7

INTEGRATION

INDEFINITE INTEGRATION

INTEGRATION OF FUNCTION

The integral or primitive of a function f (x) with respect to x is that function f (x) whose derivative with respect to x is the given function f (x). It is expressed symbolically as

$$\int f(x)\,dx = \phi(x)$$

Thus $\int f(x)\,dx = \phi(x) \Leftrightarrow \frac{d}{dx}[\phi(x)] = f(x)$

The process of finding the integral of a function is called integration and the given function is called integrand. It is obvious to note that the operation of integration is inverse operation of differentiation. Hence integral of a function is also named as anti-derivative of that function. Further observe that $\frac{d}{dx}[\phi(x)+c] = f(x)$ $\therefore$

$$\int f(x)dx = \phi(x)+c$$

This constant number is generally denoted by c and it is called constant of integration. Due to the presence of this constant such an integral is called an indefinite integral.

BASIC THEOREMS ON INTEGRATION

If f (x), g (x) are two function of a variable x and k is constant, then

(i) $\int K\,f(x)dx = K\int f(x)dx$

(ii) $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)\,dx$

(iii) $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$

(iv) $\int \left(\frac{d}{dx} f(x)\right) dx = f(x)$

STANDARD INTEGRALS

1. $\int 0.dx = c$

2. $\int 1.dx = x + c$

3. $\int k.dx = kx + c (k \in R)$

4. $\int x^n.dx = \frac{x^{n+1}}{n+1} + c\ (n \neq -1)$

5. $\int \frac{1}{x}dx = \log_e x + c$

6. $\int e^x dx = e^x + c$

7. $\int a^x dx = \frac{a^x}{\log_e a} + c = a^x \log_a e + c$

8. $\int \sin x\,dx = -\cos x + c$

9. $\int \cos x\,dx = \sin x + c$

10. $\int \tan x\,dx = \log\sec x + c = -\log\cos x + c$

11. $\int \sec x\,dx = \log(\sec x + \tan x)$

$= -\log(\sec x - \tan x) = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) + c$

12. $\int \cot x\,dx = \log\sin x + c$

13. $\int \operatorname{cosec} x\,dx = -\log(\operatorname{cosec} x + \cot x)$

$= \log(\operatorname{cosec} x - \cot x) = \log\tan\left(\frac{x}{2}\right) + C$

14. $\int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$

15. $\int \sec x \tan x\,dx = \sec x + c$

16. $\int \sec^2 x dx = \tan x + c$

17. $\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

18. $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

19. $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) + c$

20. $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + c$

21. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c = -\cos^{-1}\left(\frac{x}{a}\right) + c$

22. $\int \frac{1}{\sqrt{x^2+a^2}}dx = \log\left(x+\sqrt{x^2+a^2}\right) + C$

23. $\int \frac{1}{\sqrt{x^2-a^2}}dx = \log\left(x+\sqrt{x^2-a^2}\right) + c$

24. $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}.\sin^{-1}\frac{x}{a} + c$

25. $\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}.\log\left(x + \sqrt{x^2 + a^2}\right)$

26. $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}.\log\left(x + \sqrt{x^2 - a^2}\right)$

27. $\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

28. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + c$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}}\sin\left\{bx - \tan^{-1}\left(\frac{b}{a}\right)\right\} + c$$

29. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + c$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}}\cos\left\{bx - \tan^{-1}\frac{b}{a}\right\} + c$$

Example 1 :

Evaluate $\int e^{x\log a} + e^{a\log x} + e^{a\log a}\,dx$

Sol. $\int e^{x\log a} + e^{a\log x} + e^{a\log a}\,dx$

$$= \int e^{\log a^x} + e^{\log x^a} + e^{\log a^a}\,dx \quad [\because e^{\log\lambda} = \lambda]$$

$$= \int (a^x + x^a + a^a)\,dx$$

$$= \int a^x dx + \int x^a dx + \int a^a dx = \frac{a^x}{\log a} + \frac{x^{a+1}}{a+1} + a^a.x + C$$

Example 2 :

Evaluate $\int \frac{1}{\sin^2 x\cos^2 x}\,dx$

Sol. $\int \frac{1}{\sin^2 x\cos^2 x}\,dx$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x\cos^2 x}\,dx = \int \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\,dx$$

$$= \int \sec^2 x\,dx + \int \operatorname{cosec}^2 x\,dx = \tan x - \cot x + C$$

Example 3 :

Evaluate $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}\,dx$

Sol. We have $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}\,dx$

$$= \int \frac{(2\cos^2 x - 1) - (2\cos^2\alpha - 1)}{\cos x - \cos\alpha}\,dx$$

$$= \int \frac{2(\cos^2 x - \cos^2\alpha)}{\cos x - \cos\alpha}\,dx = 2\int (\cos x + \cos\alpha)\,dx$$

$$= 2\int \cos x\,dx + 2\int \cos\alpha\,dx$$

$$= 2\int \cos x\,dx + 2\cos\alpha\int 1.dx = 2\sin x + 2x\cos\alpha + C$$

Example 4 :

Evaluate $\int a^x.e^x\,dx$

Sol. $I = \int a^x.e^x\,dx \Rightarrow \int (ae)^x\,dx \Rightarrow \frac{(ae)^x}{\log(ae)} + C = \frac{a^x e^x}{\log a + 1} + C$

Example 5 :

Evaluate $\int \sin^{-1}(\cos x)dx$

Sol. Here $I = \int \sin^{-1}(\cos x)dx \Rightarrow \int \sin^{-1}\sin\left(\frac{\pi}{2} - x\right)dx$

$$\Rightarrow \int \left(\frac{\pi}{2} - x\right)dx = \frac{\left(\frac{\pi}{2} - x\right)^2}{-2} + C$$

Example 6 :

Evaluate $\int \frac{1}{1 + \sin x}\,dx$

Sol. $\int \frac{1}{1 + \sin x}\,dx$

$$= \int \frac{1}{1 + \sin x}.\frac{(1 - \sin x)}{(1 - \sin x)}\,dx = \int \frac{1 - \sin x}{1 - \sin^2 x}\,dx$$

$$= \int \frac{1 - \sin x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x}\,dx - \int \frac{\sin x}{\cos^2 x}\,dx$$

$$= \int \sec^2 x\,dx - \int \tan x\sec x\,dx = \tan x - \sec x + C$$

COMPARISON BETWEEN DIFFERENTIATION & INTEGRATION

(i) The operations of differentiation and integration are defined on functions.

