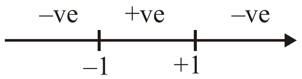
Thus, f (x) is increasing for $(-\infty, 0) \cup (1, 2)$ and decreasing for $(0,1) \cup (2, \infty)$.

(5) We have, $f(x) = \frac{x^2 + x + 1}{x^2 - x + 1}, x \in R$

and $f'(x) = \frac{(x^2 - x + 1)(2x + 1) - (x^2 + x + 1)(2x - 1)}{(x^2 - x + 1)^2}$

$$= \frac{-2(x + 1)(x - 1)}{(x^2 - x + 1)^2}$$

Now, from the sign scheme for f' (x), we have



$\Rightarrow$ f (x) strictly decreases in $(-\infty, -1)$

Strictly increases in $(-1, 1)$

Strictly decreases in $(1, \infty)$

(6) **(A).** f (x) is monotonic decreasing when

$f'(x) < 0 \;\forall\; x$

$\Rightarrow$ $\sin x - 2\lambda < 0$

$\Rightarrow$ $2\lambda > -\sin x \;\Rightarrow\; 2\lambda > 1 \Rightarrow \lambda > 1/2$

CHAPTER-6 :
APPLICATION OF DERIVATIVES
EXERCISE-1

(1) **(D).** $y = \log_e x \; ; \; \frac{dy}{du} = \frac{1}{x}$

Eq. of tangent $y - y_1 = \frac{1}{x_1}(x - x_1)$

Passing through (0, 0), $-y_1 = \frac{1}{x_1}(-x_1)$

$\Rightarrow y_1 = 1 \;\; \therefore \;\; x_1 = e \qquad \therefore$ Point is (e, 1)
Equation of normal at (e, 1) is $y - 1 = -e(x - e)$
$\Rightarrow ex + y - 1 - e^2 = e$

$$L = \left|\frac{-c}{\sqrt{a^2 + b^2}}\right| = \frac{e^2 + 1}{\sqrt{e^2 + 1}} = \sqrt{e^2 + 1}$$

(2) **(C).** $4x^5 = 5y^4 \Rightarrow 20x^4 = 20y^3 \,.\, y' \Rightarrow y' = x^4/y^3$

$$\therefore \frac{(SN)^3}{(ST)^2} = \frac{(y^4/x^4)^3}{(x^4/y^2)^2} = \frac{y^{12}}{x^{12}} \times \frac{y^4}{x^8} = \frac{y^{16}}{x^{20}} = \left(\frac{4}{5}\right)^4$$

(3) **(B).** $2yy_1 = 4 \qquad\qquad 2x + 2yy_1 = 0$

$y_1 = \frac{4}{2y} \qquad\qquad yy_1 = -x$

$y_1 = \frac{2}{y} \qquad\qquad y_1 = -\frac{x}{y}$

$m_1 = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \qquad\qquad m_2 = -\frac{2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}}$

$$\tan\theta = \left|\frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{1 + \left(-\frac{1}{2}\right)}\right| = \left(\frac{\sqrt{2}}{\frac{1}{2}}\right) = 2\sqrt{2}$$

(4) **(C).** $x^n y^m = a^{m+1} \quad \therefore \;\; n \log x + m \log y = (m + 1)\, a$

$\therefore \;\; \frac{n}{x} + \frac{m}{y}.y' = 0$ (diff. wr to x)

$\therefore \;\; y' = -\frac{n}{n}.\frac{y}{x}$ ST at (x_1, y_1)

$= -\frac{m}{n}.\frac{x_1}{y_1}.y_1 \; ; (x_1, y_1) = \frac{m}{n}|x_1|$

(5) **(D).** y/y', y, yy' are in GP

(6) **(C).** Slope of first curve $m_1 = 0$
Slope of second curve $m_2 = -1$, therefore angle is 45°
$A = \sin 45° = 1/\sqrt{2}$

(7) **(A).** Differentiating $xy^n = a$, we get $y' = \frac{-y}{nx}$
ST = –nx. Since it is proportional to x, n can be any non-zero real number.

(8) **(A).** $y' = 2x + \frac{2}{x^3} = -2 - 2 = -4$ = slope of the tangent.

$\therefore$ Slope of the normal = 1/4

(9) **(A).** Given curve $y^2 = px^3 + q$ …..(i)

Differentiate with respect to x, $2y.\frac{dy}{dx} = 3px^2$

$$\Rightarrow \frac{dy}{dx} = \frac{3p}{2}\left(\frac{x^2}{y}\right) \;\; \therefore \left|\frac{dy}{dx}\right|_{2,3} = \frac{3p}{2} \times \frac{4}{3} = 2p$$

For given line, slope of tangent $= 4$

$\therefore 2p = 4 \;\; \Rightarrow \;\; p = 2$

From equation (i), $9 = 2 \times 8 + q \;\; \Rightarrow q = -7$.

(10) **(D).** $y^2 = \alpha x^3 - \beta \Rightarrow \frac{dy}{dx} = \frac{3\alpha x^2}{2y}$

$\Rightarrow$ Slope of the normal at (2, 3) is

$$\left(-\frac{dx}{dy}\right)_{(2,3)} = -\frac{2 \times 3}{3\alpha(2)^2} = -\frac{1}{2\alpha} = -\frac{1}{4} \Rightarrow \alpha = 2$$

Also (2, 3) lies on the curve.
$\Rightarrow 9 = 8\alpha - \beta \Rightarrow \beta = 16 - 9 = 7 \Rightarrow \alpha + \beta = 9$

(11) **(B).** $m \log x + n \log y = (m + n) \log a$

$$\Rightarrow \frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-m}{n}\frac{y}{x}$$

$$\therefore \;\; \text{Subtangent} = \frac{y}{(dy/dx)} = \frac{y}{-\frac{m}{n}\frac{y}{x}} = \frac{-n}{m}x$$

$\therefore$ Subtangent $\propto x$

(12) **(B).** The point on the curve corresponding to x = 0 is (0, 1)

$$\frac{dy}{dx} = (2e^{2x} + 2x) \Rightarrow \left(\frac{dy}{dx}\right)_{x=0} = (2e^0 + 0) = 2$$

Hence the equation of the normal at the point (0, 1) is

$$(y - 1) = -\frac{1}{2}(x - 0) \Rightarrow 2y + x - 2 = 0$$

Distance of the point (0, 0) from this line is $= \frac{2}{\sqrt{5}}$

(13) **(B).** Given curve is $xy = 1$ .... (1)
Line is $ax + by + c = 0$ .... (2)

From (1), $\frac{dy}{dx} = -\frac{1}{x^2}$

$\therefore$ Slope of the normal $= x^2 =$ positive .... (3)

$ax + by + c = 0$ is normal

$\therefore$ Slope $= -\frac{a}{b}$ .... (4)

From eq. (3) & (4), $-\frac{a}{b}$ = positive
i.e. a, b have opposite signs.

(14) **(C).** Equation of the curve $x^2y^2 = a^4$.

Differentiating the given equation, $x^2 2y\frac{dy}{dx} + y^2 2x = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x} \Rightarrow \left(\frac{dy}{dx}\right)_{(-a,a)} = -\left(\frac{a}{-a}\right) = 1$$

Therefore sub-tangent $= \frac{y}{\left(\frac{dy}{dx}\right)} = a$.

(15) **(A).** Solving the line and curve, we get x = 4 and $y = -\sqrt{6}$.

Thus point of contact is $(4, -\sqrt{6})$.

(16) **(C).** Given $y^2 = 2(x-3)$ .....(i)

Differentiate w.r.t. x, $2y.\frac{dy}{dx} = 2 \Rightarrow \frac{dy}{dx} = \frac{1}{y}$

Slope of the normal $= \frac{-1}{\left(\frac{dy}{dx}\right)} = -y$

Slope of the given line = 2 $\therefore y = -2$

From equation (i), x = 5

$\therefore$ Required point is (5, –2).

(17) **(C).** $y = 2\cos x$

At $x = \frac{\pi}{4},\ y = \frac{2}{\sqrt{2}} = \sqrt{2}$ and $\frac{dy}{dx} = -2.\sin x$

$$\therefore \left(\frac{dy}{dx}\right)_{x=\pi/4} = -\sqrt{2}$$

$\therefore$ Equation of tangent at $\left(\frac{\pi}{4}, \sqrt{2}\right)$ is

$$y - \sqrt{2} = -\sqrt{2}\left(x - \frac{\pi}{4}\right).$$

(18) **(B).** $\frac{dy}{dx} = \tan\frac{\pi}{4} = 1 \Rightarrow \frac{1}{2y} = 1 \Rightarrow y = \frac{1}{2}, x = \frac{1}{4}$

(19) **(A).** $f(x) = \frac{x}{\log x}$; $f'(x) = \frac{\log x.1 - x.\frac{1}{x}}{(\log x)^2} = \frac{\log x - 1}{(\log x)^2}$

f (x) is increasing $\Rightarrow f'(x) > 0$

$$\Rightarrow \frac{\log x - 1}{(\log x)^2} > 0 \Rightarrow \log x - 1 > 0 \Rightarrow x > e$$

(20) **(A).** $f(x) = \frac{x}{3} + \frac{3}{x}$; $f'(x) = \frac{1}{3} - \frac{3}{x^2} < 0$

$$\frac{1}{3} < \frac{3}{x^2} \Rightarrow x^2 < 9 \Rightarrow x \in (-3, 3)$$

(21) **(B).** $f(x) = x^3 - 6x^2 + 9x + 3$, For decreasing $f'(x) < 0$

$\Rightarrow 3x^2 - 12x + 9 < 0 \Rightarrow x^2 - 4x + 3 < 0$

$\Rightarrow (x-3)(x-1) < 0,\ \therefore\ x \in (1, 3)$.

(22) **(C).** $f(x) = y = \tan^{-1}\left(\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)\right)$

$$\Rightarrow \tan y = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) \Rightarrow \sec^2 y\frac{dy}{dx} = \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$$

$$\frac{dy}{dx} > 0 \Rightarrow \cos\left(x + \frac{\pi}{4}\right) > 0\ .\ \therefore\ x \in \left(0, \frac{\pi}{4}\right).$$

(23) **(A).** $f'(x) = 6x^2 + 36x - 96 > 0$, for increasing

$\Rightarrow f'(x) = (x+8)(x-2) \geq 0 \Rightarrow x \geq 2, x \leq -8$.

(24) **(A).** $f(x) = (x-2)^5(x+1)^4$

$f'(x) = 5(x-2)^4(x+1)^4 + (x-2)^5(4)(x+1)^3$

$= (x+1)^3(x-2)^4(5x+5+4x-8)$

$f'(x) = 3(x+1)^3(x-2)4(3x-1) < 0$

decreasing in (–1, 1/3)

(25) **(A).**
$$f'(x) = \frac{(c\sin x + d\cos x)(a\cos x - b\sin x)}{(c\sin x + d\cos x)^2} - \frac{(a\sin x + b\cos x)(c\cos x - d\sin x)}{(c\sin x + d\cos x)^2}$$

$$= \frac{ad - bc}{(c\sin x + d\cos x)^2}$$

Function f (x) is monotonic decreasing if f'(x) < 0

$\Rightarrow ad - bc < 0$

(26) **(D).** f (x) is monotonic increasing $\Rightarrow f'(x) > 0$

$\Rightarrow 3kx^2 - 18x + 9 > 0 \Rightarrow kx^2 - 6x + 3 > 0$

which is positive only when k > 0 and $b^2 - 4ac \leq 0$

i.e. when $(-6)^2 - 4(k)(C) \leq 0$ or when $k \geq 3$

(27) **(B).** We have, $f(x) = x^{1/x}$

$$\Rightarrow f'(x) = \frac{1}{x^2}(1 - \log x)\, x^{1/x}.$$

$f'(x) > 0$ if $1 - \log x > 0$, i.e. $\log x < 1$

$\Rightarrow x < e$, $\therefore$ f (x) is increasing in the interval $(-\infty, e)$

(28) **(C).** $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$

$\Rightarrow f'(x) = 3x^2 + 2ax + b + 5\sin 2x$

since f (x) is an monotonically increasing function

$\Rightarrow 3x^2 + 2ax + b - 5 > 0\ \forall\ x \in R$

$\Rightarrow 4a^2 - 4.3(b-5) < 0$

$\Rightarrow a^2 - 3b + 15 < 0$

(29) **(C).** Let $f(x) = \sin x - bx + c$

$\therefore f'(x) = \cos x - b > 0$ or $\cos x > b$ or $b < -1$.

(30) **(B).** $P = 2r + s$; $A = \frac{1}{2}rs = \frac{1}{2}r.(20 - 2r)$; $2r + s = 20$;

$$A = 10r - r^2;\ \frac{dA}{dr} = 10 - 2r\ ;\ \frac{dA}{dr} = 0 \Rightarrow r = 5$$

$A_{max} = 50 - 25 = 25$

(31) **(B).** Given a + b = k, k const $\therefore$ b = k – a

$S = a^3 + b^3 = a^3 + (k-a)^3 = a^3 + k^3 - a^3 - 3k^2a + 3ka^2$

$S = k^3 - 3k^2a + 3ka^2$

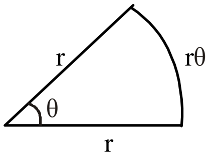
$\therefore \frac{dS}{da} = 0 \Rightarrow 3k^2 + 6ak \therefore 3k = 6a \quad (\because k \neq 0)$

$a = k/2 \quad \therefore b = k - k/2 = k/2 \Rightarrow a = b$

(32) **(A).** $P = r + r + r\theta \Rightarrow 2r + r\theta$

$A = \frac{1}{2}r^2\theta = \frac{1}{2}r^2\left(\frac{P-2r}{r}\right) = \frac{1}{2}[Pr - 2r^2]$

$\frac{dA}{dr} = \frac{1}{2}[P - 4r]$



$\frac{dA}{dr} = 0 \Rightarrow r = \frac{P}{4}$

When $r = \frac{P}{4}; \theta = \frac{P - 2\left(\frac{P}{4}\right)}{P/4} = \frac{P/2}{P/4} = 2^c$

(33) **(C).** $y = xe^{-x} = \frac{x}{e^x}$

$\frac{dy}{dx} = \frac{e^x.1 - x.e^x}{e^{2x}} = \frac{e^x(1-x)}{e^{2x}}; \ \frac{dy}{dx} = 0$

$\Rightarrow x = 1$ must be a point of maxima. $\therefore y(1) = 1/e$

(34) **(D).** r = 2; maximum rectangle is a square with each side

$a = \sqrt{2}r = 2\sqrt{2}$, therefore area $= a^2 = 8$

(35) **(B).** Let $f(x) = x^{25}(1-x)^{75}$. Then,

$f'(x) = x^{24}(1-x)^{74}(1-4x)$

Now, $f'(x) = 0 \Rightarrow x = 0, 1, 1/4$

Clearly, $f'(x) > 0$ in the left neighborhood of 1/4 and $f'(x) < 0$ in the right neighborhood of 1/4. So f'(x) changes its sign form positive to neghbourhood of 1/4. Hence, it attains maximum at $x = 1/4$.

(36) **(C).** $y = \frac{(x-2)(x-1)}{(x+3)(x-1)} = \frac{(x-2)}{(x+3)}, x \neq 1, x \neq -3$

or $y = \frac{x+3-5}{x+3} = 1 - \frac{5}{x+3}$

$\frac{dy}{dx} = \frac{5}{(x+3)^2} = +ive$

always for all values of x in its domain.

$\therefore y = f(x)$ is an increasing function in its domain.

(37) **(A).** We have ; $f(x) = \sin x - \cos x - ax + b$

$\Rightarrow f'(x) = \cos x + \sin x - a$

$\Rightarrow f'(x) < 0 \ \forall \ x \in R$

$\Rightarrow (\cos x + \sin x) < a \ \forall \ x \in R$

As the max. value of $(\cos x + \sin x)$ is $\sqrt{2}$

The above is possible when $a \geq \sqrt{2}$

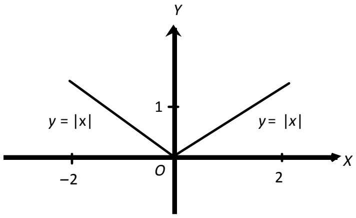
(38) **(A).** For $0 < x \leq \pi/2$; $[\cos x] = 0$

Hence, f(x) = 1 for all $(0, \pi/2]$

Trivially f(x) is continuous on $(0, \pi/2)$

This function is neither strictly increasing nor strictly decreasing and its global maximum is 1.

(39) **(A).** The graph of the function is as given below:



$f(0) = 1, f(0-h) < 1, f(0+h) < 1$

$\therefore$ f(x) has a maximum at x = 0

(40) **(D).** $R(x) = 3x^2 + 36x + 5, \ \therefore \frac{dR}{dx} = 6x + 36;$

$\frac{dR}{dx} = 6 \times 15 + 36 = 126 \qquad [\because x = 15]$

(41) **(A).** Suppose that two numbers are x and y.

$x + y = s \Rightarrow y = s - x$

Then $f(x) = xy = x(s-x) = xs - x^2$

$\therefore f'(x) = s - 2x$

$f'(x) = 0$ for maximum value of $f(x)$

$\therefore x = \frac{s}{2}$ and $y = \frac{s}{2}$

Thus each number is half of the sum.

(42) **(C).** Given $f(x) = 7 - 20x + 11x^2$

$f'(x) = -20 + 22x$

Put $f'(x) = 0$ i.e., $-20 + 22x = 0$

$\Rightarrow x = 10/11$ and $f''(x) = 22 > 0$

Hence at $x = 10/11$, $f(x)$ will have minimum value,

$\therefore f\left(\frac{10}{11}\right) = 7 - \frac{200}{11} + \frac{100 \times 11}{121} = 7 - \frac{200}{11} + \frac{100}{11} = -\frac{23}{11}.$

(43) **(D).** Consider $f(x) = 20x - x^2$ defined for all $x \in R$

$f'(x) = 20 - 2x, \ f'(x) = 0 \Rightarrow x = 10 \ \& \ f''(x) = -2 < 0$

Hence, x = 10 is a point of maximum.

$\therefore$ Maximum value = 100

(44) **(D).** $V = \frac{4}{3}\pi r^3 \qquad S = 4\pi r^2$

$\frac{dV}{dt} = 4\pi.r^2\frac{dr}{dt} \qquad \frac{dS}{dt} = 4\pi.2r\frac{dr}{dt}$

$\pi = 4\pi\frac{dr}{dt} \qquad = 2\pi$

$\Rightarrow \frac{dr}{dt} = \frac{1}{4}$

(45) **(A).** $s = 22t - 11t^2$; $ds/dt = 22 - 22t$
$v = ds/dt = 0 \Rightarrow t = 1$
$\therefore$ Distance = $s = 22 - 11 = 11$ units
Total distance = $11 + 11 = 22$

(46) **(D).** Given $\frac{dr}{dt} = 5\text{ cm / sec}$

$$A = \pi r^2 \Rightarrow \frac{dA}{dt} = \pi \times 2r\frac{dr}{dt} = 2\pi \times 8 \times 5 = 80\pi\text{ cm}^2/\text{s}$$

(47) **(A).** $\frac{dv}{dt} = 10\text{cc / sec}, \ \frac{dr}{dt} = ?$ when $r = 15$

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dv}{dt} = 4\pi^2\frac{dr}{dt} \Rightarrow 10 = 4\pi \times 225\frac{dr}{dt} = \frac{1}{90}\pi$$

(48) **(B).** Let $f(x) = \sin x + x\cos x$

Consider $g(x) = \int_0^x (\sin t + t\cos t)\,dt = [t\sin t]_0^x = x\sin x$

$g(x) = x\sin x$ which is differentiable
now $g(0) = 0$ and $g(\pi) = 0$, using Rolles Theorem
hence $\exists$ atleast one $x \in (0, \pi)$ where $g'(x) = 0$
i.e. $x\cos x + \sin x = 0$ for atleast one $x \in (0, \pi)$

(49) **(A).** $f(x)$ is monotonic decreasing when
$f'(x) < 0 \ \forall\ x \quad \Rightarrow \sin x - 2\lambda < 0$
$\Rightarrow 2\lambda > -\sin x \Rightarrow 2\lambda > 1 \Rightarrow \lambda > 1/2$

(50) **(D)** If the sum of two positive quantities is a constant, then their product is maximum, when they are equal .
$\therefore a^2x^4.\, b^2y^2$ is maximum when $a^2x^4 = b^2y^4$

$$= \frac{1}{2}(a^2x^4 + b^2y^4) = \frac{c^4}{2}$$

$\therefore$ maximum value of $a^2x^4.\, b^2y^4 = \frac{c^4}{2}.\frac{c^4}{2} = \frac{c^8}{4}$

Maximum value of $xy = \sqrt{\frac{c^8}{4a^2b^2}} = \frac{c^2}{2ab}$.