(ii) The derivative of a function, when it exists is a unique function whereas the integral of a function is not unique. In fact there are infinitely many integrals of a function such that any two integrals differ by a constant.

(iii) The derivative of a function at a point (if it exists) is meaningful but the integral of a function at a point does not have any sense.

(iv) Every function is not differentiable. Similarly, every function is not integrable.

(v) The derivative of a function at a point determines, the slope of the tangent to the corresponding curve at that point. The integral of a function represents a family of curves having parallel tangents at the points of intersection of the curves of the family with the lines orthogonal to the axis representing the variable of integration.

(vi) The processes of differentiation and integration are inverse of each other

(vii) The operations of differentiation and integration are operations on functions

(viii) Both the operations are linear i.e.

$$\frac{d}{dx}(f(x)+g(x))=\frac{d}{dx}(f(x))+\frac{d}{dx}(g(x))$$

and, $\int (f(x)+g(x))\,dx=\int f(x)\,dx+\int g(x)\,dx$

(ix) The constant can be taken out side the differential as well as integral sign. i.e.

$$\frac{d}{dx}(k(f(x))=k\frac{d}{dx}(f(x)) \text{ and } \int k.f(x)\,dx=k\int f(x)\,dx$$

(x) Differentiation and integration both are processes involving limits.

METHODS OF INTEGRATION

When integration can not be reduced into some standard form then integration is performed using following methods.

(i) Integration by substitution

(ii) Integration by parts

(iii) Integration using partial fractions

(iv) Integration by reduction formulae

INTEGRATION BY SUBSTITUTION:

Integrals of the form $\int f(ax+b)\,dx$ **:**

If $\int f(x)dx=\phi(x)$, then $\int f(ax+b)dx=\frac{1}{a}\phi(ax+b)$

(i) $\int (ax+b)^n dx=\frac{1}{a}.\frac{(ax+b)^{n+1}}{n+1}+C,\quad n\neq -1$

(ii) $\int \frac{1}{ax+b}dx=\frac{1}{a}\log|ax+b|+C$

(iii) $\int e^{ax+b}dx=\frac{1}{a}e^{ax+b}+C$

(iv) $\int a^{bx+c}dx=\frac{1}{b}.\frac{a^{bx+c}}{\log a}+C,\ a>0 \text{ and } a\neq 1$

(v) $\int \sin(ax+b)dx=-\frac{1}{a}\cos(ax+b)+C$

(vi) $\int \cos(ax+b)dx=\frac{1}{a}\sin(ax+b)+C$

(vii) $\int \sec^2(ax+b)dx=\frac{1}{a}\tan(ax+b)+C$

(viii) $\int \operatorname{cosec}^2(ax+b)dx=-\frac{1}{a}\cot(ax+b)+C$

(ix) $\int \sec(ax+b)\tan(ax+b)dx=\frac{1}{a}\sec(ax+b)+C$

(x) $\int \sec(ax+b)dx=\frac{1}{a}\log|\sec(ax+b)+\tan(ax+b)|+C$

(xi) $\int \operatorname{cosec}(ax+b)\cot(ax+b)dx=-\frac{1}{a}\operatorname{cosec}(ax+b)+C$

(xii) $\int \operatorname{cosec}(ax+b)dx=\frac{1}{a}\log|\operatorname{cosec}(ax+b)-\cot(ax+b)|+C$

(xiii) $\int \tan(ax+b)dx=-\frac{1}{a}\log|\cos(ax+b)|+C$

(xiv) $\int \cot(ax+b)dx=\frac{1}{a}\log|\sin(ax+b)|+C$

Example 7 :

Evaluate $\int \frac{8x+13}{\sqrt{4x+7}}dx$

Sol. $\int \frac{8x+13}{\sqrt{4x+7}}dx=\int \frac{8x+14-1}{\sqrt{4x+7}}dx=\int \frac{2(4x+7)-1}{\sqrt{4x+7}}dx$

$$=2\int \sqrt{4x+7}\,dx-\int \frac{1}{\sqrt{4x+7}}dx=2\left\{\frac{(4x+7)^{3/2}}{4\times\frac{3}{2}}\right\}-\left\{\frac{(4x+7)^{1/2}}{4\times\frac{1}{2}}\right\}+C$$

$$=\frac{1}{3}(4x+7)^{3/2}-\frac{1}{2}(4x+7)^{1/2}+C$$

Example 8 :

Evaluate $\int \frac{x+1}{\sqrt{2x-1}}dx$

Sol. $\int \frac{x+1}{\sqrt{2x-1}}dx=\frac{1}{2}\int \frac{2x+2}{\sqrt{2x-1}}dx=\frac{1}{2}\int \frac{2x-1+3}{\sqrt{2x-1}}dx$

$$=\frac{1}{2}\left[\int \sqrt{2x-1}\,dx+3\int \frac{1}{\sqrt{2x-1}}dx\right]$$

$$=\frac{1}{2}\left[\frac{(2x-1)^{3/2}}{2\times\frac{3}{2}}+3\frac{(2x-1)^{1/2}}{2\times\frac{1}{2}}\right]+C$$

$$=\frac{1}{2}\left[\frac{(2x-1)^{3/2}}{3}+3(2x-1)^{1/2}\right]+C$$

Integrals of the form $\int \frac{f'(x)}{f(x)}dx$:

$$\int \frac{f'(x)}{f(x)}dx = \log\{f(x)\}$$

Example 9 :

Evaluate $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$

Sol. Let $e^x + e^{-x} = t$; $(e^x - e^{-x}).\, dx = dt$

$$\therefore\quad I = \int \frac{dt}{t} = \log t + C \Rightarrow \log(e^x + e^{-x}) + C$$

Example 10 :