(51) **(D)** $x = 2\ln\ \cot t + 1,\ y = \tan t + \cot t$
Slope of tangent

$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \left(\frac{\sec^2 t - \operatorname{cosec}^2 t}{-\frac{2}{\cot t}\operatorname{cosec}^2 t}\right)_{t=\frac{\pi}{4}} = 0$$

(52) **(A).** We have, $z = (x-p)^2 + (x-q)^2 + (x-r)^2$ .... (1)
Differentiating equation (1) with respect to x, we get

$$\frac{dz}{dx} = 2(x-p) + 2(x-q) + 2(x-r). \qquad .... (2)$$

For minima or maxima, put $\frac{dz}{dx} = 0$

From equation (2), we get
$2(x-p) + 2(x-q) + 2(x-r) = 0 \Rightarrow 3x - (p+q+r) = 0$

or $x = \frac{1}{3}(p+q+r)$

$\Rightarrow$ The given equation is maximum at $x = (1/3)(p+q+r)$

(53) **(A).** Here $a^2 = 27,\ b^2 = 1,\ a = 3\sqrt{3}\ ,\ b = 1$

The point $(a\cos\theta, b\sin\theta)$ is $(3\sqrt{3}\cos\theta, \sin\theta)$.

Tangent at the above point is $\frac{x^3\ 3\sqrt{3}\cos\theta}{9} + \frac{y\sin\theta}{1} = 1$

$\therefore$ Sum of intercepts $= \frac{1}{3\sqrt{3}\cos\theta} + \frac{1}{\sin\theta}$

or $s = 3\sqrt{3}\sec\theta + \operatorname{cosec}\theta$

$\Rightarrow \frac{ds}{d\theta} = 3\sqrt{3}\sec\theta\tan\theta - \operatorname{cosec}\theta\cot\theta = 0$

$\Rightarrow \tan^3\theta = \frac{1}{3\sqrt{3}} \quad \therefore \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

$\frac{d^2s}{d\theta^2}$ is positive at $\theta = \frac{\pi}{6}$.

Therefore, sum is minimum at $\theta = \pi/6$.

(54) **(B).** In $-1 \le x < 0,\ f'(x) = 3x^2 + 2x - 10$
$= 2x^2 + (x+1)^2 - 11 < 0$
$\therefore f(x)$ is monotonically decreasing in the interval $-1 \le x < 0$

In $0 < x < \frac{\pi}{2}$, $f'(x) = -\sin x < 0,\ \therefore f(x)$ is m.d.

In $\frac{\pi}{2} < x \le \pi,\ f'(x) = \cos x < 0,\ \therefore f(x)$ is m.d.

$$\therefore\ f\left(\frac{\pi}{2} - h\right) < \left(2 = f\left(\frac{\pi}{2}\right)\right),\ f\left(\frac{\pi}{2} + h\right) < f\left(\frac{\pi}{2}\right)$$

$\therefore f(x)$ has a local maximum at $x = \pi/2$

(55) **(C).** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{dy}{dx} = \frac{-b^2x}{a^2y}$ and $x^2 - y^2 = c^2$

$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$

The two curves will cut at right angles if

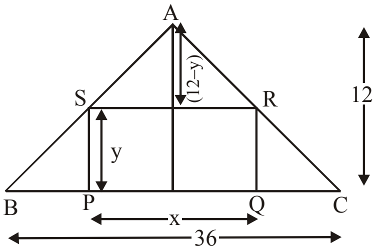
$$\left(\frac{dy}{dx}\right)_{c_1}.\left(\frac{dy}{dx}\right)_{c_2} = -1$$

$$\Rightarrow -\frac{b^2x}{a^2y}.\frac{x}{y} = -1 \Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} \Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{1}{2}$$

[Using $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$]

Substituting these values in $x^2 - y^2 = c^2$, we get $\frac{a^2}{2}-\frac{b^2}{2}=c^2 \Rightarrow a^2-b^2=2c^2$.

(56) **(B).**



Area of rectangle = A = xy .... (1)

Also, $\frac{36}{x}=\frac{12}{12-y} \Rightarrow 3y=(36-x)$ .... (2)

$\therefore \; A=\frac{x}{3}(36-x)=\frac{1}{3}(36x-x^2)$

Now, $A'(x)=0 \Rightarrow 36-2x=0 \Rightarrow x=18$

$A''(x)=\frac{1}{3}(-2)<0$. Also, $y=\frac{36-x}{3}=\frac{36-18}{3}=6$

$\therefore A_{max}=18\times 6=108$ sq. feet

(57) **(B).** $f'(x) \;=\frac{\pi}{2}\cos\frac{\pi}{2}x, 0\le x<1$

$=-2,\; x\ge 1$

$\therefore\;$ $f'(x)$ changes sign from positive to negative from the left side of $x=1$ to the right side of $x=1$.

$f(x)$ changes from an increasing function to a decreasing function. $\therefore\;$ $f(x)$ has a local maximum at $x=1$.

(58) **(C).** $m_1=\frac{x^2-y^2}{2xy}$ and $m_2=\frac{2xy}{y^2-x^2}$

Hence, $m_1m_2=-1$ and the curves intersect orthogonally. Therefore (i) is true. Replacing x by – x in the first equation, we get a new equation. Therefore, the first curve is not symmetrical with respect to the y-axis. Similarly, the second curve. Therefore (ii) is false.

If x and y are interchanged in the first equation, we get the second. Therefore (iii) is true.

(59) **(B).** As (4, 3) lies on the circle, the normal to $y=f(x)$ is the tangent to the circle (4, 3) so that the two curves intersect orthogonally at (4, 3).

$f'(4)$ = the slope of $y=f(x)$ at (4, 3) = – the reciprocal of the slope of the circle (4, 3) = 3/4.

(60) **(A).** We have $f(x)=x^3+x^2f'(1)+xf''(2)+f'''(3)$

$f'(x)=3x^2+2xf'(1)+f''(2)$

$\Rightarrow f'(1)=3+2f'(1)+f''(2)$

$\Rightarrow f'(1)+f''(2)+3=0$ .... (1)

$f''(x)=6x+2f'(1)$

$f''(2)=12+2f'(1)$

$f'''(x)=6 \Rightarrow f'''(3)=6$

$f'(1)=-5\;;\; f''(2)=2$

Hence, $f(x)=x^3-5x^2+2x+6$ .... (2)

Now, $f(-1)=-1-5-2+6=-2$

$f(0)=6 \Rightarrow$ one root $\in(-1,0)$

$f(2)=8-20+4+6=-2 \Rightarrow$ One root $\in(0,2)$

$\Rightarrow$ Two roots are real $\Rightarrow$ All three roots are real.

Also $f(5)=125-125+10+6=16$

$\Rightarrow$ One root $\in(2,5)$

$\Rightarrow$ Roots lies in $(-1,0),(0,2),(2,5)$

Hence, all roots are real and distinct.

(61) **(C).** $f'(x)=2\cos x+\sec^2 x-3>0$

$\Rightarrow\; 2\cos^3 x-3\cos^2 x+1>0$

$\Rightarrow\; (\cos x-1)^2(2\cos x+1)>0$

$\Rightarrow\; \cos x>-1/2 \Rightarrow\; x\in(-\pi/2, 2\pi/3)$

Among the given intervals, $(-\pi/2, \pi/2)$ lies within the above interval.

(62) **(B).** We have, $g(t)=\frac{\pi}{2}-2\cot^{-1}(3^{-t})$

$\therefore\; g(-t)=\frac{\pi}{2}-2\cot^{-1}(3^{t})=\frac{\pi}{2}-2\tan^{-1}(3^{-t})$

(As $\cot^{-1}x=\tan^{-1}\frac{1}{x}, x>0$)

$=\frac{\pi}{2}-2\left(\frac{\pi}{2}-\cot^{-1}(3^{-t})\right)=-\frac{\pi}{2}+2\cot^{-1}(3^{-t})$

(As $\cot^{-1}x+\tan^{-1}x=\frac{\pi}{2}\;\forall\, x\in R$)

$=-g(t)$

Hence $g(-t)=-g(t) \Rightarrow$ g is an odd function

Also, $g'(t)=\frac{-2.3^{-t}.\ln 3}{1+(3^{-t})^2}$

$\therefore\; g'(t)<0,\;\forall\, t\in R$

$\Rightarrow$ g is strictly decreasing in $(-\infty,\infty)$.

(63) **(B).** We have, $f(x)=\cot^{-1}(g(x))$

$\Rightarrow\; f'(x)=\frac{-1}{1+[g(x)]^2}\times g'(x)<0$ for $0<x<\pi$

[$\because$ g (x) is increasing for $0<x<\pi$, $\therefore g'(x)>0$]

Thus, f (x) is decreasing in $(0,\pi)$

(64) **(B).** $f'(x)=6(x^2-3ax+2a^2)=6(x-2a)(x-a)=0$

$\Rightarrow\;\; x=2a$ or a

$f''(x)=6(2x-3a)$

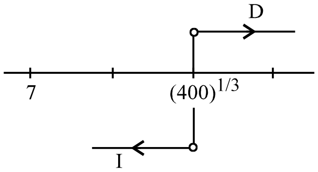
$$\left.\begin{aligned} f''(2a)&=a\\ f''(a)&=-a\end{aligned}\right] \Rightarrow \begin{aligned} &\text{If } a>0 \text{ then } x_1=a\\ &\qquad\qquad\quad\;\; x_2=2a\\ &\text{If } a<0 \text{ then } x_1=2a\\ &\qquad\qquad\quad\;\; x_2=a\end{aligned}$$

Now $x_1^2=x_2 \;\;\Rightarrow\;\; a^2=2a \;\;\Rightarrow\;\; a=2$

other option not valid]

(65) **(B).** Let $y=\frac{x^2}{x^3+200}\;;\;\frac{dy}{dx}=\frac{x(400-x^3)}{(x^3+200)^2}$

Now if $x>(400)^{1/3}$, y is decreasing and if $x < (400)^{1/3}$, y is increasing hence y is greatest at $x = (400)^{1/3}$.



But $x \in N$ hence practical maxima occurs at

$$x = 7 \text{ or } x = 8 \;;\; a_7 = \frac{49}{543} \;;\; a_8 = \frac{64}{712}$$

(66) **(B).** Domain is [2, 3], now $\frac{dy}{dx} = \frac{1}{2\sqrt{x-2}} - \frac{1}{\sqrt{3-x}}$

for maximum or minimum $\frac{dy}{dx} = 0$

$4(x-2) = 3 - x \Rightarrow 5x = 11 \Rightarrow x = 11/5$

now, $f(2) = 2, f(3) = 1, f\left(\frac{11}{5}\right) = \frac{1}{\sqrt{5}} + 2.\frac{2}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5}$

(67) **(D).** Slope of normal to $y = f(x)$ at (3, 4) is $\frac{-1}{f'(3)}$.

Thus, $\frac{-1}{f'(3)} = \tan\left(\frac{3\pi}{4}\right) = \tan\left(\frac{\pi}{2} + \frac{\pi}{4}\right) = -\cot\frac{\pi}{4} = -1$

$\Rightarrow f'(3) = 1$.

(68) **(D).** Row operations will give $f(x) = \begin{vmatrix} x-1 & x+1 & 2x+1 \\ 2 & 2 & 2 \\ 3 & -3 & -1 \end{vmatrix}$

which is a linear function of x and hence has no extreme points.

(69) **(A).** (2, 6) corresponds to t = 1 on the curve.
Slope at t = 1 is 8.

Normal at (2, 6) is $y - 6 = -\frac{1}{8}(x-2)$ whose intercept form is $\frac{x}{50} + \frac{y}{25/4} = 1$.

(70) **(C).** We have, $f(x) = \cot^{-1} x + x$

$\Rightarrow f'(x) = -\frac{1}{1+x^2} + 1 = \frac{x^2}{1+x^2}$. Clearly, $f'(x) > 0$ for all x.

Therefore f (x) increases in $(-\infty, \infty)$

(71) **(C).** Slope of tangent

$$= -\frac{f_x}{f_y} = \frac{-x^{-2/3}}{y^{-2/3}} = -\left(\frac{y}{x}\right)^{2/3} = -1 \text{ at } (a/8, a/8)$$

Tangent is $y - a/8 = -1(x - a/8)$ or $x + y = a/4$
Its intercepts on axes are A = a/4, B = a/4
Portion of tangent intercepted between the axes is

$\sqrt{A^2 + B^2} = \sqrt{2}$ (given)

$\therefore \frac{a^2}{16} + \frac{a^2}{16} = 2$ or $a^2 = 16 \Rightarrow a = 4$

(72) **(A).** $xy = -5$

$\Rightarrow x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x} > 0$ (as $xy = -5 < 0$)

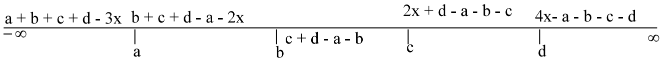
$\Rightarrow$ The slope of the normal is negative

$\Rightarrow -\frac{a}{b} < 0 \Rightarrow \frac{a}{b} > 0 \Rightarrow a > 0, b > 0$ or $a < 0, b < 0$

(73) **(A).** $f'(x) = 3x^2 + 2ax + b + 5\sin 2x > 0$ for all x
$\Rightarrow 3x^2 + 2ax + b + 5\sin 2x > 0$
$\Rightarrow 3x^2 + 2ax + b > -5\sin 2x > -5$
$\Rightarrow 3x^2 + 2ax + (b+5) > 0$
$\Rightarrow 4a^2 - 12(b+5) < 0 \Rightarrow a^2 - 3b - 15 < 0$

(74) **(D).** $f(x) = 2\left(x^2 - \frac{5}{4}\right)^2 + \frac{31}{8} \geq \frac{31}{8}$ attained at $x = -\frac{\sqrt{5}}{2}$

(75) **(C).** $|x-a| + |x-b| + |x-c| + |x-d|$



So minimum value is (c + d – a – b)

(76) **(B).** Since $-1 \leq \sin x \leq 1, -1 \leq \cos x \leq 1$,
$\therefore [\sin x] \leq 1$ and $[\cos x] \leq 1$
$[\sin x] + [\cos x] \leq 2$
But $[\sin x] + [\cos x] = 2$ if $\sin x = 1$ and $\cos x = 1$ which is not possible.
$\therefore$ maximum value of $[\sin x] + [\cos x]$ is 1

(77) **(B).** $\frac{dy}{dx} = 0$ as tangent is x - axis

$\therefore 2x + a = 0$ or $x = -a/2$. But point lies on the curve.

$y = x^2 + ax + 25 = \frac{100 - a^2}{4} = 0$ as it lies on x - axis.

$\therefore a = \pm 10$

(78) **(C).** $\because \frac{d}{dx}\cos|x| = -\sin x$, for $x \in R$

$\therefore f'(x) = -\sin x - 2a$
Now f(x) is an increasing function, therefore
$f'(x) > 0 \Rightarrow -\sin x - 2a > 0$

$\Rightarrow a < -\frac{1}{2}\sin x \Rightarrow a \leq -\frac{1}{2}$

(79) **(B).** Let $f(x) = x^3 + 2x^2 + x + 5$, then $f(-2) = 3$ and $f(-3) = -7$
$\Rightarrow$ f(x) has an odd number of real roots –2 and –3. But, if the given equation has an odd number of real roots between –2 and –3, then their product will be less than –8.
However, product of the roots is –5. So, there is exactly one real root between –3 and –2
Let it be α. Then $-3 < \alpha < -2 \Rightarrow [\alpha] = -3$
Also, $f'(x) = 3x^2 + 4x + 1$, where $D = 16 - 12 > 0$

(80) **(B).** $\because 0 < x < \frac{\pi}{2} \quad \therefore \tan x > 0, \ \cot x > 0$

Now, $f(x) = \dfrac{ab\,(a^2 - b^2)\sin x \cos x}{a^2 \sin^2 x + b^2 \cos^2 x}$

$$= \frac{ab\,(a^2 - b^2)}{a^2 \tan x + b^2 \cot x} = \frac{ab\,(a^2 - b^2)}{(a\sqrt{\tan x} - b\sqrt{\cot x})^2 + 2ab}$$

f(x) will be max. when $(a\sqrt{\tan x} - b\sqrt{\cot x})^2$ is minimum. But its minimum value is zero.

$\therefore$ Max. value of $f(x) = \dfrac{ab\,(a^2 - b^2)}{2ab} = \dfrac{a^2 - b^2}{2}$

(81) **(B).** Time taken by the truck $= \dfrac{300}{x}$ hours

$\therefore$ petrol consumed $= \left(2 + \dfrac{x^2}{600}\right)\dfrac{300}{x}$ litre

$\therefore$ Expenses on travelling,

$$e = 200 \times \frac{300}{x} + \left(2 + \frac{x^2}{600}\right)\frac{3000}{x}$$

$$= \frac{60000}{x} + \frac{6000}{x} + 5x = \frac{66000}{x} + 5x$$

$\therefore \dfrac{dE}{dx} = -\dfrac{66000}{x^2} + 5 < 0$ for all $x \in [30, 60]$

$\therefore$ most economical speed is 60 kmph.

(82) **(D).** Given, $V = \pi r^2 h$

Differentiating both sides

$$\frac{dV}{dt} = \pi\left(r^2\frac{dh}{dt} + 2r\frac{dr}{dt}h\right) = \pi r\left(r\frac{dh}{dt} + 2h\frac{dr}{dt}\right)$$

$$\frac{dr}{dt} = \frac{1}{10} \text{ and } \frac{dh}{dt} = -\frac{2}{10}$$

$$\frac{dV}{dt} = \pi r\left(r\left(-\frac{2}{10}\right) + 2h\left(\frac{1}{10}\right)\right) = \frac{\pi r}{5}(-r + h)$$

Thus, when r = 2 and h = 3,

$$\frac{dV}{dt} = \frac{\pi\,(2)}{5}(-2 + 3) = \frac{2\pi}{5}$$

(83) **(B).** $x_1 = (\tan\theta)^{\cot\theta}, x_2 = (\cot\theta)^{\cot\theta}$,

$x_3 = (\tan\theta)^{\tan\theta}$ and $x_4 = (\cot\theta)^{\tan\theta}$

$0 < \theta < \pi/4 \ \Rightarrow\ \tan\theta < \cot\theta$

$\therefore x_1 < x_2 ; \ x_3 < x_2$
$x_1 < x_3$
$x_1 < x_3 < x_2$
$x_3 < x_4 < x_2$
$x_1 < x_3 < x_4 < x_2$

(84) **(D).** Let $f(x) = x^2 \ln\dfrac{1}{x} = -x^2 \ln x$

$\therefore f'(x) = -2x\ln x - x = 0 \Rightarrow x = e^{-1/2}$

$\lim\limits_{x\to 0} f(x) = 0, \ \lim\limits_{x\to\infty} f(x) = -\infty$

and $f(e^{-1/2}) = \dfrac{1}{2e}$ $\therefore$ Maximum value of f(x) is $\dfrac{1}{2e}$.