Evaluate $\int \frac{\sin 2x}{a^2\sin^2 x + b^2\cos^2 x}dx$

Sol. Let $I = \int \frac{\sin 2x}{a^2\sin^2 x + b^2\cos^2 x}dx$

Putting $a^2 \sin^2 x + b^2 \cos^2 x = t$ and,
$(2a^2 \sin x \cos x - 2b^2 \cos x \sin x)\, dx = dt$

or $dx = \frac{dt}{2\sin x\cos x(a^2 - b^2)}$, we get

$$I = \int \frac{\sin 2x}{a^2\sin^2 x + b^2\cos^2 x}dx$$

$$= \int \frac{\sin 2x}{t}\,\frac{dt}{2\sin x\cos x(a^2 - b^2)}$$

$$= \frac{1}{(a^2 - b^2)}\int \frac{1}{t}dt = \frac{1}{(a^2 - b^2)}\log|t| + C$$

$$= \frac{1}{(a^2 - b^2)}\log|a^2\sin^2 x + b^2\cos^2 x| + C$$

Integrals of the form $\int \{f(x)\}^n\, f'(x)\,dx$:

$$\int \{f(x)\}^n\, f'(x)\, dx = \frac{\{f(x)\}^{n+1}}{n+1},\quad n \neq -1$$

Example 11 :

Evaluate $\int \frac{(\log x)^3}{x}dx$

Sol. Let $I = \int \frac{(\log x)^3}{x}dx$

Putting $\log x = t$ and $(1/x)\, dx = dt$ or, $dx = x\, dt$, we get

$$I = \int \frac{t^3}{x} x\, dt = \int t^3\, dt = \frac{t^4}{4} + C = \frac{(\log x)^4}{4} + C$$

Example 12 :

Evaluate $\int \frac{4(\sin^{-1} x)^3}{\sqrt{1 - x^2}}dx$

Sol. Let $I = \int \frac{4(\sin^{-1} x)^3}{\sqrt{1 - x^2}}dx$

Putting $\sin^{-1} x = t$ and $\frac{1}{\sqrt{1 - x^2}}\, dx = dt$ or, $dx = \sqrt{1 - x^2}\, dt$,

$$I = \int \frac{4t^3}{\sqrt{1 - x^2}}\sqrt{1 - x^2}\, dt = 4\int t^3 dt = t^4 + C = (\sin^{-1} x)^4 + C$$

Example 13 :

$$\int 2^{2^{2^x}}\; 2^{2^x}\; 2^x\, dx$$

Sol. $I = \int 2^{2^{2^x}}\; 2^{2^x}\; 2^x\, dx$

Putting $2^{2^{2^x}} = t$ and $2^{2^{2^x}}\; 2^{2^x}\; 2^x\,(\log 2)^3\, dx = dt$, we get

$$I = \frac{1}{(\log 2)^3}dt = \frac{1}{(\log 2)^3}t + C = \frac{1}{(\log 2)^3}2^{2^{2^x}} + C$$

Integrals of the form $\int \frac{f'(x)}{\sqrt{f(x)}}dx$:

$$\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c$$

Example 14 :

Evaluate $\int \frac{\sec^2 x}{\sqrt{\tan x}}dx$

Sol. Let $t = \tan x$; $dt = \sec^2 x\, dx$

$$\therefore\quad I = \int \frac{dt}{\sqrt{t}} = 2t^{1/2} + c \Rightarrow 2\sqrt{\tan x}$$

Integrals of the form $\int \sin^m x\, dx$, $\int \cos^m x\, dx$:

To evaluate integrals of the form

$\int \sin^m x\, dx, \int \cos^m x\, dx$ where $m \le 4$, we express $\sin^m x$ and $\cos^m x$ in terms of sines and cosines of multiples of x by using trigonometrical identities given below

(1) $\sin^2 x = \frac{1 - \cos 2x}{2}$ (2) $\cos^2 x = \frac{1 + \cos 2x}{2}$

(3) $\sin 3x = 3\sin x - 4\sin^3 x$ (4) $\cos 3x = 4\cos^3 x - 3\cos x$

Example 15 :

Evaluate $\int \sin^3 x \, dx$

Sol. $\int \sin^3 x \, dx = \int \frac{3\sin x - \sin 3x}{4} dx$

$$= \frac{1}{4}\int (3\sin x - \sin 3x)dx = \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right] + C$$

Example 16 :

Evaluate $\int \cos^4 x \, dx$

Sol. $\int \cos^4 x \, dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx$

$$= \frac{1}{4}\int 1 + 2\cos 2x + \cos^2 2x \, dx$$

$$= \frac{1}{4}\int 1 + 2\cos 2x + \frac{1+\cos 4x}{2} dx$$

$$= \frac{1}{8}\int 3 + 4\cos 2x + \cos 4x \, dx = \frac{1}{8}\left[3x + 2\sin 2x + \frac{\sin 4x}{4}\right] + C$$

Integrals of the form $\int \sin mx \cos nx \, dx$ **,**

$\int \sin mx \sin nx \, dx$ **,** $\int \cos mx \cos nx \, dx$ **,**

and $\int \cos mx \sin nx \, dx$ **:**

To evaluate integrals of the form $\int \sin mx \cos nx \, dx$ **,**

$\int \sin mx \sin nx \, dx$ **,** $\int \cos mx \cos nx \, dx$ **,**

and $\int \cos mx \sin nx \, dx$

we use the following trigonometrical identities :

2 sin A cos B = sin (A + B) + sin (A – B)

2 cos A sin B = sin (A+ B) – sin (A – B)

2 cos A cos B = cos (A + B) + cos (A – B)

2 sin A sin B = cos (A – B) – cos (A + B)

Example 17 :

Evaluate $\int \sin 3x \cos 4x \, dx$

Sol. $\int \sin 3x \cos 4x \, dx$

$$= \frac{1}{2}\int 2\sin 3x \cos 4x \, dx = \frac{1}{2}\int \sin 7x + \sin(-x) \, dx$$

$$= \frac{1}{2}\int (\sin 7x - \sin x)dx = \frac{1}{2}\left[-\frac{\cos 7x}{7} + \cos x\right] + C$$

Example 18 :

Evaluate $\int \sin x \sin 2x \sin 3x \, dx$

Sol. $\int \sin x \sin 2x \sin 3x \, dx$

$$= \frac{1}{2}\int (2\sin 2x \sin x) \sin 3x \, dx = \frac{1}{2}\int (\cos x - \cos 3x) \sin 3x \, dx$$

$$= \frac{1}{4}\int (2\sin 3x \cos x - 2\sin 3x \cos 3x) \, dx$$

$$= \frac{1}{4}\int (\sin 4x + \sin 2x - \sin 6x) \, dx$$

$$= \frac{1}{4}\left[-\frac{\cos 4x}{4} - \frac{\cos 2x}{2} + \frac{\cos 6x}{6}\right] + C$$

Integrals of the form $\int \sin^m x \cos^n x \, dx$ **;** $m, n \in N$**:**

In order to evaluate the integrals of the form $\int \sin^m x \cos^n x \, dx$, we may use the following algorithm.

(i) Obtain the integral, say, $\int \sin^m x \cos^n x \, dx$

(ii) Check the exponents of sin x and cosx.

(iii) If the exponent of sinx is an odd positive integer put cosx= t
If the exponent of cosx is an odd positive integer put sinx=t.
If the exponents of sin x and cos x both are odd positive integers put either sinx = t or cosx = t.
If the exponents of sinx and cosx both are even positive integers, then express $\sin^m x \cos^n x$ in terms of sines and cosines of multiples of x by using trigonometric results or De' Moivere's theorem.

(iv) Evaluate the integral obtained in step (iii)

Example 19 :

Evaluate $\int \sin^3 x \cos^4 x \, dx$

Sol. Let $I = \int \sin^3 x \cos^4 x \, dx$

Here, power of sinx is odd, so we substitute

$$\cos x = t \Rightarrow -\sin x dx = dt \Rightarrow dx = -\frac{dt}{\sin x}$$

$$\therefore \quad I = \int \sin^3 x \; t^4 \left(-\frac{dt}{\sin x}\right)$$

$$= -\int \sin^2 x \; t^4 dt = -\int (1-t^2)t^4 dt - \int (t^4 - t^6) dt$$

$$= -\frac{t^5}{5} + \frac{t^7}{7} + C = -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$$

Example 20 :

Evaluate $\int \sin^2 x \cos^5 x \, dx$

Sol. Let $I = \int \sin^2 x \cos^5 x \, dx$

Here, power of cos x is odd, so we substitute

$$\sin x = t \Rightarrow \cos x dx = dt \Rightarrow dx = \frac{dt}{\cos x}$$

$$\therefore I = \int t^2 \cos^5 x \frac{dt}{\cos x} = \int t^2 (1-\sin^2 x)^2 dt = \int t^2 (1-t^2)^2 dt$$

$$I = \int (t^2 - 2t^4 + t^6) dt = \frac{t^3}{3} - 2\frac{t^5}{5} + \frac{t^7}{7} + C$$

$$= \frac{\sin^3 x}{3} - 2\frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + C$$

Example 21 :

Evaluate $\int \cos^3 x\, e^{\log \sin x} dx$

Sol. $I = \int \cos^3 x\, e^{\log \sin x} dx = \int \cos^3 x \sin x\, dx$

Putting cos x = t and – sin x dx = dt or, sin x dx = – dt, we get

$$I = -\int t^3 dt = \frac{-t^4}{4} + C = -\frac{\cos^4 x}{4} + C$$

Some special integrals:

In this section, we will introduce some important formulae of integrals and apply them to evaluate many integrals. Following are some substitutions useful in evaluating integrals

Expression	**Substitution**
$a^2 + x^2$	$x = a\tan\theta$ or $a\cot\theta$
$a^2 - x^2$	$x = a\sin\theta$ or $a\cos\theta$
$x^2 - a^2$	$x = a\sec\theta$ or $a\,\mathrm{cosec}\theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(x-\beta)}$	$x = \alpha\cos^2\theta + \beta\sin^2\theta$

Example 22 :

Evaluate $\int \frac{1}{9x^2-4} dx$

Sol. $\int \frac{1}{9x^2-4} dx = \frac{1}{9}\int \frac{1}{x^2-(2/3)^2} dx$

$$= \frac{1}{9}\cdot\frac{1}{2\times\frac{2}{3}}\log\left|\frac{x-\frac{2}{3}}{x+\frac{2}{3}}\right| + C = \frac{1}{12}\log\left|\frac{3x-2}{3x+2}\right| + C$$

Example 23 :

Evaluate $\int \frac{dx}{\sqrt{x(a-x)}}$

Sol. Let $x = a\sin^2\theta$ then $dx = 2a\sin\theta\cos\theta\, d\theta$

$$\therefore I = \int \frac{2a\sin\theta.\cos\theta}{\sqrt{a\sin^2\theta.a\cos^2\theta}} = 2\int d\theta = 2\theta + c$$

$$= 2\sin^{-1}\left(\sqrt{x/a}\right) + C$$

Integration of rational and irrational functions :

Integrals of the type $\int \frac{1}{ax^2+bx+c} dx$ **or reducible to the**

$$\int \frac{1}{ax^2+bx+c} dx$$

To evaluate this type of integrals we express $ax^2 + bx + c$ as the sum or difference of two squares by using the following algorithm

(i) Make the coefficient of x^2 unity, if it is not, by multiplying and dividing by it.

(ii) Add and subtract the square of the half of coefficient of x to express $ax^2 + bx + c$ in the form $a\left[\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right]$

(iii) Use the suitable formula from the following formulas

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$$

$$\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$$

Example 24 :

Evaluate $\int \frac{1}{2x^2+x-1} dx$

Sol. $\int \frac{1}{2x^2+x-1} dx = \frac{1}{2}\int \frac{1}{x^2+\frac{x}{2}-\frac{1}{2}} dx$

$$= \frac{1}{2}\int \frac{1}{x^2+x/2+(1/4)^2-(1/4)^2-1/2} dx$$

$$= \frac{1}{2}\int \frac{1}{(x+1/4)^2-(3/4)^2} dx$$

$$= \frac{1}{2}\cdot\frac{1}{2(3/4)}\log\left|\frac{x+1/4-3/4}{x+1/4+3/4}\right| + C$$

$$= \frac{1}{3}\log\left|\frac{x-1/2}{x+1}\right| + C = \frac{1}{3}\log\left|\frac{2x-1}{2(x+1)}\right| + C$$

Example 25 :

Evaluate $\int \frac{x}{x^4+x^2+1}\,dx$

Sol. $I = \int \frac{x}{x^4+x^2+1}\,dx = \int \frac{x}{(x^2)^2+x^2+1}$

Let $x^2 = t$. Then $d(x^2) = dt \Rightarrow 2x\,dx = dt \Rightarrow dx = \frac{dt}{2x}$

$$I = \int \frac{x}{t^2+t+1}\cdot\frac{dt}{2x} = \frac{1}{2}\int \frac{1}{t^2+t+1}dt$$

$$= \frac{1}{2}\int \frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dt = \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)+C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+C$$

Integrals of the type $\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$ **or reducible to the**

$$\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

In order to evaluate this type of integrals, we may use the following algorithm

(i) Make the coefficient of x^2, if it is not.