Alt. : Check with f''(x)

(85) **(D).** $f(1) = -6$

For maximum at $x = 1$

$\lim\limits_{x\to 1^-} f(x) = \tan^{-1}\alpha - 5 < -6$

$\Rightarrow \tan^{-1}\alpha - 1 \Rightarrow \alpha < -\tan 1$

(86) **(C).** $y - e^{xy} + x = 0$

$$\therefore \frac{dy}{dx} - e^{xy}\left(y + x\frac{dy}{dx}\right) + 1 = 0$$

$$\text{i.e., } \frac{dy}{dx} - y(x+y) - x(x+y)\frac{dy}{dx} + 1 = 0$$

$$\text{i.e., } [1 - x\,(x+y)]\,\frac{dy}{dx} = y\,(x+y) - 1$$

for the vertical tangents

$1 - x\,(x+y) = 0$ i.e., $y = \dfrac{1 - x^2}{x}$

i.e., $\dfrac{1}{x} - x - e^{1-x^2} + x = 0$ i.e., $e^{1-x^2} = \dfrac{1}{x}$

$\therefore x = 1$ and $y = 0$

(87) **(A).** $y\,(\text{at } t = \frac{\pi}{4}) = a\left(\dfrac{1}{\sqrt{2}} - \dfrac{\pi}{4}\dfrac{1}{\sqrt{2}}\right) = \dfrac{a}{\sqrt{2}}\left(1 - \dfrac{\pi}{4}\right)$

$$\left|\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \frac{a\left(\cos\frac{\pi}{4} - \cos\frac{\pi}{4} + \frac{\pi}{4}\sin\frac{\pi}{4}\right)}{a\left(-\sin\frac{\pi}{4} + \frac{\pi}{4}\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)}$$

$$= \frac{\frac{\pi a}{4}\sin\frac{\pi}{4}}{\frac{\pi a}{4}\cos\frac{\pi}{4}} = \tan\frac{\pi}{4} = 1$$

$\therefore$ length of the tangent

$$= \frac{y}{\frac{dy}{dx}}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{\frac{a}{\sqrt{2}}\left(1 - \frac{\pi}{4}\right)}{1}\sqrt{1+1} = a\left(1 - \frac{\pi}{4}\right)$$

(88) **(B).** t = length of tangent = $\frac{|y_1|}{\sin\psi}$

n = length of normal = $\frac{|y_1|}{\cos\psi}$, t' = sub tangent = $\frac{|y_1|}{\tan\psi}$,

n' = subnormal = $\frac{|y_1|}{\cot\psi}$

(i) $t'n' = \frac{|y_1|}{\tan\psi}.\frac{|y_1|}{\cot\psi} = y_1^2$

(ii) $\frac{1}{t^2}+\frac{1}{n^2} = \frac{\cos^2\psi}{y_1^2}+\frac{\sin^2\psi}{y_1^2} = \frac{1}{y_1^2}$

(iii) $nt' = \frac{|y_1|}{\cos\psi}.\frac{|y_1|}{\tan\psi} = \frac{|y_1||y_1|}{\sin\psi} = |y_1|t$

(89) **(C).** $x^2y = c^3$

$$x^2\frac{dy}{dx}+2xy=0 \Rightarrow \frac{dy}{dx} = -\frac{2y}{x}$$

equation of tangent at (x,y)

$$Y-y = -\frac{2y}{x}(X-x)$$

$Y=0$, gives, $X = \frac{3x}{2} = a$

and $X = 0$, gives, $Y = 3y = b$

Now $a^2b = \frac{9x^2}{4}.3y = \frac{27}{4}x^2y = \frac{27}{4}c^3 \Rightarrow$ (C)

(90) **(A).** $\because x = -2$

$\Rightarrow |x| = -x$

$\Rightarrow f(x) = |x^2+x|$ $\quad\because x^2+x>0$

$\therefore f(x) = x^2+x$ .... (1)

$\because \frac{dy}{dx} = f'(x) = 2x+1$

$\therefore$ Slope of normal at x = – 2

$$= \frac{-1}{2(-2)+1} = \frac{-1}{-4+1} = \frac{1}{3}$$

(91) **(B).** $y = x(\ln x - 2)$

$$y' = x\left(\frac{1}{x}\right) + (\ln x - 2) = \ln x - 1$$

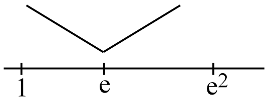
$$\frac{dy}{dx} = \ln x - 1 = 0 \Rightarrow x = e$$

now f(1) = – 2

f(e) = – e (least)

$f(e^2) = 0$ (greatest)

$\therefore$ difference = 0 – (–e) = e Ans.



(92) **(D).** $f(x) = x\ln x - x + 1$ $\quad\therefore f(1) = 0$

$f'(x) = 1 + \ln x - 1 = \ln x$

$\therefore f'(x) < 0$ if $0 < x < 1$

$f'(x) > 0$ if $1 < x$

$\therefore f'(x) > 0$ for all $x \in (0, 1) \cup (1, \infty)$

(93) **(C).** $f(x) = x^4 + ax^3 + \frac{3x^2}{2} + 1$

$f'(x) = 4x^3 + 3ax^2 + 3x$

$f''(x) = 12x^2 + 6ax + 3 \geq 0 = 3(4x^2 + 2ax + 1) \geq 0 \quad \forall x \in R$

$4a^2 - 16 \leq 0 \Rightarrow a^2 \leq 4 \Rightarrow -2 \leq a \leq 2$

(94) **(B).** $y = \frac{ax}{b-x}$ .... (1)

Since (1, 1) lies on (1)

$\therefore b - 1 = a$ .... (2)

From (1), $\frac{dy}{dx} = \frac{ab}{(b-x)^2}$

At (1, 1), $\frac{dy}{dx} = \frac{ab}{(b-1)^2} = 2$ .... (3)

From (2) and (3), $\frac{b(b-1)}{(b-1)^2} = 2 \Rightarrow b = 2b-2 \Rightarrow b = 2$

(95) **(D).** $x^2+y^2-6x+8y+24=0$

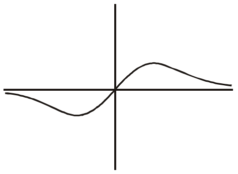
it centre is (3, – 4) and radius = 1

$\therefore$ least distance of (0, 0) from the circle = 5 – 1 = 4

$\therefore \sqrt{x^2+y^2} = 4$ i.e., $\therefore x^2+y^2 = 16$

$\therefore$ minimum value of $\log_2(x^2+y^2) = \log_2 16 = 4$

(96) **(A).** Graph of $y = \frac{2x}{1+x^2}$ is



from the graph it is clear that there are three points of inflection separated by a point of minimum

Alternate :

$$\frac{dy}{dx} = \frac{2(1+x^2)-2x.2x}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}$$

$\therefore$ x = – 1 is a minimum and x = 1 is a maximum.

$$\frac{d^2y}{dx^2} = \frac{(1+x^2)^2(-4x)-2(1-x^2)\,2(1+x^2).2x}{(1+x^2)^4}$$

$$= \frac{-4x(1+x^2)-8x(1-x^2)}{(1+x^2)^3}$$

$$= \frac{-4x-4x^3-8x+8x^3}{(1+x^2)^3} = \frac{-12x+4x^3}{(1+x^2)^3} = \frac{4x(x^2-3)}{(1+x^2)^3}$$

$\therefore$ There are 3 points of inflection : $x = 0, -\sqrt{3}, \sqrt{3}$

($\because \frac{d^2y}{dx^2}$ changes sign while x passes through these points)

(97) **(B).** $f(x) = 2x^3 - 3(a+1)x^2 + 6ax - 12$

$f'(x) = 6\{x^2 - (a+1)x + a\} = 0$

$= 6(x-1)(x-a) = 0$

$f''(x) = 12x - 6(a+1)$

$f''(1) = 6 - 6a > 0$

$f''(a) = 12a - 6a - 6 = 6a - 6 = 6(a-1)$

$\therefore$ If $a < 1$ then $x = a$ is a local max.

$\therefore 2a = 1 \quad \therefore a = 1/2$

If $a > 1$ then $x = a$ is a least minimum $\quad \therefore 2 = a$

(98) **(C).** Point of intersection of curves

$y = |x^2 - 1|$ and $y = \sqrt{7 - x^2}$ is $(\pm\sqrt{3}, 2)$

$y = x^2 - 1$ and $y = \sqrt{7 - x^2}$

$\frac{dy}{dx} = 2x$ and $\frac{dy}{dx} = -\frac{x}{y}$

$m_1 = \left.\frac{dy}{dx}\right|_{(\sqrt{3},2)} = 2\sqrt{3}$ and $m_2 = \left.\frac{dy}{dx}\right|_{(\sqrt{3},2)} = -\frac{\sqrt{3}}{2}$

$\tan\theta = \left|\frac{5\sqrt{3}}{4}\right|$

(99) **(C).** $f(x) = 2\sin x + \sin 2x$

$f'(x) = 2\cos x + 2\cos 2x = 2(\cos x + \cos 2x)$

$\therefore f'(x) = 0 \Leftrightarrow 2\cos^2 x + \cos x - 1 = 0$

$\cos x = \frac{-1 \pm 3}{4} = -1, \frac{1}{2} \qquad \therefore x = \pi, \frac{\pi}{3}$

Now, $f(0) = 0, f\left(\frac{3\pi}{2}\right) = -2$;

$f(\pi) = 0, f\left(\frac{\pi}{3}\right) = 2\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

$\therefore$ difference between greatest value and least value

$= \frac{3\sqrt{3}}{2} + 2$

(100) **(B).** Given curve is $x^{3/2} + y^{3/2} = 2a^{3/2}$ .... (1)

$\therefore \frac{3}{2}\sqrt{x} + \frac{3}{2}\sqrt{y}\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = -\frac{\sqrt{x}}{\sqrt{y}}$

Since tangent is equally inclined to the axes

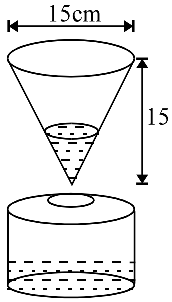
$\therefore \frac{dy}{dx} = \pm 1 \quad \therefore -\frac{\sqrt{x}}{\sqrt{y}} = \pm 1 \quad \Rightarrow -\frac{\sqrt{x}}{\sqrt{y}} = -1$

$\Rightarrow \sqrt{x} = \sqrt{y} \qquad [\because \sqrt{x} > 0, \sqrt{y} > 0]$

Putting $\sqrt{y} = \sqrt{x}$ in (1), we get

$2x^{3/2} = 2a^{3/2} \quad \Rightarrow x^3 = a^3$

$\therefore x = a$ and so $y = a$



(101) **(D).** For cylindrical pot $V = \pi r^2 h$

$$\frac{dV}{dt} = \pi\left[r^2\frac{dh}{dt} + h \cdot 2r\frac{dr}{dt}\right]$$

$(r = \text{constant}, \frac{dr}{dt} = 0)$

hence, $100 = \pi r^2 \frac{dh}{dt}$

$100 = \pi \cdot \frac{225}{4} \cdot \frac{dh}{dt} \qquad (r = 15/2 \text{ cm})$

$\frac{dh}{dt} = \frac{400}{225\pi} = \frac{16}{9\pi}$ cm/min

EXERCISE-2

(1) **(B).** $xy = 25$; $xy' + y(1) = 0$; $y' = -y/x$

Consider a point (5, 5) on $xy = 25$; $m = -5/5 = -1$

Equation of tangent at (5, 5) is $y - 5 = -1(x - 5)$

$y - 5 = -x + 5$; $x + y = 10$

$\frac{x}{10} + \frac{y}{10} = 1$; Area $= \frac{1}{2} \times 10 \times 10 = 50$

$\therefore$ In general, area = 50 sq. units

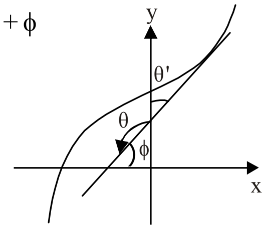
(2) **(A).** Clearly $\theta = 90 + \phi$; $\theta = 90 + \phi$

$\tan\theta = \tan(90 + \phi)$

$= -\cot\phi$

$\tan\phi = \frac{dy}{dx} = 3x^2\Big|_{(1,2)} = 3$

$\therefore$ Required $= -\cot\phi = -1/3$



(3) **(B).** Diff. $x^3 - 3xy^2 + 2 = 0$ w.r.t. x to get $\frac{dy}{dx} = -\frac{3x^2 - 3y^2}{-6xy}$

Diff. $3x^2y - y^3 - 2 = 0$ w.r.t. x to get $\frac{dy}{dx} = -\frac{6xy}{3x^2 - 3y^2}$

Clearly, $m_1 \times m_2 = -1$

(4) **(C).** $f'(x) = \frac{1}{1+x^2} - \frac{1}{2x} = \frac{-(x-1)^2}{2x(1+x^2)}$

$\therefore f'(x) = 0 \Rightarrow x = 1$

Now, $f(1) = \tan^{-1} 1 - \frac{1}{2}\log 1 = \frac{\pi}{4}$

$f\left(\frac{1}{\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \frac{1}{4}\log 3 = \frac{\pi}{6} + \frac{1}{4}\log 3$

and $f(\sqrt{3}) = \tan^{-1}(\sqrt{3}) - \frac{1}{4}\log 3 = \frac{\pi}{3} - \frac{1}{4}\log 3$

Hence, the greatest value of f (x) is $\frac{\pi}{3} - \frac{1}{4}\log 3$

(5) $f\left(\frac{\pi}{4}\right)=\frac{\pi}{4}+1$

$f'(x)=1+3\tan^2x\sec^2x$

$f'\left(\frac{\pi}{4}\right)=1+3\times2=7;\ \ g'\left(\frac{\pi}{4}+1\right)=\frac{1}{f'\left(\frac{\pi}{4}\right)}=\frac{1}{7}$

(6) **(D).** $y^3+3x^2=12y \Rightarrow 3y^2.\frac{dy}{dx}+6x=12.\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}(3y^2-12)+6x=0 \Rightarrow \frac{dy}{dx}=\frac{6x}{12-3y^2}$

$\Rightarrow \frac{dx}{dy}=\frac{12-3y^2}{6x}$

Since tangent is parallel to y-axis

$\therefore \frac{dx}{dy}=0 \Rightarrow 12-3y^2=0$ or $y=\pm2$.

Then $x=\pm\frac{4}{\sqrt{3}}$. At $\left(\pm\frac{4}{\sqrt{3}},-2\right)$; the equation of curve doesn't satisfy.

(7) **(C).** $\frac{dy}{dx}=x^3-3x+\lambda$

We must have $\frac{dy}{dx}=0$ for three values.

$\Rightarrow$ Equation $x^3-3x+\lambda=0$ has three real roots.
Let $g(x)=x^3-3x+\lambda$ then $g'(x)=0$ for $x=1,-1$
For three roots of $g(x)=0$
$g(1)\,g(-1)<0 \Rightarrow -2<\lambda<2 \Rightarrow k=2$

(8) **(B).** We have, $xy=c^2$

$\Rightarrow x\frac{dy}{dx}+y.1=0 \Rightarrow \frac{dy}{dx}=-\frac{y}{x};$

$\therefore \left.\frac{dy}{dx}\right]_{(ct_1,c/t_1)}=-\frac{1}{t_1^2}$

The equation of the normal at

$\left(ct_1,\frac{c}{t_1}\right)$ is $y-\frac{c}{t_1}=t_1^{\,2}(x-ct_1)$.

since, this normal passes through

$\left(ct_2,\frac{c}{t_2}\right)$ therefore, $\frac{c}{t_2}-\frac{c}{t_1}=t_1^{\,2}(ct_2-ct_1)$

$\Rightarrow t_1^{\,3}t_2=-1$ (as $t_1-t_2\neq0$)

(9) **(A).** Let P (x_1, y_1) be the point of intersection of the two curves.

We have, $y^2-2x \Rightarrow 2y\frac{dy}{dx}=2 \Rightarrow m_1=\left.\left(\frac{dy}{dx}\right)\right]_{(x_1,y_1)}=\frac{1}{y_1}$

and $2xy=k \Rightarrow x\frac{dy}{dx}+y=0$

$\Rightarrow m_2=\left.\left(\frac{dy}{dx}\right)\right]_{(x_1,y_1)}=-\frac{y_1}{x_1}$

Since the two curves intersect at right angles,

$\therefore m_1m_2=-1 \Rightarrow \left(\frac{1}{y_1}\right)\left(-\frac{y}{x_1}\right)=-1 \Rightarrow x_1=1$

and hence from $y_1^{\,2}=2x_1$, we get $y_1^{\,2}=2$
since (x_1, y_1) also lies on $2xy=k$
$\therefore k^2=4x_1^{\,2}y_1^{\,2}=4\times1\times2=8$

(10) **(A).** $f(x)=xe^{x(1-x)}$
$\Rightarrow f'(x)=e^{x(1-x)}+xe^{x(1-x)}(1-2x)$
$=e^{x(1-x)}[1+x-2x^2]=-e^{x(1-x)}(x-1)(2x+1)$
Now f (x) is increasing when $f'(x)>0$
$\Rightarrow -e^{x(1-x)}(x-1)(2x+1)>0$
$\Rightarrow (x-1)(2x+1)<0 \qquad [\because e^{x(1-x)}>0]$
$\Rightarrow -1/2<x<1$. Also f (x) is decreasing when $f'(x)<0$
$\Rightarrow (x-1)(2x+1)>0 \Rightarrow x<-1/2$ or $x>1$

(11) **(D).** $f'(x)=\frac{\cos x-\sin x}{1+(\sin x+\cos x)^2}$

f (x) is monotonic increasing when $f'(x)>0$

$\Rightarrow \frac{\cos x-\sin x}{1+(\sin x+\cos x)^2}>0 \Rightarrow \cos x-\sin x>0$

$\Rightarrow \sqrt{2}\cos(x+\pi/4)>0$
$\Rightarrow -\pi/2<x+\pi/4<\pi/2$
($\because \cos\theta$ is positive when $-\pi/2<\theta<\pi/2$)
$\therefore -3\pi/4<x<\pi/4$

(12) **(D).** $f'(x)=4x-1/x$
f '(x) is monotonic increasing when $f'(x)>0$

$\Rightarrow 4x-1/x>0 \Rightarrow \frac{4x^2-1}{x}>0$

$$\Rightarrow \begin{cases} 4x^2-1>0 & \text{when} \quad x>0 \\ 4x^2-1<0 & \text{when} \quad x<0 \end{cases}$$

But $x>0,\ 4x^2-1>0 \Rightarrow x^2>1/4 \Rightarrow |x|>1/2$
$\Rightarrow x\in(1/2,\infty)$
and $x<0,\ 4x^2-1<0 \Rightarrow x^2<1/4 \Rightarrow |x|<1/2$
$\Rightarrow x\in(-1/2,0)$
$\therefore x\in(-1/2,0)\cup(1/2,\infty)$

(13) **(D).** $f(x)=2\log(x-2)-x^2+4x+1$

$\Rightarrow f'(x)=\frac{2}{x-2}-2x+4$

$\Rightarrow f'(x)=2\left[\frac{1-(x-2)^2}{x-2}\right]=-2\frac{(x-1)(x-3)}{x-2}$

$\Rightarrow f'(x)=\frac{2(x-1)(x-3)(x-2)}{(x-2)^2}$

$\therefore f'(x)>0 \Rightarrow -2(x-1)(x-3)(x-2)>0$
$\Rightarrow (x-1)(x-2)(x-3)<0 \Rightarrow x\in(-\infty,1)\cup(2,3)$
Thus, f (x) is increasing on $(-\infty,1)\cup(2,3)$.

(14) **(C).** Here, $by^2 = (x+a)^3$ .... (1)

Differentiating both the sides, we get

$$2by\frac{dy}{dx} = 3\,(x+a)^2 \Rightarrow \frac{dy}{dx} = \frac{3\,(x+a)^2}{2by}$$

∴ Length of subnormal

$$SN = y\frac{dy}{dx} = \frac{3}{2}\frac{(x+a)^2}{b} \quad ....(2)$$

∴ Length of subtangent

$$ST = y.\frac{dx}{dy} = \frac{2by^2}{3\,(x+a)^2} \quad ....(3)$$

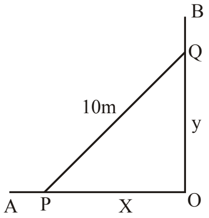
$\therefore\ p\,(SN) = q\,(ST)^2$

$$\Rightarrow \frac{p}{q} = \frac{(ST)^2}{(SN)} = \frac{8}{27}\frac{b^3y^4}{(x+a)^6} = \frac{8b}{27}$$

$$\left(\because \frac{b^2y^4}{(x+a)^6} = 1\right)$$

(15) **(C).** $\because\ x^2 + y^2 = 102$

Given that $\frac{dx}{dt} = 2$



We have to find $\frac{dy}{dt}$ when $x = 6$ in $x^2 + y^2 = 100$,

we get $y = 8$.

Now, $x^2 + y^2 = 10^2$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow 6\frac{dx}{dt} + 8\frac{dy}{dt} = 0$$

$$\Rightarrow 6 \times 2 + 8\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{3}{2} \text{ metres/min}$$

(16) **(A).** $f'(x) = -12\cos^3 x \sin x - 30\cos^2 x \sin x - 12\cos x \sin x$
$= -6 \sin x \cos x\,(\cos x + 2)\,(2\cos x + 1)$

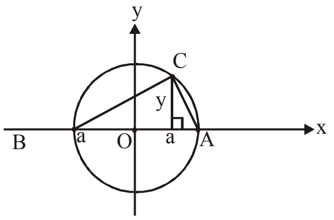
$f'(x) = 0$, for $x = 0, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$

Clearly, $f'(x) > 0$ for $\frac{\pi}{2} < x < \frac{2\pi}{3}$

And $f'(x) < 0$; for $0 < x < \frac{\pi}{2}$ or $\frac{2\pi}{3} < x < \pi$

(17) **(B).** Let $f(x) = 2x^3 + 15$ and $g(x) = 9x^2 - 12$ then
$f'(x) = 6x^2 \ \forall\ x \in R$
∴ $f(x)$ is increasing function $\forall\ x \in R$
Also, $g'(x) > 0 \Rightarrow 18x - 12 > 0 \Rightarrow x > 2/3$
Thus, $f(x)$ and $g(x)$ both increases for $x > 2/3$
Let $f(x) = f(x) - g(x)$, $F'(x) < 0$
($\because$ $f(x)$ increases less rapidly than the function $g(x)$)
$\Rightarrow 6x^2 - 18x + 12 < 0 \Rightarrow 1 < x < 2$

(18) **(A).** Let the equation of the circle by $x^2 + y^2 = a^2$
Let A (a, 0), B (–a, 0) be the ends of the diameter and C (x, y) be any point on the circle.



Area of $\Delta\, ABC = A = \frac{1}{2} \times AB \times y = ay = a\sqrt{a^2 - x^2}$

∴ A is maximum if $x = 0$

i.e. c lies on y-axis and then Δ CAB is an isosceles triangle.

(19) **(B).** $f'(x) = \frac{d}{dx}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \frac{2x}{|x|\,(1+x^2)}$

$f'(x) < 0$ if $x < 0$
$x \in (-\infty, 0)$

or $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1} x, \text{ if } 0 \le x < \infty \\ -2\tan^{-1} x, \text{ if } -\infty < x < 0 \end{cases}$

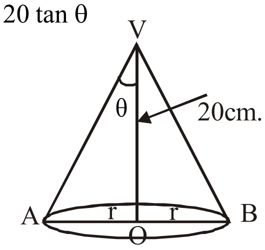
(20) **(B).** Let $f(x) = x^{1/x} \Rightarrow \log f(x) = \left(\frac{1}{x}\right)\log x$

Diff. both the sides, we get

$$f'(x) = f(x)\left(\frac{x.(1/x) - \log x}{x^2}\right)$$

So, $f'(x) = 0 \Rightarrow x = e$
when $0 < x < e \Rightarrow f'(x) > 0$ and $e < x < \infty \Rightarrow f'(x) < 0$
Thus, $f(x)$ has a maximum at $x = e$ and max. $f(x) = e^{1/e}$.

(21) **(B).** Let θ be the semi-vertical angle and r be the radius of the cone at time t. Then, $r = 20\tan\theta$



$$\frac{dr}{dt} = 20\sec^2\theta\frac{d\theta}{dt}$$

$$\Rightarrow \frac{dr}{dt} = 20 \times \sec^2 30° \times 2$$

$$= \frac{160}{3} \text{ cm/sec}$$

(22) **(B).** Since $\log|\,x\,| = \begin{cases} \log(x); & x > 0 \\ \log(-x); & x < 0 \end{cases}$

$$\therefore \frac{d}{dx}\log|\,x\,| = \left[\begin{array}{l} \frac{1}{x};\ x > 0 \\ \frac{1}{(-x)}(-1) = \frac{1}{x};\ x < 0 \end{array}\right.$$

y has extreme values at $x = -1, 2,$

So $\left(\frac{dy}{dx}\right)_{(-1)} = \left(\frac{dy}{dx}\right)_{(2)} = 0$. Now $\frac{dy}{dx} = \frac{a}{x} + 2bx + 1$

$$\therefore \left(\frac{dy}{dx}\right)_{(-1)} = -a - 2b + 1 = 0$$

$$\left(\frac{dy}{dx}\right)_{(2)} = \frac{a}{2} + 4b + 1 = 0 \;;\; \therefore a = 2,\; b = -\frac{1}{2}.$$

(23) **(D).** Since $f(-x) = \frac{(e^{-2x}-1)}{(e^{-2x}+1)} = \frac{1-e^{2x}}{1+e^{2x}} = -\left[\frac{e^{2x}-1}{e^{2x}+1}\right] = -f(x)$

$\therefore f(x)$ is an odd function.

Also $f'(x) = \frac{(e^{2x}+1).2e^{2x} - 2e^{2x}(e^{2x}-1)}{(e^{2x}+1)^2}$

$$= \frac{2e^{2x}(e^{2x}+1-e^{2x}+1)}{(e^{2x}+1)^2} = \frac{4e^{2x}}{(e^{2x}+1)^2} > 0$$

$\Rightarrow f'(x)$ is $+ve$, $\therefore f(x)$ is an increasing function.

(24) **(D).** If $f(x) = (a+2)x^3 - 3ax^2 + 9ax - 1$ decreases monotonically for all $x \in R$, then $f'(x) \le 0$ for all $x \in R$

$\Rightarrow 3(a+2)x^2 - 6ax + 9a \le 0$ for all $x \in R$

$\Rightarrow (a+2)x^2 - 2ax + 3a \le 0$ for all $x \in R$

$\Rightarrow a + 2 < 0$ and Discriminant ≤ 0

$\Rightarrow a < -2, -8a^2 - 24a \le 0 \Rightarrow a < -2$ and $a(a+3) \ge 0$

$\Rightarrow a < -2,\; a \le -3$ or $a \ge 0 \Rightarrow a \le -3 \Rightarrow -\infty < a \le -3$.

(25) **(A).** Let (x, y) be the one point of parabola,
$y = x^2 + 7x + 2$
its distance from the line $y = 3x - 3$ or $3x - y - 3 = 0$ is

$$D = \left|\frac{3x - y - 3}{\sqrt{(10)}}\right| = \left|\frac{3x - (x^2 + 7x + 2) - 3}{\sqrt{(10)}}\right|$$

$$= \left|\frac{-x^2 - 4x - 5}{\sqrt{(10)}}\right|$$

$$D = \left|\frac{x^2 + 4x + 5}{\sqrt{(10)}}\right| = \left|\frac{(x+2)^2 + 1}{\sqrt{(10)}}\right|$$

$$= \frac{(x+2)^2 + 1}{\sqrt{(10)}} \text{ as } \frac{N^r}{D^r} \text{ is + ive}$$

$$\frac{dD}{dx} = \frac{2(x+2)}{\sqrt{(10)}} = 0 \therefore x = -2$$

and hence y is – 8 i.e. point is (–2, – 8)

$$\frac{d^2D}{dx^2} = \frac{2}{\sqrt{(10)}} = \text{+ ive and hence min. at } (-2, -8)$$

(26) **(B).** $z = y + f(v)$, (given $v = \frac{x}{y}$)

$$z = y + f\left(\frac{x}{y}\right) \qquad \text{……(i)}$$

Partially differentiate w.r.t. x and y respectively,

$$\frac{\partial z}{\partial x} = \frac{f'(x/y)}{y} \qquad \text{…..(ii)}$$

$$\frac{\partial z}{\partial y} = 1 + f'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right) \qquad \text{…..(iii)}$$

Now, $v\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{x}{y}\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$

$$= \frac{x}{y} f'\left(\frac{x}{y}\right) \times \frac{1}{y} + 1 - \frac{x}{y^2} f'\left(\frac{x}{y}\right) \text{ or } v\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1 .$$

(27) **(D).** $y = x \log x \Rightarrow \frac{dy}{dx} = 1 + \log x$

The slope of the normal $= -\frac{1}{(dy/dx)} = \frac{-1}{1 + \log x}$

The slope of the line $2x - 2y = 3$ is 1.

$\therefore \frac{-1}{1 + \log x} = 1 \Rightarrow \log x = -2 \Rightarrow x = e^{-2}$

$\therefore y = -2e^{-2}$ $\therefore$ Co-ordinate of the point is $(e^{-2}, -2e^{-2})$.

(28) **(D).** Curve is $y = be^{-x/a}$

Since the curve crosses y-axis (i.e., x = 0) $\therefore$ y = b

Now $\frac{dy}{dx} = \frac{-b}{a}e^{-x/a}$. At point (0, b), $\left(\frac{dy}{dx}\right)_{(0,b)} = \frac{-b}{a}$

Equation of tangent is, $y - b = \frac{-b}{a}(x - 0) \Rightarrow \frac{x}{a} + \frac{y}{b} = 1$.

(29) **(D).** $f(x) = 3x^2 + 15x + 5$; $f'(x) = 6x + 15$
Let $x = 3$ and $x + \Delta x = 3.02$; $f(3.02) = f(x + \Delta x)$
$\Delta y = f(x + \Delta x) - f(x)$
$\therefore f(x + \Delta x) = f(x) + \Delta y = f(x) + f'(x) . \Delta x$
$[\because \Delta y = f'(x) . \Delta x]$
$= (3x^2 + 15x + 5) + (6x + 15) . \Delta x$
$= [3 \times 3^2 + 15 \times 3 + 5] + [6 \times 3 + 15] (0.02)$
$= [27 + 45 + 5] + [18 + 15] (0.02) = 77 + 33 \times 0.02 = 77 + 0.66$
$= 77.66$

(30) **(B).** Curve $x + y = e^{xy}$

Differentiating with respect to x

$$1 + \frac{dy}{dx} = e^{xy}\left(y + x\frac{dy}{dx}\right) \text{ or } \frac{dy}{dx} = \frac{ye^{xy} - 1}{1 - xe^{xy}}$$

$$\frac{dy}{dx} = \infty \Rightarrow 1 - xe^{xy} = 0 \Rightarrow 1 - x(x + y) = 0$$

This hold for $x = 1$, $y = 0$.

(31) **(D).** Let $y = \frac{1}{40}(3x^4 + 8x^3 - 18x^2 + 60)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{40}(12x^3 + 24x^2 - 36x)$$

and $\frac{d^2y}{dx^2} = \frac{1}{40}(36x^2+48x-36)$

Now $\frac{dy}{dx} = 0 \Rightarrow x^3+2x^2-3x=0$

or $x(x-1)(x+3)=0$ or $x = 0, 1, -3$

At $x=0$, $\frac{d^2y}{dx^2} = -36 < 0$

∴ y is maximum at $x = 0$

⇒ The given function i.e. $1/y$ is minimum at $x = 0$

∴ minimum value of the function $= \frac{40}{60} = \frac{2}{3}$

(32) **(C).** Consider the function

$f(x) = \frac{x^2}{(x^3+200)}$ .....(i) ; $f'(x) = x\frac{(400-x^3)}{(x^3+200)^2} = 0$

When $x = (400)^{1/3}$, $(\because x \neq 0)$

$x = (400)^{1/3} - h \Rightarrow f'(x) > 0$

$x = (400)^{1/3} + h \Rightarrow f'(x) < 0$

∴ $f(x)$ has maxima at $x = (400)^{1/3}$

Since $7 < (400)^{1/3} < 8$, either a_7 or a_8 is the greatest term of the sequence.

$\because a_7 = \frac{49}{543}$ and $a_8 = \frac{8}{89}$ and $\frac{49}{543} > \frac{8}{89}$

∴ $a_7 = \frac{49}{543}$ is the greatest term.

(33) **(A).** $y = x^5 - 5x^4 + 5x^3 - 10$

∴ $\frac{dy}{dx} = 5x^4 - 20x^3 + 15x^2 = 5x^2(x^2-4x+3)$

$= 5x^2(x-3)(x-1)$

$\frac{dy}{dx} = 0$, gives $x = 0, 1, 3$

Now, $\frac{d^2y}{dx^2} = 20x^3 - 60x^2 + 30x = 10x(2x^2-6x+3)$

and $\frac{d^3y}{dx^3} = 10(6x^2 - 12x + 3)$

For $x = 0$: $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} = 0, \frac{d^3y}{dx^3} \neq 0$

∴ Neither minimum nor maximum

For $x = 1, \frac{d^2y}{dx^2} = -10 =$ negative .

∴ Maximum value $y_{\text{max.}} = -9$

For $x = 3, \frac{d^2y}{dx^2} = 90 =$ positive

∴ Minimum value $y_{\text{min.}} = -37$.

(34) **(C).** $f(x) = \frac{x}{4+x+x^2}$

Differentiate, $f'(x) = \frac{4+x+x^2-x(1+2x)}{(4+x+x^2)^2}$

For maximum $f'(x) = 0 \Rightarrow \frac{4-x^2}{(4+x+x^2)^2} = 0$

$\Rightarrow x = 2, -2$

Both values of x are out of interval

$\therefore f(-1) = \frac{-1}{4-1+1} = \frac{-1}{4}$, $f(1) = \frac{1}{4+1+1} = \frac{1}{6}$

(maximum).

(35) **(A).** Let co-ordinate of R (x, 0)

Given $P(1,1)$ and $Q(3,2)$

$$PR + RQ = \sqrt{(x-1)^2+(0-1)^2} + \sqrt{(x-3)^2+(0-2)^2}$$

$$= \sqrt{x^2-2x+2} + \sqrt{x^2-6x+13}$$

For minimum value of PR + RQ, $\frac{d}{dx}(PR+RQ) = 0$

$$\Rightarrow \frac{d}{dx}(\sqrt{x^2-2x+2}) + \frac{d}{dx}(\sqrt{x^2-6x+13}) = 0$$

$$\Rightarrow \frac{(x-1)}{\sqrt{x^2-2x+2}} = -\frac{(x-3)}{\sqrt{x^2-6x+13}}$$

Squaring both sides, $\frac{(x-1)^2}{(x^2-2x+2)} = \frac{(x-3)^2}{x^2-6x+13}$

$\Rightarrow 3x^2 - 2x - 5 = 0 \Rightarrow (3x-5)(x+1) = 0$, $x = \frac{5}{3}, -1$.

Also $1 < x < 3$. ∴ $R = (5/3, 0)$.

(36) **(A).** $f(x) = x^4 - \frac{x^3}{3} \Rightarrow f'(x) = 4x^3 - x^2$

For increasing $4x^3 - x^2 > 0 = x^2(4x-1) > 0$

Therefore, the function is increasing for $x > \frac{1}{4}$

Similarly decreasing for $x < \frac{1}{4}$.

(37) **(B).** Let y = 64 secx + 27 cosecx

$\Rightarrow \frac{dy}{dx} = 64\sec x \tan x - 27 \operatorname{cosec} x \cot x$

$$\frac{d^2y}{dx^2} = 64\sec^3 x + 64 \sec x \tan^2 x + 27\operatorname{cosec}^3 x + 27 \operatorname{cosec} x \cot^2 x$$

Now $\frac{dy}{dx} = 0 \Rightarrow 64 \sec x \tan x = 27 \operatorname{cosec} x \cot x$

$\Rightarrow \tan^3 x = 27/64 \Rightarrow \tan x = 3/4$

Also then $\frac{d^2y}{dx^2} > 0 \qquad (\because 0 < x < \frac{\pi}{2})$

So y is minimum when $x = \tan^{-1}(3/4)$ and its min. value $= 64\,(5/4) + 27\,(5/3) = 125$

(38) **(B).** $$f'(x) = \frac{\frac{1}{\sqrt{1-x^2}} \times (\cos^{-1}x - \sin^{-1}x) \times \left(-\frac{1}{\sqrt{1-x^2}}\right)}{(\cos^{-1}x)^2}$$

$$= \frac{\cos^{-1}x + \sin^{-1}x}{\sqrt{1-x^2}\,(\cos^{-1}x)^2} = \frac{\pi}{\sqrt{1-x^2}\,(\cos^{-1}x)^2} = +\text{ve}$$

$= f'(x) > 0$ st. Increasing function.