(ii) Find half of the coefficient of x.

(iii) Add the subtract $\left(\frac{1}{2}\text{coeff.of } x\right)^2$ inside the square root to express the quantity inside the square root in the form

$$\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2} \text{ or } \frac{4ac-b^2}{4a^2}-\left(x+\frac{b}{2a}\right)^2.$$

(iv) Use the suitable formula from the following formulas

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \log\left|x+\sqrt{a^2+x^2}\right|+C$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x+\sqrt{x^2-a^2}\right|+C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C$$

Example 26 :

Evaluate $\int \frac{1}{\sqrt{9+8x-x^2}}dx$

Sol. $\int \frac{1}{\sqrt{9+8x-x^2}}dx = \int \frac{1}{\sqrt{-\{x^2-8x-9\}}}dx$

$$= \int \frac{1}{\sqrt{-\{x^2-8x+16-25\}}}dx$$

$$= \int \frac{1}{\sqrt{-\{(x-4)^2-5^2\}}}dx = \sin^{-1}\left(\frac{x-4}{5}\right)+C$$

Example 27 :

Evaluate $\int \frac{a^x}{\sqrt{1-a^{2x}}}\,dx$

Sol. $I = \int \frac{a^x}{\sqrt{1-a^{2x}}}\,dx = \int \frac{a^x}{\sqrt{1^2-(a^x)^2}}\,dx$

Let $a^x = t$. Then, $d(a^x) = dt$

$\Rightarrow a^x \log_e a\,dx = dt \Rightarrow dx = \frac{dt}{a^x \log_e a}$

$$\therefore \quad I = \int \frac{a^x}{\sqrt{1^2-t^2}}\frac{dt}{a^x \log a} = \frac{1}{\log a}\int \frac{dt}{\sqrt{1^2-t^2}}$$

$$= \frac{1}{\log a}\sin^{-1} t + C = \frac{1}{\log a}\sin^{-1}(a^x)+C$$

Integrals of the form $\int \frac{px+q}{ax^2+bx+c}\,dx$

To evaluate this type of integrals, we use the following algorithm

(i) Write the numerator px + q in the following form

$$px+q = \lambda\left\{\frac{d}{dx}(ax^2+bx+c)\right\}+\mu$$

i.e. $px + q = \lambda\,(2ax + b) + \mu$

(ii) Obtain the values of λ and μ by equating the coefficient of like powers of x on both sides.

(iii) Replace $px + q$ by $\lambda\,(2ax + b) + \mu$ in the given integral to get

$$\int \frac{px+q}{ax^2+bx+c}\,dx$$

$$= \lambda\int \frac{2ax+b}{ax^2+bx+c}\,dx + \mu\int \frac{1}{ax^2+bx+c}\,dx$$

(iv) Integrate RHS in step (iii) and put the values of λ and μ obtained is step (ii).

Example 28 :

Evaluate $\int \frac{x}{x^2+x+1}dx$

Sol. Let $x = \lambda.\frac{d}{dx}(x^2+x+1)+\mu$. Then, $x = \lambda(2x+1)+\mu$

Comparing the coefficients of like powers of x, we get

$1 = 2\lambda$ and $\lambda + \mu = 0 \Rightarrow \lambda = \frac{1}{2}$ and $\mu = -\lambda = -\frac{1}{2}$.

$$\therefore \int \frac{x}{x^2+x+1}dx = \int \frac{1/2(2x+1)-1/2}{x^2+x+1}dx$$

$$= \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx - \frac{1}{2}\int \frac{1}{x^2+x+1}dx$$

$$= \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx - \frac{1}{2}\int \frac{1}{\left(x^2+x+\frac{1}{4}\right)+\frac{3}{4}}dx$$

$$= \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx - \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$= \frac{1}{2}\log|x^2+x+1| - \frac{1}{2}.\frac{1}{(\sqrt{3}/2)}\tan^{-1}\left(\frac{x+1/2}{\sqrt{3}/2}\right)+C$$

$$= \frac{1}{2}\log|x^2+x+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

Example 29 :

Evaluate $\int \frac{1}{2e^{2x}+3e^x+1}dx$

Sol. $I = \int \frac{1}{2e^{2x}+3e^x+1}dx$

$$= \int \frac{1}{\frac{2}{e^{-2x}}+\frac{3}{e^{-x}}+1}dx = \int \frac{e^{-2x}}{2+3e^{-x}+e^{-2x}}dx$$

Let $e^{-x} = t$. Then, $d(e^{-x}) = dt \Rightarrow -e^{-x}dx = dt \Rightarrow dx = -\frac{dt}{e^{-x}}$

$$\therefore \quad I = \int \frac{-t\,dt}{2+3t+t^2} = -\int \frac{t}{t^2+3t+2}dt$$

Let $t = \lambda(2t+3)+\mu$

Comparing the coefficient of like powers of t, we get

$2\lambda = 1$, $3\lambda + \mu = 0 \Rightarrow \lambda = 1/2$, $\mu = -3/2$

$$\therefore \quad I = -\int \frac{\lambda(2t+3)+\mu}{t^2+3t+2}dt$$

$$= -\lambda\int \frac{2t+3}{t^2+3t+2}dt - \mu\int \frac{1}{t^2+3t+2}dt$$

$$= -\frac{1}{2}\int \frac{2t+3}{t^2+3t+2}dt + \frac{3}{2}\int \frac{t}{(t+3/2)^2-(1/2)^2}dt$$

$$= -\frac{1}{2}\log|t^2+3t+2| + \frac{3}{2}\times\frac{1}{2\left(\frac{1}{2}\right)}\log\left|\frac{t+\frac{3}{2}-\frac{1}{2}}{t+\frac{3}{2}+\frac{1}{2}}\right|+C$$

$$= -\frac{1}{2}\log|e^{-2x}+3e^{-x}+2| + \frac{3}{2}\log\left|\frac{e^{-x}+1}{e^{-x}+2}\right|+C$$

Integrals of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

In order to evaluate this type of integrals, we use the following algorithm

(i) Write the numerator px + q in the following form

$$px+q = \lambda\left\{\frac{d}{dx}(ax^2+bx+c)\right\}+\mu$$

i.e. $px+q = \lambda(2ax+b)+\mu$

(ii) Obtain the values of λ and μ by equating the coefficient of like powers of x on both sides.