(39) **(A).** $$g'(x) = \frac{x^2 - 2x - 1}{(x-1)^2}$$

$$g''(x) = \frac{2\,(x-1)^2(x-1) - (x^2-2x-1)\,2\,(x-1)}{(x-1)^4}$$

$= \frac{4}{(x-1)^3} \neq 0 \Rightarrow$ No point of inflexion

(40) **(D).** At y-axis $\frac{dy}{dx} = 3$, $3ax^2 + 2bx + c = 3$ at $(0, 5)$

$\Rightarrow c = 3$

$\Rightarrow$ Again $\frac{dy}{dx} = 0$ at $(-2, 0)$ $\therefore$ $12a - 4b + c = 0$

$\Rightarrow 12a - 4b + 3 = 0$, also P lies on curve

$-8a + 4b - 2c + 5 = 0$ or $-8a + 4b - 1 = 0$

$\Rightarrow 4a + 2 = 0 \Rightarrow a = -1/2,\ b = -3/4$

(41) **(A).** (i) $$\frac{dx}{d\theta} = -\frac{-a\,(\cos 2\theta \sin\theta + \cos\theta \sin 2\theta)}{\sqrt{\cos 2\theta}} = \frac{-a\sin 3\theta}{\sqrt{\cos 2\theta}}$$

(ii) $$\frac{dy}{d\theta} = a\sqrt{\cos 2\theta}.\cos\theta - \frac{a\sin\theta\sin 2\theta}{\sqrt{\cos 2\theta}} = \frac{a\cos 3\theta}{\sqrt{\cos 2\theta}}$$

$\Rightarrow \frac{dy}{dx} = -\cot 3\theta \Rightarrow \left(\frac{dy}{dx}\right)\left(\theta = \frac{\pi}{6}\right) = 0 \Rightarrow$ parallel to x-axis.

(42) **(C).** For curve, $y^2 = 6x$, $\frac{dy}{dx} = \frac{3}{y} \Rightarrow m_1$ .... (1)

For $9x^2 + by^2 = 16$, $\frac{dy}{dx} = \frac{-9x}{by} \Rightarrow m_2$ .... (2)

For orthogonally, $1 + m_1 m_2 = 0$, $by^2 = 27x$ .... (3)

Putting value of $y^2 = 6x$ in eq. (3)

We get $b = 9/2$

(43) **(D).** We have $$f(x) = \frac{|x-1|}{x^2} = \begin{cases} \frac{-x+2}{x^2}, & x \geq 1 \\ \frac{1-x}{x^2}, & x < 1 \end{cases}$$

Clearly, f (x) is not differentiable at x = 0 and x = 1.

So, by definition, these are two of the critical points.

For points other than these two, we have

$$f'(x) = \begin{cases} \frac{-x+2}{x^3}, & x > 1 \\ \frac{1-x}{x^2}, & x < 1 \end{cases}$$

Clearly, $f'(x) = 0$ at $x = 2$. So $x = 2$ is also a critical points.

Hence, f (x) has three critical points, viz. 0, 1 and 2.

(44) **(B).** $$f'(x) = \frac{c\,(x-1)(x-4) - (cx+d)\,(2x-5)}{(x-1)^2(x-4)^2}$$

So, $0 = f'(2) = \frac{-2c + (2c+d)}{4} = \frac{d}{4} \Rightarrow d = 0$

Also, $-1 = f(2) = \frac{2c+d}{-2} = -c \Rightarrow c = 1$

(45) **(C).** $$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{nx^{n-1}y}{x^n} = -\frac{ny}{x}$$

Equation of tangent is $Y - y = -\frac{ny}{x}(X - x)$

Putting y = 0 then X = 0 the intercepts on axes are

$A = \frac{x(1+n)}{n}$, $B = y\,(1+n)$

$\therefore \Delta = \frac{1}{2}AB = \frac{1}{2}\frac{(1+n)^2}{n}xy = \frac{1}{2}\frac{(1+n)^2}{n}\ x\ \frac{a^n}{x^n}$

It will be constant if $x^{n-1} = 1$ i.e. $n - 1 = 0$ or $n = 1$

(46) **(C).** Given curve is $x^3 - y^2 = 0$

Differentiating w.r.t. x,

$$3x^2 - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \left(\frac{3x^2}{2y}\right)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(m^2, -m^3)} = \frac{3 \times m^4}{-2m^3} = -\frac{3m}{2}$$

$\therefore$ Slope of the normal $= \frac{2}{3m}$

Slope of given line = m $\quad \therefore \frac{2}{3m} = m \Rightarrow m^2 = \frac{2}{3}$

(47) **(B).** At any point of the curve $\frac{dy}{dx} > 0$

$\Rightarrow 3x^2 + 2\lambda x + 1 > 0$

It is possible only when $B^2 - 4AC < 0$

$\Rightarrow 4\lambda^2 - 12 < 0 \Rightarrow \lambda^2 - 3 < 0 \quad \therefore -\sqrt{3} < \lambda < \sqrt{3}$

(48) **(B).** Given curves meet at x = 1

For first curve $\frac{dy}{dx}=\frac{3^{x-1}}{x}+3^{x-1}\log 3\log x$

$$\left(\frac{dy}{dx}\right)_{x=1}=1+0=1$$

For second curve $\left(\frac{dy}{dx}\right)=x^{x}(1+\log x)$

$$\left(\frac{dy}{dx}\right)_{x=1}=1.(1+0)=1$$

If θ be the angle between these curves at x = 1, then
$\tan\theta=0\Rightarrow\cos\theta=1$.

(49) **(D).** $e^{y}=1+x^{2}$ .... (1)
Differentiating w.r.t. x,

$$\Rightarrow e^{y}\frac{dy}{dx}=2x\Rightarrow\frac{dy}{dx}=\frac{2x}{e^{y}}=\left(\frac{2x}{1+x^{2}}\right)=m$$

or $|m|=\frac{2|x|}{1+|x|^{2}}$ $\quad\therefore |m|\le 1$

(50) **(A).** $\because\ f'(x)=m=e^{x}[\cos x-\sin x]$

$$\frac{dm}{dx}=e^{x}(-\sin x-\cos x)+(\cos x-\sin x)\,e^{x}$$

$$\frac{dm}{dx}=-2\sin x e^{x}=0\ \Rightarrow x=0,\pi............$$

$$\frac{d^{2}m}{dx^{2}}=-2\,[e^{x}\cos x+e^{x}\sin x]$$

$$\left(\frac{d^{2}m}{dx^{2}}\right)_{x=0}=-2<0\ \Rightarrow \text{max. at } x=0$$

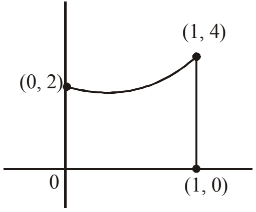
(51) **(A).** $D^{2}=z=(x-4)^{2}+(y+1/2)^{2}$
$=(x-4)^{2}+(x^{2}+1/2)^{2}$ $\quad[\because\ y=x^{2}]$
$z=x^{4}+2x^{2}-8x$

$$\frac{dz}{dx}=4x^{3}+4x-8=0\ \Rightarrow x=1$$

$$\frac{d^{2}z}{dx^{2}}=12x^{2}+4>0\text{. Minimum at } x=1$$

So point is (1, 1).

(52) **(C).** $f(x)=x^{3}+x+2$
$f'(x)=3x^{2}+1>0\ \forall\ x\in R$
$\Rightarrow f(x)$ is increasing function
$f(0)=2$; $f(1)=4$
$\therefore$ f(x) has least value = 2



(53) **(C).** (A) $|x^{2}-1|=|x^{2}-3|\Rightarrow(x^{2}-1)=\pm(x^{2}-3)$

$\Rightarrow\ x=(\pm\sqrt{2},1)$

(B) At point $(\sqrt{2},1)$

$y=x^{2}-1$	$y=-(x^{2}-3)$
$\frac{dy}{dx}=2x$	$\frac{dy}{dx}=-2x$
$\left(\frac{dy}{dx}\right)_{(\sqrt{2},1)}=2\sqrt{2}$	$\left(\frac{dy}{dx}\right)_{(\sqrt{2},1)}=-2\sqrt{2}$

$$\therefore\ \tan\alpha=\left|\frac{2\sqrt{2}-(-2\sqrt{2})}{1+2\sqrt{2}.(-2\sqrt{2})}\right|$$

$$\alpha=\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)\ \text{or}\ \pi-\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$$

(54) **(D).** $g'(x)=(f'(\tan x-1)^{2}+3)\ (2\tan x-2)\sec^{2}x$
Since $f''(x)>0\Rightarrow f'(x)$ is increasing.

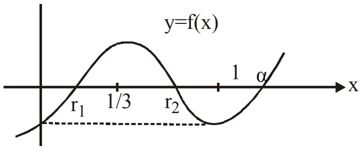
$$f'((\tan x-1)^{2}+3)>f'(3)=0\ \forall\ x\in R\left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{4},\frac{\pi}{2}\right)$$

Also $(\tan x-1)>0\ \forall\ x\in R\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

So, g (x) is increasing in $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

(55) **(A).** $r_{1}=\frac{a}{(1-r_{1})^{2}},\ r_{2}=\frac{a}{(1-r_{2})^{2}}$

$x=\frac{a}{(1-x)^{2}}$ has roots r_{1}, r_{2} and α (say)



$x^{3}-2x^{2}+x-a=0$
Let $f(x)=x^{3}-2x^{2}+x-a$
$f'(x)=3x^{2}-4x+1=(3x-1)(x-1)$

$$f\left(\frac{1}{3}\right)=\frac{4}{27}-a>0,\ f(1)=-a<0$$

$f(0)=-a<0$

(56) **(B).** $f'(x)=3\,(x-3)^{2}$; $f'(x)=0\Rightarrow x=3$
$f''(x)=6\,(x-3)$; $f''(3)=0$
$f'''(x)=6,\ f'''(3)\neq 0$
Hence, f (x) neither max. nor min. at x = 3

(57) **(B).** $\because\ f(x)=\max.\ \{(x-1)^{2}+1,|x-1|\}=(x-1)^{2}+1$
$\therefore\ f'(x)=2\,(x-1)=0$
$x=1\in[0,3]$
Greatest value of f (x) = max. { f(0). f(1), f(3)}
= max. {2, 1, 5} = 5

(58) S-1 : $f(x)=\begin{cases} x^2-5x+6, & x\le 2\\ -x^2+5x-6, & 2\le x\le 3\\ x^2-5x+6, & x\ge 3\end{cases}$

$f'(x)=\begin{cases} 2x-5, & x<2\\ -2x+5, & 2<x<3\\ 2x-5, & x>3\end{cases}$

$f'(2^-)+f'(2^+)=-1+1=0$

S-2 : $f(x)=\begin{cases} (x-a)(x-b), & x<a\\ -(x-a)(x-a), & a\le x\le b\\ (x-a)(x-b), & x>b\end{cases}$

$f'(x)=\begin{cases} 2x-a-b, & x<a\\ -2x+a+b, & a<x<b\\ 2x-a-b, & x>b\end{cases}$

$\therefore\ f'(a-)=a-b,\ f'(a+)=-a+b$

$\therefore\ f'(a-)+f'(a+)=0$

Statement 2 explains statement-1.

(59) **(B).** $f(x)=x+\cos x$

$\therefore\ f'(x)=1-\sin x>0\ \ \forall\ x\in R$, except at $x=2n\pi+\dfrac{\pi}{2}$

and $f'(x)=0$ at $x=2n\pi+\dfrac{\pi}{2}$

$\therefore$ f (x) is strictly increasing

$\therefore$ Statement 2 is true but does not explain statement-1

$\because$ Statement-2 gives f ' (x) may vanish at finite number of points but in S-1 f ' (x) vanishes at infinite no. of points

(60) **(D).** $g(x)=\sqrt{x}-\sqrt{x-1}$

$$g'(x)=\frac{-1}{2\sqrt{x}\sqrt{x-1}(\sqrt{x}+\sqrt{x+1})}<0\quad\forall\ x>1$$

g (x) is decreasing

$C+1>C\quad;\quad g(C+1)<g(C)$

$f(u)<f(v)$ (f is increasing function)

(61) **(A).** $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}$ or $b\sin A=a\sin B$

$b\cos A\, dA=a\cos B\, dB$

$$\frac{dA}{a\cos B}=\frac{dB}{b\cos A}\Rightarrow\frac{dA}{a\sqrt{1-\sin^2 B}}=\frac{dB}{b\sqrt{1-\sin^2 A}}$$

$$\Rightarrow\frac{dA}{a\sqrt{1-\dfrac{b^2\sin^2 A}{a^2}}}=\frac{dB}{b\sqrt{1-\dfrac{a^2\sin^2 B}{b^2}}}$$

$$\Rightarrow\frac{dA}{\sqrt{a^2-b^2\sin^2 A}}=\frac{dB}{\sqrt{b^2-a^2\sin^2 B}}$$

(62) **(D).** When $x=1, y=1, y'=3x^2-2x-1\Rightarrow y'|_{x=1}=0$

equation of tangent is y = 1

Solving with the curve

$x^3-x^2-x+2=1\Rightarrow x^3-x^2-x+1=0$

The tangent meets the curve again at $x=-1$

$\therefore$ Statement 1 is false and statement 2 is true.

(63) **(C).** $\dfrac{dy}{dx}=3x^2-3$

Statement 1 : $\therefore\ \left.\dfrac{dy}{dx}\right|_{at\ (0,1)}=-3$

Equation of tangent is $y-1=-3(x-0)$ i.e. $y=-3x+1$

$-3x+1=x^3-3x+1\Rightarrow x=0$

$\therefore$ The tangent meets the curve at 1 point only.

$\therefore$ statement is true.

Statement 2 : $\therefore\ \left.\dfrac{dy}{dx}\right|_{at\ (1,-1)}=0$

$\therefore$ Equation of tangent is $y+1=0(x-1)$ i.e. $y=-1$

$-1=x^3-3x+1\Rightarrow x^3-3x+2=0$

$\Rightarrow (x-1)(x^2+x-2)=0\Rightarrow (x-1)^2(x+2)=0$

$\therefore$ The tangent meets the curve at 2 point.

$\therefore$ Statement is false.

(64) **(A).** $h(x)=g(f(x))$ and $f(x)\in[0,\infty)$

$\therefore\ h(x)\ge 0$ .... (1)

$h(0)=0$ .... (2)

$h'(x)=g'(f(x))\,f'(x)\le 0$ .... (3)

From eq. (1), (2) and (3)

h (x) is a constant function.

(65) **(A).** Suppose $f(x)=0$ has a real root say $x=a$ then $f(x)<0$ for all $x<a$.

Thus $|f(x)|$ becomes strictly decreasing on $(-\infty, a)$ which is contradiction.

(66) (B), (67) (D), (68) (C).

$\dfrac{da}{dt}=2\Rightarrow a=2t+c\ \because\ c=0\ \{\because\ a=0$, when $t=0\}$

$\therefore$ the curve $y=x^2-2ax+a^2+a$ becomes

$y=x^2-4tx+4t^2+2t$

if $x=0$, then $\mathbf{y}=4t^2+2t$

$\dfrac{dy}{dx}=2x-4t\qquad\therefore\ \left.\dfrac{dy}{dx}\right|_{at\ x=0}=-4t$

$\therefore$ equation of the tangent

$y-(4t^2+2t)=-4t(x-0)$

i.e., $y=-4tx+4t^2+2t$

vertex of $y=x^2-4tx+4t^2+2t$ is $(2t, 2t)$

$\therefore$ distance of vertex from the origin $=2\sqrt{2}t$

$\therefore$ rate of change of distance of vertex from origin with respect to $t=2\sqrt{2}$ i.e. $k=2\sqrt{2}$; $c(t)=4t^2+2t$

$\therefore\ \dfrac{dc}{dt}=8t+2\ \ \therefore\ \left.\dfrac{dc}{dt}\right|_{at\ t=2\sqrt{2}}=16\sqrt{2}+2$

$\therefore\ \ell=16\sqrt{2}+2\ ;\ \ m(t)=-4t$

$\therefore \frac{dm}{dt} = -4 \quad \therefore \left.\frac{dm}{dt}\right|_{\text{at } t=\ell} = -4$

(69) **(A).** Consider a function $g(x) = \frac{f(x)}{x}$

as f (x) and x are x are differentiable hence g (x) is also differentiable.

Now $g(a) = \frac{f(a)}{a}$ and $g(b) = \frac{f(b)}{b}$

Since $\frac{f(a)}{a} = \frac{f(b)}{b} \quad \therefore \quad g(a) = g(b)$

Hence Rolle's theorem is applicable for g (x)

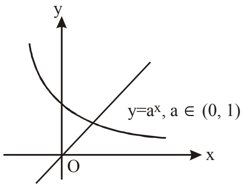
$\therefore \ \exists$ some $x_0 \in (a, b)$ where $g'(x) = 0$

but $g'(x) = \frac{xf'(x) - f(x)}{x^2}$,

$g'(x_0) = \frac{x_0 f'(x_0) - f(x_0)}{x_0^2} = 0$

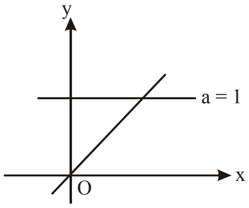
$\therefore \ x_0 f'(x_0) = f(x_0)$

(70) **(B).** For $0 < a \le 1$ the line always cuts $y = a^x$.



For a > 1, say a = e

Consider $f(x) = e^x - x$

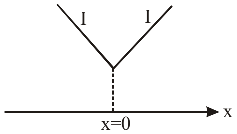


$f'(x) = e^x - 1$

$f'(x) > 0$ for $x > 0$ and $f'(x) < 0$ for $x < 0$

$\therefore$ f (x) is increasing (↑) for x > 0

and decreasing (↓) for x < 0

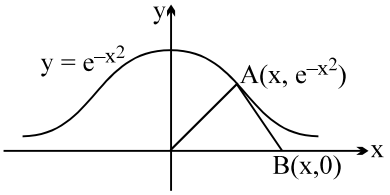


$y = e^x$ always lies above $y = x$ i.e. $e^x - x \ge 1$ for $a > 1$

Hence never cuts.

(71) **(D).** $A = \frac{x e^{-x^2}}{2}$; $A' = \frac{1}{2}[e^{-x^2} - 2x^2 \cdot e^{-x^2}]$

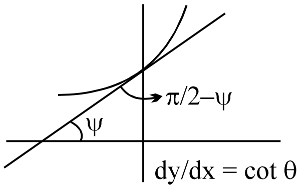
$= \frac{e^{-x^2}}{2}[1 - 2x^2] = 0 \Rightarrow x = \frac{1}{\sqrt{2}}$ gives $A_{\text{max.}}$



$\therefore A_{\max} = \frac{e^{-1/2}}{2\sqrt{2}} = \frac{1}{\sqrt{8e}}$

(72) **(B).** $\left.\frac{dy}{dx}\right]_{x=0} = k^2 \Rightarrow \tan \Psi = k^2$

$\Rightarrow \cot\left(\frac{\pi}{2} - \psi\right) = k^2$



$\Rightarrow \left(\frac{\pi}{2} - \psi\right) = \cot^{-1} k^2 = \sin^{-1} \frac{1}{\sqrt{1 + k^4}}$

(73) **(C).** Note the graph of f(x). Least value coincides with local minima

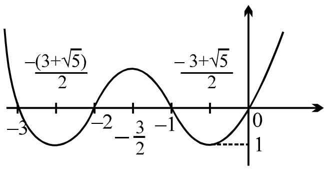
$y = (x^2 + 3x)(x^2 + 3x + 2) = z(z + 2)$

$= (z + 1)^2 - 1 = (x^2 + 3x + 1)^2 - 1$

$y_{\text{least}} = -1$; this occurs where $z = -1$ i.e. $x^2 + 3x + 1 = 0$

or $\frac{dy}{dx} = 2(2x + 3)(x^2 + 3x + 1) = 0$

$\Rightarrow x = \frac{-3 + \sqrt{5}}{2}, \ -\frac{3}{2}$ or $\frac{-3 - \sqrt{5}}{2}$



Here $x = \frac{-3 + \sqrt{5}}{2}$ & $x = \frac{-3 - \sqrt{5}}{2}$ are the points of local minima and $x = -3/2$ is the point of local maxima.

Local maximum value = 9/16

(74) **(A).** Slope of the normal at (1, 1) is – 1/a

⇒ Slope of the tangent at (1, 1) is a i.e.,

$\left.\frac{dy}{dx}\right]_{(1,1)} = a \qquad \ldots\ldots (1)$

We are given that $\frac{dy}{dx} \propto y$

$\frac{dy}{dx} = ky$, where k is some constant $\frac{dy}{y} = k\,dx$

$\log|y| = kx + c$, where c is a constant

$|y| = e^{kx+c}$

$y = \pm e^c e^{kx} = Ae^{kx}$, where A is a constant.

Since the curve passes through (1, 1),

therefore $1 = Ae^k \Rightarrow A = e^{-k}$

Therefore, $y = e^{-k} . e^{kx} = e^k (x-1)$

$$\Rightarrow \frac{dy}{dx} = ke^k (x-1) \Rightarrow \left.\frac{dy}{dx}\right]_{(1,\,1)} = k \Rightarrow a = k[\text{Using (1)}]$$

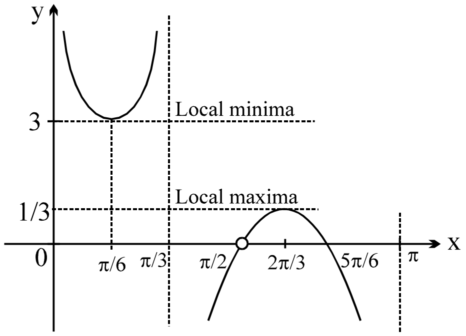
Thus, the required curve is $y = e^a (x-1)$.

(75) **(D).** f(x) has a period equal to π & can take values $(-\infty, \infty)$

$\Rightarrow$ 3 is the local minimum value.

$$y = \frac{2\sin\left(x+\frac{\pi}{6}\right)\cos x}{2\sin x\cos\left(x+\frac{\pi}{6}\right)} = \frac{\sin\left(2x+\frac{\pi}{6}\right)+\sin\frac{\pi}{6}}{\sin\left(2x+\frac{\pi}{6}\right)-\sin\frac{\pi}{6}}$$

$$= 1 + \frac{1}{\sin\left(2x+\frac{\pi}{6}\right)-\sin\frac{\pi}{6}}$$



y is minimum if $2x + \frac{\pi}{6} = \frac{\pi}{2}$

$\Rightarrow x = \frac{\pi}{6}$ $\Rightarrow y_{min} = 1 + 2 = 3$]

(76) **(C).** Let $f(x) = 1 + x\log\left(x+\sqrt{x^2+1}\right) - \sqrt{1+x^2}$

$$\Rightarrow f'(x) = 1 . \log\left(x+\sqrt{x^2+1}\right)$$

$$+ x . \frac{1}{x+\sqrt{x^2+1}} \times \left(1+\frac{x}{\sqrt{x^2+1}}\right) - \frac{x}{\sqrt{1+x^2}}$$

$$= \log\left(x+\sqrt{x^2+1}\right)$$

$$+ \frac{x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+1}} = \log\left(x+\sqrt{x^2+1}\right).$$

Clearly, $f'(x) \ge 0$, for $x \ge 0$

$\Rightarrow$ f (x) is increasing for $x \ge 0$

$\Rightarrow f(x) \ge f(0)$, for $x \ge 0$.

$$\Rightarrow 1 + x\log\left(x+\sqrt{x^2+1}\right) - \sqrt{1+x^2} \ge 0,$$

for $x \ge 0 \Rightarrow 1 + x\log\left(x+\sqrt{x^2+1}\right) \ge \sqrt{1+x^2}$, for $x \ge 0$.

(77) **(A).** Let tan x = t

$$\Rightarrow f(x) = \frac{t^n}{1+t+....+t^4+....+t^{2n}}$$

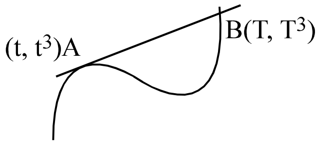
$$= \frac{1}{\left(t^n+\frac{1}{t^n}\right)+\left(t^{n-1}+\frac{1}{t^{n-1}}\right)+.......+\left(t+\frac{1}{t}\right)+1}$$

$\le \frac{1}{2n+1}$ [Equality holds at $x = \pi/4$]

also $f(0) = 0 \Rightarrow$ range of f (x) is $\left[0, \frac{1}{2n+1}\right]$

(78) **(A).** $\frac{dy}{dx} = 3x^2 = 3t^2$ at 'A'

$$\therefore 3t^2 = \frac{T^3 - t^3}{T - t} = T^2 + Tt + t^2$$



$T^2 + Tt - 2t^2 = 0$

$(T - t)(T + 2t) = 0 \Rightarrow T = t$ or $T = -2t$

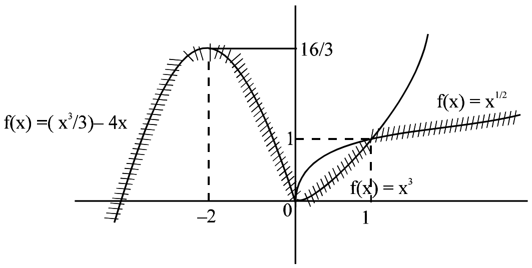
(T = t is not possible)

now, $m_A = \frac{t^3}{t} = t^2$; $m_B = T^2$

$\frac{m_B}{m_A} = \frac{T^2}{t^2} = \frac{4t^2}{t^2}$ (using T = – 2t)

$m_B = 4$

(79) **(C).**



Function is inc. in $(-\infty, -2) \cup (0, \infty)$
function is dec. in $(-2, 0)$
$x = -2 \to$ local maxima
$x = 0 \to$ local minima
Derivable

$$\forall x \in R - \{0,1\} - \left[\begin{array}{l} f'(0^+) = 0,\ f'(0^-) = -4 \\ \\ f'(1^+) = 1/2, f'(1^-) = 3 \end{array}\right.$$

Continuous $\forall x \in R$.

(80) **8.** $f(x) = 7e^{\sin^2 x} - e^{\cos^2 x} + 2$

Let $e^{\sin^2 x} = t \Rightarrow t \in [1, e]$

$g(t) = 7t - \frac{e}{t} + 2$; $g'(t) = 7 + \frac{e}{t^2} = 0 \Rightarrow$ no critical point

$g(1) = 9 - e =$ minimum value
$g(e) = 7e + 1 =$ maximum value

$\sqrt{7f_{min} + f_{max}} = 8$

(81) **5040.** We have $F(x) = \frac{x^3}{3} + (a-3)x^2 + x - 13$.

$\therefore$ For F (x) to have negative point of local minimum, the equation F '(x) = 0 must have two distinct negative roots.
Now, $F'(x) = x^2 + 2(a-3)x + 1$
Following condition(s) must be satisfied simultaneously.
(i) Discriminant > 0; (ii) Sum of roots < 0 ; (iii) Product of roots > 0
Now, $D > 0$
$\Rightarrow 4(a-3)^2 > 4 \Rightarrow (a-3)^2 - 1 > 0 \Rightarrow (a-2)(a-4) > 0$
$\therefore\ a \in (-\infty, 2) \cup (4, \infty)$ .... (i)
Also, $-2(a-3) < 0 \Rightarrow a - 3 > 0 \Rightarrow a > 3$ .... (ii)
And product of root(s) $= 1 > 0\ \forall\ a \in R$
$\therefore\ (i) \cap (ii) \cap (iii) \Rightarrow a \in (4, \infty)$ .... (iii)
Hence sum of value(s) of $a = 5 + 6 + 7 + ... + 100 = 5040$

(82) **12.** $F(x) = \begin{cases} -2x + \log_{1/2}(k^2 - 6k + 8), & -2 \le x < -1 \\ x^3 + 3x^2 + 4x + 1, & -1 \le x \le 3 \end{cases}$

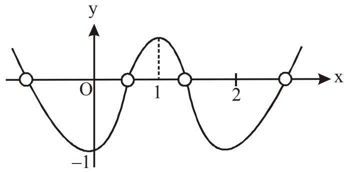
Also F (x) is increasing on [–1, 3] because
$F'(x) > 0\ \forall\ x \in [-1, 3]$
And $F'(x) = -2\ \forall\ x \in [-2, -1)$, so F (x) is decreasing on $[-2, -1)$.
$\therefore$ If F (x) has smallest value at $x = -1$, then we must have

$\lim_{h \to 0} F(-1-h) \ge F(-1)$

$\Rightarrow 2 + \log_{1/2}(k^2 - 6k + 8) \ge -1$
$\Rightarrow \log_{1/2}(k^2 - 6k + 8) \ge -3$
$\Rightarrow k^2 - 6k + 8 \le 8 \Rightarrow k^2 - 6k \le 0 \Rightarrow k \in [0, 6]$ ..... (1)
But in order to define $\log_{1/2}(k^2 - 6k + 8)$
We must have $k^2 - 6k + 8 > 0$
$\Rightarrow (k-2)(k-4) > 0 \Rightarrow k < 2$ or $k > 4$ .... (2)
$\therefore$ From (1) and (2), we get $k \in [0, 2) \cup (4, 6]$
$\Rightarrow$ Possible integer(s) in the range of k are 0, 1, 5, 6

Hence the sum of all possible positive integer(s) in the range of $k = 1 + 5 + 6 = 12$

(83) **4.**



$\lim_{x \to \infty} P(x) \to \infty$ and $\lim_{x \to -\infty} P(x) \to \infty$

minimum number of zeroes using IVT is 4.
Hence, $R \ge 4$
The roots using lie in $(-\infty, 0)$; $(0, 1)$; $(1, 2)$; $(2, \infty)$

(84) **4000.** Given $S = x^2 + 4xh = 1200$ and $V = x^2h$

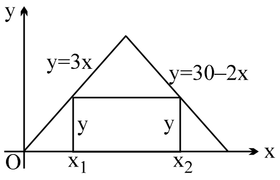
$V(x) = \frac{x^2(1200 - x^2)}{4x}$; $V(x) = \frac{1}{4}(1200x - x^3)$

Put $V'(x) = 0$ gives $x = 20$
If $x = 20$, $h = 10$
Hence, $V_{max.} = x^2h = (400)(10) = 4000$ cubic cm.

(85) **2.** $2A = xy\sin\theta$; $4A^2 = x^2y^2\sin^2\theta$;

$f(x) = \frac{x^4}{2x-1}$; $f'(x) = 0 \Rightarrow x = \frac{2}{3}$

(86) **2.** $A = (x_2 - x_1)y$



$y = 3x_1$ and $y = 30 - 2x_2$

$A(y) = \left(\frac{30-y}{2} - \frac{y}{3}\right)y$

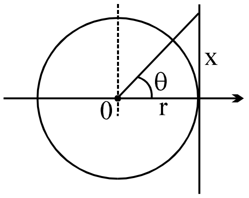
$6A(y) = (90 - 3y - 2y)y$
$= 90y - 5y^2$
$6A'(y) = 90 - 10y = 0$
$\Rightarrow y = 9$; $A''(y) = -10 < 0$
$x_1 = 3$; $x_2 = 21/2$

$A_{max} = \left(\frac{21}{2} - 3\right)9 = \frac{15 \cdot 9}{2} = \frac{135}{2}$

(87) **10.** $\frac{dv}{dx} = 3x(10 - x) = 0 \Rightarrow x = 0$; $x = 10$;

$\left.\frac{d^2v}{dx^2}\right]_{x=10} < 0 \Rightarrow$ v is max at $x = 10 \Rightarrow$ EF = 10 cm.

(88) **40.** $\tan\theta = x/r \Rightarrow x = r\tan\theta$
$\Rightarrow dx/dt = r\sec^2\theta\,(d\theta/dt) = r\,\omega\sec^2\theta = v\sec^2\theta$



where $\theta = \pi/8$, $dx/dt = v\sec^2(\pi/4) = 2v = 40$ km/hr ;
$\theta = 45°$

(89) **33.** $\frac{dr}{dt} = c$ and $h = ar + b$. Also $\frac{dh}{dt} = 3\frac{dr}{dt}$ (given)

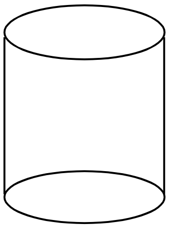
$\therefore\ a\frac{dr}{dt}=3\frac{dr}{dt} \Rightarrow a=3.$ Hence $h=3r+b$

when $r=1$; $h=6 \Rightarrow 6=3+b \Rightarrow b=3$

$\therefore\ h=3(r+1)$

$V=\pi r^2 h=3\pi r^2(r+1)$

$=3\pi(r^3+r^2)$



$\frac{dV}{dt}=3\pi(3r^2+2r)\frac{dr}{dt}$

where $r=6$; $\frac{dV}{dt}=1$ cc/sec

$\therefore\ 1=3\pi(108+12)\frac{dr}{dt} \Rightarrow 360\pi\frac{dr}{dt}=1$

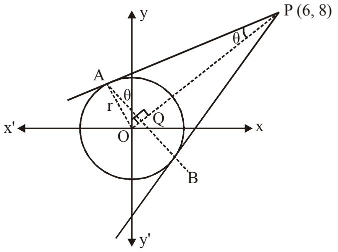
again when $r=36$, $\frac{dV}{dt}=n$

$n=3\pi((3.36)^2+2.36)\frac{dr}{dt}$

$n=3\pi.36(110).\frac{1}{360\pi}$; $n=33$

(90) **1.** The maximum value of $f(x)=\cos x+\cos(\sqrt{2x})$ is 2 which occurs at $x=0$. Also, there is no value of x for which this value will be attained again.

(91) **5.** To maximise area of Δ APB; we know, OP = 10 and $\sin\theta=r/10$, where $\theta\in(0,\pi/2)$ ...... (1)



Area $=\frac{1}{2}(2AQ)(PQ)=AQ.PQ=(r\cos\theta)(10-OQ)$

$=(r\cos\theta)(10-r\sin\theta)$

$=10\sin\theta\cos\theta(10-10\sin^2\theta)$ [From eq. (1)]

$\Rightarrow\ A=100\cos^3\theta\sin\theta$

$\Rightarrow\ \frac{dA}{d\theta}=100\cos^4\theta-300\cos^2\theta.\sin^2\theta$

Put $\frac{dA}{d\theta}=0 \Rightarrow \cos^2\theta=3\sin^2\theta$

$\Rightarrow\ \tan\theta=\frac{1}{\sqrt{3}} \Rightarrow \theta=\frac{\pi}{6}$

At which $\frac{dA}{d\theta}<0$, thus when $\theta=\frac{\pi}{6}$, area is maximum

From eq. (1), $r=10\sin\frac{\pi}{6}=5$ unit.

(92) **1.** At $x=0, y=1$

$\log(x+y)-2xy=0$

$\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)\frac{dy}{dx}=0$

$\Rightarrow\ \frac{dy}{dx}=\frac{2y(x+y)-1}{1-2(x+y)x} \Rightarrow \left.\frac{dy}{dx}\right|_{(0,1)}=1$

(93) **6.** $g(x)=\frac{d}{dx}(f(x)f(x))$

To get the zero of g (x) we take function

$h(x)=f(x).f'(x)$

between any two roots of h(x) there lies at least one root of $h'(x)=0$

$\Rightarrow\ g(x)=0$

$h(x)=0$

$\Rightarrow\ f(x)=0$ or $f'(x)=0$

$f(x)=0$ has 4 minimum solutions

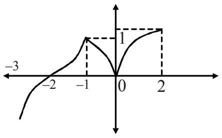
$f'(x)=0$ minimum three solution

$h(x)=0$ minimum 7 solution

$\Rightarrow\ h'(x)=g(x)=0$ has minimum 6 solutions.

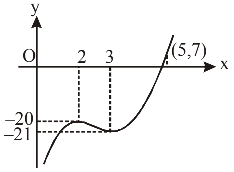
(94) **2.** Local maximum at $x=-1$

and local minimum at $x=0$



Hence total number of local maxima and local minima is 2.

(95) **7.** $A=\{x|x^2+20\le 9x\}=\{x|x\in[4,5]\}$



Now, $f'(x)=6(x^2-5x+6)$

$f'(x)=0 \Rightarrow x=2,3$

$f(2)=-20, f(3)=-21, f(4)=-16, f(5)=7$

From graph, maximum of f(x) on set A is $f(5)=7$.

(96) **1.** $f(x)=\ln\{g(x)\}$

$g(x)=e^{f(x)}$

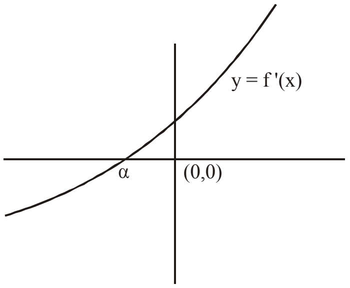
$g'(x)=e^{f(x)}.f'(x)$

$g'(x)=0 \Rightarrow f'(x)=0$ as $e^{f(x)}\ne 0$

$\Rightarrow\ 2010(x-2009)(x-2010)^2(x-2011)^3(x-2012)^4=0$

so there is only one point of local maxima.

(97) **2.**



$f(x) = x^4 - 4x^3 + 12x^2 + x - 1$

$f'(x) = 4x^3 - 12x^2 + 24x + 1$

$f''(x) = 12x^2 - 24x + 24$

$= 12(x^2 - 2x + 2) > 0 \ \forall\ x \in R$

$\therefore$ $f'(x)$ is S.I. function

Let α is a real root of the equation $f'(x) = 0$

$\therefore$ $f(x)$ is MD for $x \in (-\infty, \alpha)$ and M.I. for $x \in (\alpha, \infty)$ where $\alpha < 0$

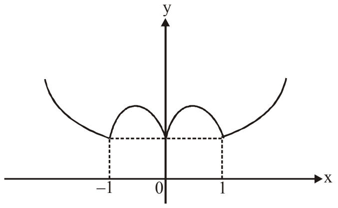
$\therefore$ $f(0) = -1$ and $\alpha < 0$

$\Rightarrow$ $f(\alpha)$ is also negative

$\therefore$ $f(x) = 0$ has two real & distinct roots.

(98) **5.** $f(x) = |x| + |(x+1)(x-1)|$

$$\Rightarrow f(x) = \begin{cases} x^2 - x - 1 & x < -1 \\ -x^2 - x + 1 & -1 \le x < 0 \\ -x^2 + x + 1 & 0 \le x < 1 \\ x^2 + x - 1 & x \ge 1 \end{cases}$$



$\therefore$ f has 5 points where it attains either a local maximum or local minimum.

(99) **9.** Let $P'(x) = k(x-1) + (x-3) = k(x^2 - 4x + 3)$

$$\Rightarrow P(x) = k\left(\frac{x^3}{3} - 2x^2 + 3x\right) + c \qquad \because P(1) = 6$$

$\frac{4k}{3} + c = 6$ .... (1)

$P(3) = 2 \Rightarrow c = 2$ .... (2)

By eq. (1) and (2), $k = 3$

$\therefore$ $P'(x) = 3(x-1)(x-3)$

$\Rightarrow$ $P'(0) = 9$

(100) **2.** Let $f(x) = x^2 - x\sin x - \cos x$

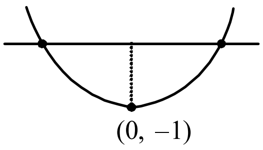
$\Rightarrow$ $f'(x) = 2x - x\cos x$

$\lim_{x\to\infty} f(x) \to \infty$

$\lim_{x\to-\infty} f(x) \to \infty$

$f(0) = -1$

Hence 2 solutions.