(iii) Replace px + q by $\lambda(2ax+b)+\mu$ in the given integral to get

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$$

$$= \lambda\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}dx + \mu\int \frac{1}{\sqrt{ax^2+bx+c}}dx$$

(iv) Integrate RHS in step (iii) and put the values of λ and μ obtained is step (ii).

Example 30 :

Evaluate $\int \frac{2x+3}{\sqrt{x^2+4x+1}}dx$

Sol. Let $2x+3 = \lambda.\frac{d}{dx}(x^2+4x+1)+\mu$. Then

$2x+3 = \lambda(2x+4)+\mu$.

Comparing the coefficient of like powers of x, we get

$2\lambda = 2$ and $4\lambda + \mu = 3 \Rightarrow \lambda = 1$ and $\mu = -1$

$$\therefore \quad \int \frac{2x+3}{\sqrt{x^2+4x+1}}dx = \int \frac{(2x+4)-1}{\sqrt{x^2+4x+1}}dx$$

$$= \int \frac{2x+4}{\sqrt{x^2+4x+1}}dx - \int \frac{1}{\sqrt{x^2+4x+1}}dx$$

$$= \int \frac{dt}{\sqrt{t}} - \int \frac{1}{\sqrt{(x+2)^2 - (\sqrt{3})^2}} dx \text{ where } t = x^2 + 4x + 1$$

$$= 2\sqrt{t} - \log|(x+2) + \sqrt{x^2+4x+1}| + C$$

$$= 2\sqrt{x^2+4x+1} - \log|x+2+\sqrt{x^2+4x+1}| + C$$

Example 31 :

Evaluate $\int \sqrt{\frac{a-x}{a+x}} dx$

Sol. $I = \int \sqrt{\frac{a-x}{a+x}} dx = \int \sqrt{\frac{a-x}{a+x} \times \frac{a-x}{a-x}} dx = \int \frac{a-x}{\sqrt{a^2-x^2}} dx$

$$\Rightarrow I = \int \frac{a}{\sqrt{a^2-x^2}} dx - \int \frac{x}{\sqrt{a^2-x^2}} dx$$

$$\Rightarrow I = a\int \frac{1}{\sqrt{a^2-x^2}} dx + \frac{1}{2}\int \frac{-2x}{\sqrt{a^2-x^2}} dx$$

Putting $a^2 - x^2 = t$, and $-2x\,dx = dt$, we get

$$I = a\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\int \frac{dt}{\sqrt{t}} = a\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\left(\frac{t^{1/2}}{1/2}\right) + C$$

$$\Rightarrow I = a\sin^{-1}\left(\frac{x}{a}\right) + \sqrt{t} + C = a\sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2-x^2} + C$$

Integrals of the form $\int \frac{P(x)}{ax^2+bx+c} dx$ Where P (x) is a Polynomial of Degree greater than or equal to 2:

To evaluate this type of integrals we divide the numerator by the denominator and express the integrand as

$Q(x) + \frac{R(x)}{ax^2+bx+c}$, where R (x) is a linear function of x.

$$\therefore \int \frac{P(x)}{ax^2+bx+c} dx = \int Q(x)dx + \int \frac{R(x)}{ax^2+bx+c} dx$$

Example 32 :

Evaluate $\int \frac{x^2+5x+3}{x^2+3x+2} dx$

Sol. $\int \frac{x^2+5x+3}{x^2+3x+2} dx = \int \left(1 + \frac{2x+1}{x^2+3x+2}\right) dx$

$$= \int 1.dx + \int \frac{2x+3-2}{x^2+3x+2}.dx$$

$$= \int 1.dx + \int \frac{2x+3}{x^2+3x+2}.dx - 2\int \frac{1}{x^2+3x+2}.dx$$

$$= x + \log|x^2+3x+2| - 2\int \frac{1}{\left(x+\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}.dx$$

$$= x + \log|x^2+3x+2| - 2.\frac{1}{2\left(\frac{1}{2}\right)}\log\left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right| + c$$

$$= x + \log|x^2+3x+2| - 2\log\left|\frac{x+1}{x+2}\right| + c.$$

Integrals of the form $\int \sqrt{ax^2+bx+c}\, dx$:

In order to evaluate the above type of integrals, we use the following algorithm

(i) Make coefficient of x^2 as one by taking 'a' common to obtain

$x^2 + \frac{b}{a}x + \frac{c}{a}$.

(ii) Add and subtract $\left(\frac{b}{2a}\right)^2$ in $x^2 + \frac{b}{a}x + \frac{c}{a}$ to obtain

$\left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}$

After applying these two steps the integral reduces to one of the following three forms

$\int \sqrt{a^2+x^2}\, dx$, $\int \sqrt{a^2-x^2}\, dx$, $\int \sqrt{x^2-a^2}\, dx$

(iii) Use the appropriate formula

Example 33 :

Evaluate $\int \sqrt{x^2+2x+5}\, dx$

Sol. $\int \sqrt{x^2+2x+5}\, dx = \int \sqrt{x^2+2x+1+4}\, dx$

$$= \frac{1}{2}(x+1)\sqrt{(x+1)^2+2^2} + \frac{1}{2}(2)^2 \log|(x+1) + \sqrt{(x+1)^2+2^2}| + C$$

$$= \frac{1}{2}(x+1)\sqrt{x^2+2x+5} + 2\log|(x+1) + \sqrt{x^2+2x+5}| + C$$

Integrals of the form $\int (px+q)\sqrt{ax^2+bx+c}\, dx$:

In order to evaluate this type of integrals, we use the following algorithm

(i) Express $px + q$ as $px + q = \lambda \frac{d}{dx}(ax^2+bx+c) + \mu$

i.e. $px + q = \lambda(2ax + b) + \mu$

(ii) Obtain the values of λ and μ by equating the coefficient of x and constant terms on both sides

(iii) Replace px + q by λ (2ax + b) + μ in the integral to obtain

$$\int (px+q)\sqrt{ax^2+bx+c}\,dx = \lambda$$

$$\int (2ax+b)\sqrt{ax^2+bx+c}\,dx + \mu \int \sqrt{ax^2+bx+c}\,dx$$

(iv) To evaluate first integral on RHS, use the formula

$$\int (f(x))^n\ f'(x)\ dx = \frac{(f(x))^{n+1}}{n+1}.$$

Example 34 :

Evaluate $\int x\sqrt{1+x-x^2}\,dx$

Sol. Let $x = \lambda.\frac{d}{dx}(1+x-x^2)+\mu$ Then,

$x = \lambda(1-2x)+\mu$

Comparing the coefficients of like powers of x , we get

$$1 = -2\lambda \text{ and } \lambda+\mu = 0 \Rightarrow \lambda = -\frac{1}{2} \text{ and } \mu = \frac{1}{2}$$

$$\therefore \int x\sqrt{1+x-x^2}\,dx$$

$$= \int\left\{-\frac{1}{2}(1-2x)+\frac{1}{2}\right\}\sqrt{1+x-x^2}\,dx$$

$$= -\frac{1}{2}\int (1-2x)\sqrt{1+x-x^2}\,dx + \int\sqrt{1+x-x^2}\,dx$$

$$= -\frac{1}{2}\int (1-2x)\sqrt{1+x-x^2}\,dx + \frac{1}{2}dx\int\sqrt{1+x-x^2}\,dx$$

$$= -\frac{1}{2}\int\sqrt{t}\,dt + \frac{1}{2}\int\sqrt{-\left(x^2-x+\frac{1}{4}-\frac{1}{4}-1\right)}\,dx$$

$$= -\frac{1}{2}.\left(\frac{t^{3/2}}{3/2}\right)+\frac{1}{2}\int\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\,dx,$$

$$= -\frac{1}{3}t^{3/2}+\frac{1}{2}\left[\left(x-\frac{1}{2}\right)\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}\right.$$

$$\left.+\frac{1}{2}\left(\frac{\sqrt{5}}{2}\right)^2\sin^{-1}\left(\frac{x-1/2}{\sqrt{5}/2}\right)\right]+C$$

$$= -\frac{1}{3}(1+x-x^2)^{3/2}$$

$$+\frac{1}{2}\left[\frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{1+x-x^2}+\frac{5}{8}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)\right]+C$$

Integration of trigonometric function:
Integrals of the form

$$\int\frac{1}{a\sin^2x+b\cos^2x}dx,\ \int\frac{1}{a+b\sin^2x}dx, \int\frac{1}{a+b\cos^2x}dx,$$

$$\int\frac{1}{(a\sin x+b\cos x)^2}dx\ , \int\frac{1}{a+b\sin^2x+c\cos^2x}dx:$$

To evaluate this type of integrals we use the following algorithm.

(i) Divide numerator and denominator both by $\cos^2 x$

(ii) Replace $\sec^2 x$, if any, in denominator by $1 + \tan^2 x$

(iii) Put tan x = t so that $\sec^2 x\,dx = dt$.
This substitution reduces the integral in the form

$$\int\frac{1}{at^2+bt+c}dt$$

(iv) Evaluate the integral obtained in step III by using the methods discussed earlier.

Example 35 :

Evaluate $\int\frac{1}{a^2\sin^2x+b^2\cos^2x}\,dx$

Sol. Dividing the numerator and denominator of the given integrand by $\cos^2x$, we get

$$I = \int\frac{1}{a^2\sin^2x+b^2\cos^2x}dx = \int\frac{\sec^2x}{a^2\tan^2x+b^2}dx$$

Putting tan x = t and $\sec^2x\,dx = dt$, we get

$$I = \int\frac{dt}{a^2t^2+b^2} = \frac{1}{a^2}\int\frac{dt}{t^2+(b/a)^2} = \frac{1}{a^2}\frac{1}{b/a}\tan^{-1}\left(\frac{t}{b/a}\right)+C$$

$$\Rightarrow I = \frac{1}{ab}\tan^{-1}\left(\frac{at}{b}\right)+C = \frac{1}{ab}\tan^{-1}\left(\frac{a\tan x}{b}\right)+C$$

Example 36 :

Evaluate $\int\frac{1}{2-3\cos 2x}\,dx$

Sol. $I = \int\frac{1}{2-3\cos 2x}\,dx = \int\frac{1}{2-3(\cos^2x-\sin^2x)}dx$

$$= \int\frac{\sec^2x}{2\sec^2x-3+3\tan^2x}dx = \int\frac{\sec^2x}{5\tan^2x-1}dx$$

Putting tan x = t and $\sec^2 x\,dx$ =dt ,we get

$$\therefore I = \int\frac{dt}{5t^2-1} = \frac{1}{5}\int\frac{dt}{t^2-\left(\frac{1}{\sqrt{5}}\right)^2} = \frac{1}{5}.\frac{1}{2\left(\frac{1}{\sqrt{5}}\right)}\log\left|\frac{t-\frac{1}{\sqrt{5}}}{t+\frac{1}{\sqrt{5}}}\right|+C$$

$$= \frac{1}{2\sqrt{5}}\log\left|\frac{\sqrt{5}t-1}{\sqrt{5}t+1}\right|+C = \frac{1}{2\sqrt{5}}\log\left|\frac{\sqrt{5}\tan x-1}{\sqrt{5}\tan x+1}\right|+C$$

Integrals of the form

$$\int\frac{1}{a\sin x+b\cos x}dx,\ \int\frac{1}{a+b\sin x}dx,\int\frac{1}{a+b\cos x}dx,$$

$$\int\frac{1}{a\sin x+b\cos x+c}dx$$

To evaluate this type of integrals we use the following algorithm

(i) $x=\frac{2\tan x/2}{1+\tan^2 x/2}$, $\cos x=\frac{1-\tan^2 x/2}{1+\tan^2 x/2}$,

(ii) Replace $1+\tan^2\frac{x}{2}$ in the numerator by $\sec^2\frac{x}{2}$

(iii) Put $\tan\frac{x}{2}=t$ so that $\frac{1}{2}\sec^2\frac{x}{2}dx=dt$.

This substitution reduces the integral in the form

$$\int\frac{1}{at^2+bt+c}dt$$

(iv) Evaluate the integral in step III by using methods discussed earlier.