EXERCISE-3

(1) **(D).** $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$; $a > 0$

$f(x) = 6x^2 - 18ax + 12a^2$

For maxima and minima

$f'(x) = 0$

$\Rightarrow 6x^2 - 18ax + 12a^2 = 0$

$\Rightarrow 6(x-a)(x-2a) = 0 \Rightarrow x = a, 2a$ and $f'(x) = 12 - 8a$

$\Rightarrow 6(2x - 3a)$

also $f''(a) = 6(2a - 3a) = -6a < 0$

and $f''(2a) = 6(4a - 3a) = 6a > 0$

$\therefore$ f has a local maximum at a and local minimum at 2a

$\Rightarrow$ p = a and q = 2a

$\because p^2 = q \Rightarrow a^2 = 2a$

$\Rightarrow a(a-2) = 0 \Rightarrow a = 0$ or $a = 2$ but $a > 0$ $\therefore$ $a = 2$

(2) **(C).** According to question,

$$x + \frac{1}{x} = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 + 2 \ge 2 \quad \forall\ x > 0$$

This means that minimum value of $x + \frac{1}{x}$ is 2 and it

occurs when $\sqrt{x} - \frac{1}{\sqrt{x}} = 0$ i.e. when $x = 1$

if $x < 0$, $x + \frac{1}{x} \le -2$

$\therefore$ minimum of $x + \frac{1}{x}$ is $-\infty$

or $x + \frac{1}{x} \to -\infty$ as $x \to -\infty$

(3) **(D).**

$$u = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$u^2 = a^2\cos^2\theta + b^2\sin^2\theta + a^2\sin^2\theta + b^2\cos^2\theta$$

$$+ 2\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$= a^2 + b^2 + 2\sqrt{(a^4 + b^4)\sin^2\theta\cos^2\theta + a^2b^2(\sin^4\theta + \cos^4\theta)}$$

$$= a^2 + b^2 + 2\sqrt{(a^4 + b^4)\sin^2\theta\cos^2\theta + a^2b^2[(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta + \cos^2\theta]}$$

$$= a^2 + b^2 + 2\sqrt{\sin^2\theta\cos^2\theta + [a^4 + b^4 - 2a^2b^2] + a^2b^2}$$

$$u^2 = a^2 + b^2 + 2\sqrt{(a^2 - b^2)^2\sin^2\theta\cos^2\theta + a^2b^2}$$

$\because$ a & b are constant

$\Rightarrow u^2$ will be minimum or maximum if $\sin^2\theta\cos^2\theta$ is min. or max. respectively but we know that

$\sin^2\theta\cos^2\theta \ge 0$

$$\therefore u^2_{min} = a^2 + b^2 + 2\sqrt{(a^2 - b^2)^2 \times (0) + a^2b^2}$$

$$= a^2 + b^2 + 2\sqrt{a^2b^2} = a^2 + b^2 + 2ab = (a+b)^2$$

We know that A.M. ≥ G.M.

$$\therefore \quad \frac{\sin^2\theta+\cos^2\theta}{2} \ge \sqrt{\sin^2\theta\cos^2\theta}$$

$$\Rightarrow \frac{1}{2} \ge \sqrt{\sin^2\theta\cos^2\theta} \Rightarrow \sin^2\theta\cos^2\theta \le \frac{1}{4}$$

$$\therefore \quad (\sin^2\theta\cos^2\theta)_{max} = \frac{1}{4}$$

$$\therefore \quad u^2_{max} = a^2+b^2+2\sqrt{(a^2-b^2)^2\frac{1}{4}+a^2b^2}$$

$$= a^2+b^2+2.\frac{1}{2}\sqrt{(a^2+b^2)^2}$$

$$= a^2+b^2+a^2+b^2 = 2(a^2+b^2)$$

$$= u^2_{max}-u^2_{min} = 2(a^2+b^2)-(a+b)^2$$

$$= a^2+b^2-2ab = (a-b)^2$$

(4) **(B).** $f''(x) = 6(x-1)$

$$\therefore \quad f(x) = \frac{6x^2}{2} - 6x + c = 3x^2 - 6x + c$$

f (x) represents slope of tangent

∴ slope of tangent at (2, 1) is

$= 3 \times 2^2 - 6 \times 2 + c = c$ .... (1)

Now equation of tangent at this point is $y = 3x - 5$

∴ slope of tangent = 3 .... (2)

From (1) and (2), c = 3

$$\text{Now } f'(x) = 3x^2 - 6x + 3 \text{ and } f(x) = \frac{3x^3}{3} - \frac{6x^2}{2} + 3x + k$$

$f(x) = y = x^3 - 3x^2 + 3x + k$

∴ curve passes through (2, 1)

$\therefore 1 = 2^3 - 3.2^2 + 3 \times 2 + k \Rightarrow k = -1$

$\therefore y = f(x) = x^3 - 3x^2 + 3x - 1 = (x-1)^3$

(5) **(A).** $x = a(1+\cos\theta)$ and $y = a\sin\theta$

$$\Rightarrow \frac{dx}{d\theta} = -a\sin\theta \text{ and } \frac{dy}{d\theta} = a\cos\theta$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\cos\theta}{-a\sin\theta} = -\cot\theta$$

∵ $-\cot\theta$ is the slope of tangent at point θ

∴ slope of normal w ill be = tan θ

Now equation of normal is

$y - a\sin\theta = \tan\theta\,[x - a(1+\cos\theta)]$

$\Rightarrow x\tan\theta - y = a\tan(1+\cos\theta) - a\sin\theta$

$\Rightarrow x\tan\theta - y = a\tan\theta$

Clearly normal passes through (a, 0).

(6) **(A).** Let $f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx$ in [0, 1]

$f(0) = 0$

$$\text{and } f(1) = \frac{a}{3} + \frac{b}{2} + c = \frac{2a+3b+6c}{6}$$

$= 0 \; \{\because \; 2a+3b+6c = 0 \text{ given}\}$

$\therefore f(0) = f(1)$

Clearly f (x) is continuous in [0,1] and differentiable in (0, 1)

∴ By Rolle's theorem $f'(x) = 0$ for at least one $x \in (0, 1)$

$\Rightarrow ax^2 + bx + c = 0$ has at lest one root in (0, 1)

(7) **(B).** $a_nx^n + a_{n-1}x^{n-1} + ... + a_1x = 0$ has one root $x = \alpha$.

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x = 0$

$f(0) = 0$ and $f(\alpha) = 0$

∴ f (x) is polynomial of degree n

∴ It is continuous in [0, α) and differentiable in (0, α)

∴ $f'(x) = 0$ has at least one root in (0, α)

∴ $na_nx^{n-1} + (n-1)\,a_{n-1}\,x^{n-2} + ... + a_1 = 0$ has at least one root is (0, α)

∴ one root of $na_nx^{n-1} + (n-1)\,a_{n-1}x^{n-2} + ... + a_1 = 0$ less than α.

(8) **(BD).** $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

$$\frac{dx}{d\theta} = a[-\sin\theta + \theta\cos\theta + \sin\theta] = a[\theta\cos\theta] \text{ and}$$

$$\frac{dy}{d\theta} = a[\cos\theta + \theta\sin\theta - \cos\theta] = a[\theta\sin\theta]$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

this is slope of tangent

∴ slope of normal to the curve will be – cot θ

∴ equation of normal is

$y = -a(\sin\theta - \theta\cos\theta) = -\cot\theta\,[x - a(\cos\theta + \theta\sin\theta)]$

$x\cot\theta + y = a\cot\theta\,(\cos\theta + \theta\sin\theta) + a(\sin\theta - \theta\cos\theta)$

$$= a\left[\frac{\cos^2\theta}{\sin\theta} + \theta\cos\theta + \sin\theta - \theta\cos\theta\right]$$

$$\Rightarrow \frac{x\cos\theta}{\sin\theta} + y = a\left[\frac{\cos^2\theta}{\sin\theta} + \sin\theta\right]$$

$\Rightarrow x\cos\theta + y\sin\theta = a(\cos^2\theta + \sin^2\theta)$

$x\cos\theta + y\sin\theta = a(\cos^2\theta + \sin^2\theta)$

$x\cos\theta + y\sin\theta = a$ .... (1)

$y = -\cot\theta\, x + a = \tan(\pi/2 + \theta)\,x + a$

Clearly, $m = \tan(\pi/2 + \theta)$

Normal make angle π/2 + θ with the x-axis and ⊥ distance of normal from origin is $\left|\frac{0\cos\theta + 0\sin\theta - a}{\sqrt{\cos^2\theta + \sin^2\theta}}\right|$ = a constant

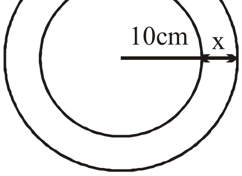
(9) **(B).** Let the thickness of ice is x

Now volume of sphere with ice layer is

$$v = \frac{4}{3}\pi(10+x)^3$$

$$\frac{dv}{dt} = \frac{4}{3}\pi\, 3\,(10+x)^2\frac{dx}{dt}$$

$$\Rightarrow 50 = 4 \times \pi \times 15 \times 15 \times \left(\frac{dx}{dt}\right)_{x=5}$$



$$\Rightarrow \left(\frac{dx}{dt}\right)_{x=5} = \frac{1}{18\pi} cm/min.$$

(10) **(D).** If f (x) is increasing ; $f'(x) \geq 0$

Now if $f(x) = x^3 - 3x^2 + 3x + 3$

$\therefore f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2$

if function is increasing

$\therefore f(x) \geq 0$

$\Rightarrow 3(x-1)^2 > 0 \ \forall \ x \in R$

If $f(x) = 2x^3 - 3x^2 - 12x + 6$

$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$

If f (x) is increasing

$\therefore f'(x) \geq 0$

$\Rightarrow 6(x^2 - x - 2)^2 \geq 0 \Rightarrow 6(x-2)(x+1) \geq 0$

$\Rightarrow x \in (-\infty, -1] \cup [2, \infty)$

If $f(x) = 3x^2 - 2x + 1$

$f'(x) = 6x - 2$ $\quad \because$ f (x) is increasing

$\therefore f(x) \geq 0 \Rightarrow 6x - 2 \geq 0 \Rightarrow x \geq 1/3$

If $f(x) = x^3 + 6x^2 + 6 \Rightarrow f'(x) = 3x^2 + 12x$ [$\because$ f is increasing]

$\therefore f'(x) \geq 0 \Rightarrow 3x^2 + 12x \geq 0 \Rightarrow 3x(x+4) \geq 0$

$\Rightarrow x \in (-\infty, -4] \cup [0, \infty)$

(11) **(A).** $f(1) = -2$ and $f'(x) \geq 2$

$$\Rightarrow \frac{dy}{dx} \geq 2 \Rightarrow dy \geq 2\,dx \Rightarrow \int dy = \geq 2\int dx$$

$$\Rightarrow \int_{f(1)}^{f(6)} dy \geq 2\int_1^6 dx$$

$\Rightarrow f(6) - f(1) \geq 2(6-1) \Rightarrow f(6) - f(1) \geq 10$

$\Rightarrow f(6) \geq 10 + f(1) \Rightarrow f(6) \geq 10 - 2 \ \{\because f(1) = -2\}$

$\Rightarrow f(6) \geq 8$

(12) **(A).** $y = x^2 - 5x + 6 ; \ \frac{dy}{dx} = 2x - 5$

At point (2, 0) $\therefore m_1 = \frac{dy}{dx} = 2 \times 2 - 5 = -1 \ \therefore m_1 = -1$

and at (3, 0), $m_2 = \frac{dy}{dx} = 2 \times 3 - 5 = 1$

$\because m_1 m_2 = (-1) \times 1 = -1$

$\therefore$ angle between the tangents at (2, 0) and (3, 0) is $\pi/2$.

(13) **(D).** $f(x) = \frac{x}{2} + \frac{2}{x}$; $f'(x) = \frac{1}{2} - \frac{2}{x^2}$

For maxima and minima $f'(x) = 0$

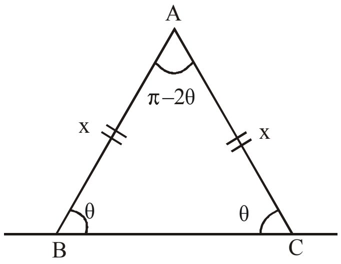
$$f'(x) = \frac{1}{2} - \frac{2}{x^2} = 0 \Rightarrow \frac{1}{2} = \frac{2}{x^2} \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

Again $f''(x) = 0 + \frac{2.2}{x^3} = \frac{4}{x^3}$; At $x = 2$ $f''(x) = \frac{4}{2^3} > 0$

$\therefore$ Function local minima at $x = 2$

(14) **(B).** Let $\angle ABC = \theta$ and $\angle ACB = \theta$

Angle opposite to equal sides are equal



$\therefore \angle BAC = \pi - 2\theta$

$$\therefore A = \frac{1}{2} x \times x \sin(\pi - 2\theta) = \frac{1}{2} x^2 \sin 2\theta$$

Area will be maximum is $\sin 2\theta$ will have it maximum value 1.

$$\therefore A = \frac{x^2}{2}$$

(15) **(A).** According to mean value theorem if f (x) is continuous in [a, b] and derivable in (a, b) then at least one point $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Here in question $f(x) = \log_e x$ holds mean value theorem on the interval [1, 3]

$c \Rightarrow [1, 3]$ and $\because f(x) = \log_e x \ \therefore \ f'(x) = \frac{1}{x}$ and $f'(c) = \frac{1}{c}$

Now, $f'(c) = \frac{f(3) - f(1)}{3 - 1}$

$$\Rightarrow \frac{1}{c} = \frac{\log 3 - \log 1}{2} \Rightarrow \frac{1}{c} = \log 3$$

$$\Rightarrow c = \frac{2}{\log_e 3} = 2\log_3 e$$

(16) **(B).** $f(x) = \tan^{-1}(\sin x + \cos x)$

$$f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} \times (\cos x - \sin x)$$

For increasing function $f'(x) > 0$

$$\therefore \frac{\cos x - \sin x}{1 + (\sin x + \cos x)^2} > 0$$

$\because \ 1 + (\sin x + \cos x)^2$ is always $> 0 \ \forall \ x \in R$

$\therefore \cos x - \sin x > 0$

$= \sqrt{2} \cos(x + \pi/4) > 0$

$$\therefore -\frac{\pi}{2} \leq x + \frac{\pi}{4} \leq \frac{x}{2} \Rightarrow -\frac{3\pi}{4} \leq x \leq \frac{\pi}{4} \quad ......(1)$$

From given option $(-\pi/2, \pi/4)$ lies in (1)

(17) **(D).** $\because$ Let $f(x) = p + q$

$\Rightarrow f(x) = p + \sqrt{1 - p^2}$ $\quad \{\because$ p and q +ve & $p^2 + q^2 = 1$

$$\Rightarrow q = \sqrt{1-p^2}$$

$$f'(x) = 1 + \frac{1}{2\sqrt{1-p^2}} \times -2p = 1 - \frac{p}{\sqrt{1-p^2}}$$

For maxima and minima

$$f'(x) = 0 \Rightarrow 1 - \frac{p}{\sqrt{1-p^2}} = 0 \Rightarrow 1 = \frac{p}{\sqrt{1-p^2}}$$

$$\Rightarrow 1 - p^2 = p^2 \Rightarrow 2p^2 = 1 \Rightarrow p = \frac{\pm 1}{\sqrt{2}}$$

$$\Rightarrow p + q = p + \sqrt{1-p^2}$$

$$= \frac{1}{\sqrt{2}} + \sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

(18) **(D).** Let $f(x) = x^3 - px + q$

$f'(x) = 3x^2 - p$

For maxima and minima $f'(x) = 0$

$$\Rightarrow 3x^2 - p = 0 \Rightarrow x = \pm\sqrt{\frac{p}{3}}$$

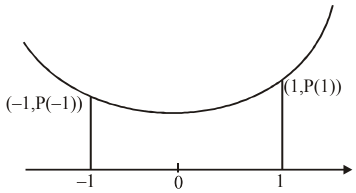
Now, $f''(x) = 6x$

at $x = \sqrt{\frac{p}{3}}$ $f''(x) > 0$ $\{\because p > 0, q > 0\}$

$\therefore$ at $x = \sqrt{\frac{p}{3}}$ f(x) has local minima and at

$x = -\sqrt{\frac{p}{3}}$ $f''(x) < 0$ $\therefore$ at $x = \sqrt{\frac{p}{3}}$ f(x) has local maxima

(19) **(B).** $P(x) = x^4 + ax^3 + bx^2 + cx + d$

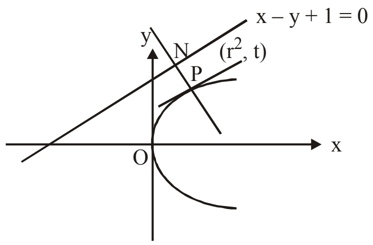


Now, $P'(0) = 0 \Rightarrow c = 0$

$\Rightarrow P(x) = x^4 + ax^3 + bx^2 + d$

Clearly P(1) is maximum but P(–1) is not minimum.

(20)



$$x = y^2 \Rightarrow y' = \frac{2}{2y}\ ;\ m = \frac{1}{2t} = 1 \Rightarrow t = \frac{1}{2}\ ;\ P\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$\text{So,}\quad PN = \frac{\left|\frac{1}{4} - \frac{1}{2} + 1\right|}{\sqrt{2}} = \frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{8}$$

(21) **(C).** Parallel to x-axis $\Rightarrow \frac{dy}{dx} = 0 \Rightarrow 1 - \frac{8}{x^3} = 0$

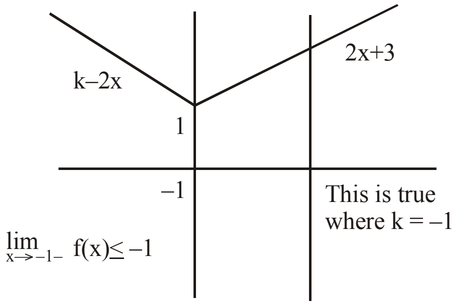
$\Rightarrow x = 2$ $\qquad\qquad \Rightarrow y = 3$

Equation of tangent is $y - 3 = 0\,(x - 2)$

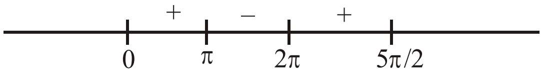
$\Rightarrow y - 3 = 0$

(22) **(C).** $f(x) = k - 2x \quad \text{if } x \le -1$

$\qquad = 2k + 3 \quad \text{if } x > -1$



(23) **(D).** $f(x) = \int_0^x \sqrt{t}\sin t\,dt\ ;\quad f(x) = \sqrt{x}\sin x$



local maximum at π and local minimum at 2π

(24) **(B).** $y - x = 1$

$$y^2 = x\ ;\ 2y\frac{dy}{dx} = 1\ ;\ \frac{dy}{dx} = \frac{1}{2y} = 1\ ;\ y = \frac{1}{2}\ ;\ x = \frac{1}{4}$$

$$\text{Tangent at } \left(\frac{1}{4}, \frac{1}{2}\right)\ ;\ \frac{1}{2}y = \frac{1}{2}\left(x + \frac{1}{4}\right)$$

$$y = x + \frac{1}{4}\ ;\ y - x = \frac{1}{4}$$

$$\text{Distance} = \left|\frac{1 - \frac{1}{4}}{\sqrt{2}}\right| = \frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{8}$$

(25) **(C).** $V = \frac{4}{3}\pi r^3;\ \ 4500\pi = \frac{4\pi r^3}{3}\ ;\ \frac{dV}{dt} = 4\pi r^2\left(\frac{dr}{dt}\right)$

$45 \times 25 \times 3 = r^3\ ;\ r = 15\text{m}$

After 49 min = $(4500 - 49.72)\,\pi = 972\,\pi\,\text{m}^3$

$$972\,\pi = \frac{4}{3}\pi r^3\ ;\ r^3 = 3 \times 243 = 3 \times 3^5\ ;\ r = 9$$

$$72\pi = 4\pi \times 9 \times 9\left(\frac{dr}{dt}\right)\ ;\ \frac{dr}{dt} = \frac{2}{9}$$

(26) **(B).** $f'(x) = \frac{1}{x} + 2bx + a$

At x = –1, $-1 - 2b + a = 0$

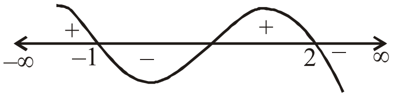
$a - 2b = 1$ .... (1)

At x = 2, $\frac{1}{2} + 4b + a = 0$

$a + 4b = -\frac{1}{2}$ .... (2)

On solving, (1) and (2), a = 1/2, b = –1/4

$$f'(x) = \frac{1}{x} - \frac{x}{2} + \frac{1}{2} = \frac{2 - x^2 + x}{2x} = \frac{-(x+1)(x-2)}{2x}$$



So maxima at x = –1, 2

(27) **(D).** $f(x) = 2x^3 + 3x + k$

$f'(x) = 6x^2 + 3 > 0 \;\forall\; x \in R$

⇒ f (x) is strictly increasing function

⇒ f (x) = 0 has only one real root, so two roots are not possible.

(28) **(A).** $\frac{dy}{dx} = |x| = 2$; $x = \pm 2$; Points $y = \int_{0}^{\pm 2} |t|\,dt = \pm 2$

∴ Equation of tangent is

$y - 2 = 2(x - 2)$ or $y + 2 = 2(x + 2)$ ⇒ x-intercept = ± 1.

(29) **(D).** Let $h(f) = f(x) - 2g(x)$ as $h(0) = h(1) = 2$

Hence, using Rolle's theorem, $h'(c) = 0$

⇒ $f'(c) = 2g'(c)$

(30) **(C).** $f'(x) = \frac{\alpha}{x} + 2\beta x + 1 = 0$ at x = –1 and 2.

$-\alpha - 2\beta + 1 = 0 \Rightarrow \alpha + 2\beta = 1$

$\frac{\alpha}{2} + 4\beta + 1 = 0 \Rightarrow \alpha + 8\beta = -2$

$6\beta = -3 \Rightarrow \beta = -\frac{1}{2} \quad \therefore \alpha = 2$

(31) **(B).** Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$

Using $\lim_{x\to 0}\left[1 + \frac{f(x)}{x^2}\right] = 3 \Rightarrow \lim_{x\to 0}\frac{f(x)}{x^2} = 2$

$$\Rightarrow \lim_{x\to 0}\frac{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4}{x^2} = 2$$

So, $a_0 = 0$, $a_1 = 0$, $a_2 = 2$

i.e., $f(x) = 2x^2 + a_3x^3 + a_4x^4$

Now, $f'(x) = 4x + 3a_3x^2 + 4a_4x^3 = x[4 + 3a_3x + 4a_4x^2]$

Given, $f'(1) = 0$ and $f'(2) = 0$

⇒ $3a_3 + 4a_4 + 4 = 0$ ..... (i)

and $6a_3 + 16a_4 + 4 = 0$ ..... (ii)

Solving, $a_4 = 1/2$, $a_3 = -2$

i.e., $f(x) = 2x^2 - 2x^3 + \frac{1}{2}x^4$ i.e., $f(2) = 0$

(32) **(C).** Curve is $x^2 + 2xy - 3y^2 = 0$. Differentiate wr.t. x,

$$2x + 2\left[x\frac{dy}{dx} + y\right] - 6y\frac{dy}{dx} = 0 \Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = 1$$

So equation of normal at (1, 1) is

$y - 1 = -1\,(x - 1) \Rightarrow y = 2 - x$

Solving it with the curve, we get

$x^2 + 2x(2 - x) - 3(2 - x)^2 = 0$

$\Rightarrow -4x^2 + 16x - 12 = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow x = 1, 3$

So points of intersections are (1, 1) & (3, –1) i.e. normal cuts the curve again in fourth quadrant.

(33) **(B).** $4x + 2\pi r = 2 \Rightarrow 2x + \pi r = 1 \Rightarrow r = \frac{1 - 2x}{\pi}$

$f(x) = x^2 + \pi r^2$

$$= x^2 + \pi \times \frac{[1 - 2x]^2}{\pi^2} = x^2 + \frac{(1 - 2x)^2}{\pi}$$

$$f'(x) = 2x - \frac{2(1 - 2x)\times(2)}{\pi} = 0\;;\; x = \frac{2(1 - 2x)}{\pi}$$

$\Rightarrow \pi x = 2 - 4x \Rightarrow \pi x = 2 - 4\left[\frac{1 - \pi r}{2}\right]$

$\Rightarrow \pi x = 2 - 2(1 - \pi r)$

$\Rightarrow \pi x = 2 - 2 + \pi r \Rightarrow \pi x = 2\pi r$

(34) **(A).** $f(x) = \tan^{-1}\left(\sqrt{\frac{1 + \sin x}{1 - \sin x}}\right)$, $x \in \left(0, \frac{\pi}{2}\right)$

$$f'(x) = \frac{1}{1 + \frac{1 + \sin x}{1 - \sin x}} \times \frac{1}{2\sqrt{\frac{1 + \sin x}{1 - \sin x}}}$$

$$\times \left\{\frac{(1 - \sin x)(\cos x) - (1 - \sin x)(-\cos x)}{(1 - \sin x)^2}\right\}$$

At $x = \frac{\pi}{6}$:

$$f'\left(\frac{\pi}{6}\right) = \frac{1}{1 + \frac{1 + \frac{1}{2}}{1 - \frac{1}{2}}} \times \frac{1}{2\sqrt{\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}}}} \times \left\{\frac{2 \times \sqrt{3}/2}{\left(\frac{1}{2}\right)^2}\right\}$$

$$= \frac{1}{1 + 3} \times \frac{1}{2\sqrt{3}} \times \frac{\sqrt{3}}{1/4} = \frac{1}{4} \times \frac{1}{2\sqrt{3}} \times 4 \times \sqrt{3} = \frac{1}{2}$$

Slope of normal = –2

Point at $x=\frac{\pi}{6}$, $f\left(\frac{\pi}{6}\right)=\tan^{-1}\sqrt{\frac{1+\frac{1}{2}}{1-\frac{1}{2}}}=\tan^{-1}\sqrt{3}=\frac{\pi}{3}$

∴ Equation $y-\frac{\pi}{3}=(-2)\left(x-\frac{\pi}{6}\right)$

$\Rightarrow y-\frac{\pi}{3}=-2x+\frac{\pi}{3}\Rightarrow y+2x=\frac{2\pi}{3}$

(35) **(D).** $6y=6$ (0, 1)

$y=1$

$(x-2)(x-3)y'+(x-3)+(x-2)y=1$

$6y'-3-2=1$; $y'=1$

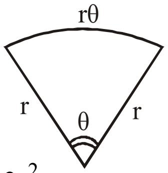
$y'_{(x=0)}=1$ ⇒ Slope of normal $=-1$

$(y-1)=-x$; $y+x=1$

(36) **(A).** Given $2r+r\theta=20$ .... (i)

Area $=\frac{1}{2}r^2\theta=A$

From (i), $\theta=\frac{20-2r}{r}$



$A=\frac{1}{2}r^2\frac{(20-2r)}{r}=\frac{20r-2r^2}{2}=10r-r^2$

$\frac{dA}{dr}=10-2r=0$; $r=5$; $\frac{d^2A}{dr^2}=-2<0$

r = 5 will given maximum area

$\theta=\frac{20-2(5)}{5}=2$ rad ; $A=\frac{1}{2}(5)^2\times2=25$

(37) **(B).** $2yy'=6$; $y'=\frac{6}{2y}=\frac{3}{y_1}$

$18x_1+2by_1y'=0$

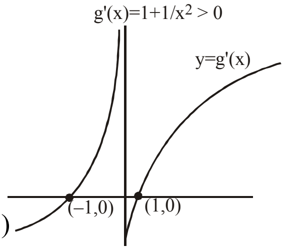
$y'=-\frac{18x_1}{2by_1}=\frac{-9x_1}{by_1}\Rightarrow-\frac{27x_1}{by_1^2}=-1\Rightarrow b=\frac{27x_1}{y_1^2}$

$y_1{}^2=6x_1$ ⇒ $b=9/2$

(38) **(B).** Let $g(x)=x-\frac{1}{x}=t$

$g'(x)=1+\frac{1}{x^2}>0$

$t\in R-\{0\}$; $t^2\in(0,\infty)$

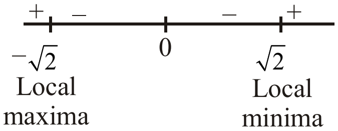


$f(x)=x^2+\frac{1}{x^2}=\left(x-\frac{1}{x}\right)^2+2=t^2+2\in(2,\infty)$

$h(x)=\frac{f(x)}{g(x)}$; $\frac{f(x)}{g(x)}=\frac{t^2+2}{t}=t+\frac{2}{t}$

Let $h(t)=t+\frac{2}{t}$

$h'(t)=1-\frac{2}{t^2}$



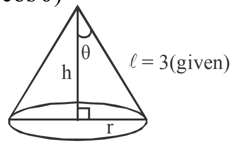
Local minimum value occurs at $t=\sqrt{2}$

Local minimum value $=h(\sqrt{2})=\sqrt{2}+\frac{2}{\sqrt{2}}=2\sqrt{2}$

(39) **(C).** $\ell=3$ (given) ∴ $h=3\cos\theta$; $r=3\sin\theta$

$V=\frac{1}{3}\pi r^2h=\frac{\pi}{3}(9\sin^2\theta)(3\cos\theta)$

∴ $\frac{dV}{d\theta}=0\Rightarrow\sin\theta=\sqrt{\frac{2}{3}}$



Also, $\left.\frac{d^2V}{d\theta^2}\right]_{\sin\theta=\sqrt{\frac{2}{3}}}$ = negative

∴ Volume is maximum, when $\sin\theta=\sqrt{2/3}$

∴ $V_{max}\left(\sin\theta=\sqrt{\frac{2}{3}}\right)=2\sqrt{3}\,\pi$ (in cu. m)

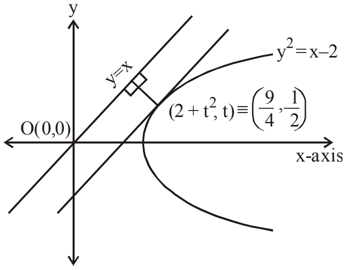
(40) **(D).** Point of intersection is P(2,6).

Also, $m_1=\left(\frac{dy}{dx}\right)_{P(2,6)}=-2x=-4$

$m_2=\left(\frac{dy}{dx}\right)_{P(2,6)}=2x=4$

∴ $|\tan\theta|=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\frac{8}{15}$

(41) **(A).**



We have,

$2y\frac{dy}{dx}=1\Rightarrow\left.\frac{dy}{dx}\right]_{P(2+t^2,t)}=\frac{1}{2t}=1$ ⇒ $t=1/2$

∴ P (9/4, 1/2)

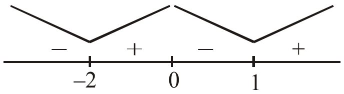
So, shortest distance $=\frac{\left|\frac{9}{4}-\frac{2}{4}\right|}{\sqrt{2}}=\frac{7}{4\sqrt{2}}$

(42) **(A).** $f(x) = 9x^4 + 12x^3 - 36x^2 + 25$

$f'(x) = 36x^3 + 36x^2 - 72x$

$= 36x\,(x^2 + x - 2)$

$= 36x\,(x - 1)(x + 2)$



Points of minima = $\{-2, 1\} = S_1$

Point of maxima = $\{0\} = S_2$

(43) **(B).** $\phi(x) = f(x) + f(2 - x)$

$\phi'(x) = f'(x) - f'(2 - x)$ ......(1)

Since $f''(x) > 0$

$\Rightarrow$ $f'(x)$ is increasing $\forall\ x \in (0, 2)$

Case-I : When $x > 2 - x \Rightarrow x > 1$

$\Rightarrow$ $\phi'(x) > 0\ \forall\ x \in (1, 2)$

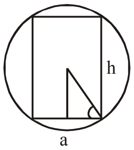
$\Rightarrow$ $f(x)$ is increasing on $(1, 2)$

Case-II : When $x < 2 - x \Rightarrow x < 1$

$\Rightarrow$ $\phi'(x) < 0\ \forall\ x \in (0, 1)$

$\therefore$ $\phi(x)$ is decreasing on (0, 1)

(44) **(A).**



$h = 2r\sin\theta$; $a = 2r\cos\theta$

$v = \pi(r\cos\theta)^2\,(2r\sin\theta)$

$v = 2\pi r^3\cos^2\theta\,\sin\theta$

$$\frac{dv}{d\theta} = \pi r^3(-2\cos\theta\sin^2\theta + \cos^3\theta) = 0$$

or $\tan\theta = \frac{1}{\sqrt{2}}$; $h = 2 \times 3 \times \frac{1}{\sqrt{3}} = 2\sqrt{3}$

(45) **(C).** $f(1) = 1 - 1 - 2 = -2$

$f(-1) = -1 - 1 + 2 = 0$

$$m = \frac{f(1) - f(-1)}{1 + 1} = \frac{-2 - 0}{2} = -1$$

$$\frac{dy}{dx} = 3x^2 - 2x - 2$$

$\Rightarrow$ $3x^2 - 2x - 2 = -1 \Rightarrow 3x^2 - 2x - 1 = 0$

$\Rightarrow$ $(x - 1)(3x + 1) = 0 \Rightarrow x = 1, -1/3$

(46) **(B).** $y = x^3 + ax - b$

(1, –5) lies on the curve

$\Rightarrow$ $-5 = 1 + a - b \Rightarrow a - b = -6$ ... (i)

Also, $y' = 3x^2 + a$

$y'(1, -5) = 3 + a$ (slope of tangent)

This tangent is $\perp$ to $-x + y + 4 = 0$

$\Rightarrow$ $(3 + a)(1) = -1 \Rightarrow a = -4$ ....(ii)

By (i) and (ii) : $a = -4, b = 2$

$\therefore$ $y = x^3 - 4x - 2$.

(2,–2) lies on this curve.

(47) **(A).** $f(x) = ax^5 + bx^4 + cx^3$

$$\lim_{x \to 0}\left(2 + \frac{ax^5 + bx^4 + cx^3}{x^3}\right) = 4$$

$\Rightarrow$ $2 + c = 4 \Rightarrow c = 2$

$f'(x) = 5ax^4 + 4bx^3 + 6x^2$

$= x^2(5ax^2 + 4bx + 6)$

$f'(1) = 0 \Rightarrow 5a + 4b + 6 = 0$

$f'(-1) = 0 \Rightarrow 5a - 4b + 6 = 0$

$b = 0$

$a = -6/5$

$$f(x) = \frac{-6}{5}x^5 + 2x^3$$

$f'(x) = -6x^4 + 6x^2$

$= 6x^2(-x^2 + 1)$

$= -6x^2(x + 1)(x - 1)$

–1 + 1–

1– 1

Minima at $x = -1$

Maxima at $x = 1$

(48) **(A).** $f(x)$ is a polynomial function

It is continuous and differentiable in [0, 1]

Here $f(0) = 11$, $f(1) = 1 - 4 + 8 + 11 = 16$

$f'(x) = 3x^2 - 8x + 8$

$$\therefore\ f'(c) = \frac{f(1) - f(0)}{1 - 0} = \frac{16 - 11}{1} = 3c^2 - 8c + 8$$

$\Rightarrow$ $3c^2 - 8c + 3 = 0$

$$C = \frac{8 \pm 2\sqrt{7}}{6} = \frac{4 \pm \sqrt{7}}{3}\ ;\quad c = \frac{4 - \sqrt{7}}{3} \in (0, 1)$$

(49) **(A).** $f(3) = f(4) \Rightarrow \alpha = 12$

$$f'(x) = \frac{x^2 - 12}{x(x^2 + 12)}\quad \therefore\quad f'(c) = 0\quad \therefore\ c = \sqrt{12}$$

$\therefore$ $f''(c) = 1/12$

(50) **(D).** $f'(x) = x(\pi - \cos^{-1}(\sin|x|))$

$$= x\left(\pi - \left(\frac{\pi}{2} - \sin^{-1}(\sin|x|)\right)\right) = x\left(\frac{\pi}{2} + |x|\right)$$

$$f(x) = \begin{cases} x\left(\dfrac{\pi}{2} + x\right) & x \ge 0 \\ x\left(\dfrac{\pi}{2} - x\right) & x < 0 \end{cases};\quad f'(x) = \begin{cases} \dfrac{\pi}{2} + 2x & x \ge 0 \\ \dfrac{\pi}{2} - 2x & x < 0 \end{cases}$$

$f'(x)$ is increasing in $(0, \pi/2)$ and decreasing in $(-\pi/2, 0)$

(51) **4.** $P \equiv (x_1, y_1)$

$$2yy' - 6x + y' = 0 \Rightarrow y' = \frac{6x_1}{1 + 2y_1}$$

$$\left(\frac{\frac{3}{2} - y_1}{-x_1}\right) = -\left(\frac{1 + 2y_1}{6x_1}\right)$$

$9 - 6y_1 = 1 + 2y_1 \Rightarrow y_1 = 1 \Rightarrow x_1 = \pm 2$

$\therefore$ Slope of tangent $= \pm 12/3 = \pm 4$ $\therefore |n| = 4$

(52) **3.00.**

Let $f(x) = ax^3 + bx^2 + cx + d$

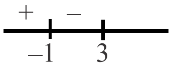
$a = \frac{1}{4}$ $d = \frac{35}{4}$

$b = \frac{-3}{4}$ $\qquad c = \frac{-9}{4}$

$f(x) = a(x^3 - 3x^2 - 9x) + d$

$f'(x) = \frac{3}{4}(x^2 - 2x - 3)$

$\Rightarrow f'(x) = 0 \Rightarrow x = 3, -1$



Local minima exist at $x = 3$

(53) **(C).** Let thickness of ice be 'h'.

Volume of ice, $v = \frac{4\pi}{3}((10+h)^3 - 10^3)$

$\frac{dv}{dt} = \frac{4\pi}{3}(3(10+h)^2)\cdot\frac{dh}{dt}$

Given $\frac{dv}{dt} = 50\text{cm}^3/\text{min}$ and $h = 5\text{cm}$

$\Rightarrow 50 = \frac{4\pi}{3}(3(10+h)^2)\cdot\frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{50}{4\pi \times 15^2} = \frac{1}{18\pi}\text{cm}/\text{min}$

(54) **(A).** $F'(x) = x^2 g(x) = x^2 \int_1^x f(u)\,du \Rightarrow F'(1) = 0$

$F''(x) = x^2 f(x) - 2x \int_1^x f(u)\,du$

$F''(1) = 1.f(1) - 2 \times 0 = 3$

$F'(1) = 0$ and $F''(1) = 3 > 0$ So, Minima