Example 37 :

Evaluate $\int\frac{1}{1+\sin x+\cos x}dx$

Sol. Putting $\sin x=\frac{2\tan x/2}{1+\tan^2 x/2}$ and $\cos x=\frac{1-\tan^2 x/2}{1+\tan^2 x/2}$

$$I=\int\frac{1}{1+\sin x+\cos x}dx$$

$$=\int\frac{1}{1+\frac{2\tan x/2}{1+\tan^2 x/2}+\frac{1-\tan^2 x/2}{1+\tan^2 x/2}}dx$$

$$\int\frac{1+\tan^2 x/2}{1+\tan^2 x/2+2\tan x/2+1-\tan^2 x/2}dx$$

$$=\int\frac{\sec^2 x/2}{2+2\tan x/2}dx$$

Putting $\tan\frac{x}{2}=t$ and $\frac{1}{2}\sec^2\frac{x}{2}dx=dt$ or,

$\sec^2\frac{x}{2}dx=2\,dt$, we get

$$I=\int\frac{2dt}{2+2t}=\int\frac{1}{t+1}dt=\log|t+1|+C=\log\left|\tan\frac{x}{2}+1\right|+C$$

Integrals of the form $\int\frac{1}{a\sin x+b\cos x}dx$, **:**

To evaluate this type of integrals, we substitute $a=r\cos\theta$, $b=r\sin\theta$ and so that

$$r=\sqrt{a^2+b^2},\ \theta=\tan^{-1}\left(\frac{b}{a}\right)$$

$\therefore a\sin x+b\cos x=r\cos\theta\sin x+r\sin\theta\cos x=r\sin(x+\theta)$

so $\int\frac{1}{a\sin x+b\cos x}dx$

$$=\frac{1}{r}\int\frac{1}{\sin(x+\theta)}dx=\frac{1}{r}\int\operatorname{cosec}(x+\theta)dx$$

$$=\frac{1}{r}\log\left|\tan\left(\frac{x}{2}+\frac{\theta}{2}\right)\right|+C$$

$$=\frac{1}{\sqrt{a^2+b^2}}\log\left|\tan\left(\frac{x}{2}+\frac{1}{2}\tan^{-1}\frac{b}{a}\right)\right|+C$$

Example 38 :

Evaluate $\int\frac{1}{\sqrt{3}\sin x+\cos x}dx$

Sol. Let $\sqrt{3}=r\sin\theta$ and $1=r\cos\theta$. Then

$$r=\sqrt{(\sqrt{3})^2+1^2}=2 \text{ and } \tan\theta=\frac{\sqrt{3}}{1}\Rightarrow\theta=\frac{\pi}{3}$$

$$\therefore\int\frac{1}{\sqrt{3}\sin x+\cos x}dx=\int\frac{1}{r\sin\theta\sin x+r\cos\theta\cos x}dx$$

$$\frac{1}{r}\int\frac{1}{\cos(x-\theta)}dx=\frac{1}{r}\int\sec(x-\theta)dx$$

$$=\frac{1}{r}\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}-\frac{\theta}{2}\right)\right|+C$$

$$=\frac{1}{2}\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}-\frac{\pi}{6}\right)\right|+C=\frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)\right|+C$$

Integrals of the form $\int\frac{a\sin x+b\cos x}{c\sin x+d\cos x}dx$,

To evaluate this type of integrals,we use the following algorithm

(i) Write Numerator = λ (Diff. of denominator) + μ (Denominator)
i.e. $a\sin x+b\cos x=\lambda(c\cos x-d\sin x)+\mu(c\sin x+d\cos x)$

(ii) Obtain the values of λ and μ by equating the coefficient of sinx and cosx on both the sides.

(iii) Replace numerator in the integrand by
$\lambda(c\cos x-d\sin x)+\mu(c\sin x+d\cos x)$ to
obtain $\int\frac{a\sin x+b\cos x}{c\sin x+d\cos x}dx$

$$= \lambda \int \frac{c\cos x - d\sin x}{c\sin x + d\cos x}dx + \mu \int \frac{c\sin x + d\cos x}{c\sin x + d\cos x}dx$$

$$= \lambda \log|c\sin x + d\cos x| + \mu x + C$$

Example 39 :

Evaluate $\int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}dx$

Sol. $I = \int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}dx$

Let 3 sinx + 2cos x

$$= \lambda . \frac{d}{dx}(3\cos x + 2\sin x) + \mu(3\cos x + 2\sin x)$$

$\Rightarrow$ 3 sin x + 2 cos x = λ (–3 sin x + 2cox) + μ (3cosx + 2sinx)

Comparing the coefficient of sin x and cos x on both sides, we get

$-3\lambda + 2\mu = 3$ and $2\lambda + 3\mu = 2 \Rightarrow \mu = \frac{12}{13}$ and $\lambda = -\frac{5}{13}$

$$\therefore \quad I = \int \frac{\mu(-3\sin x + 2\cos x) + \lambda(3\cos x + 2\sin x)}{3\cos x + 2\sin x}dx$$

$$= \lambda \int 1.dx + \mu \int \frac{-3\sin x + 2\cos x}{3\cos x + 2\sin x}dx$$

$$= \lambda x + \mu \int \frac{dt}{t}, \text{ where } t = 3\cos x + 2\sin x$$

$$= \lambda x + \mu \log|t| + C = \frac{-5}{13}x + \frac{12}{13}\log|3\cos x + 2\sin x| + C$$

Integrals of the form $\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}dx$:

To evaluate this type of integrals,we use the following algorithm

(i) Write Numerator = λ (Diff. of denominator) + μ (Denominator) + ν

i.e. a sin x + b cos x + c = λ (p cosx – q sinx) + μ (psin x + q cosx + r) + ν

(ii) Obtain the values of λ and μ by equating the coefficient of sinx and cosx and the constant terms on both the sides

(iii) Replace numerator in the integrand by λ (p cosx – q sin x) + μ (p sin x + qcosx+r) + ν to obtain

$$\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}dx = \lambda \int \frac{p\cos x - q\sin x}{p\sin x + q\cos x + r}dx$$

$$+ \mu \int \frac{p\sin x + q\cos x + r}{p\sin x + q\cos x + r} + \nu \int \frac{1}{p\sin x + q\cos x + r}dx$$

$$= \lambda \log|p\sin x + q\cos x + r| + \mu x + \nu \int \frac{1}{p\sin x + q\cos x + r}dx$$

(iv) Evaluate the integral on RHS in step III by using the method discussed earlier.

Example 40 :

Evaluate $\int \frac{3\cos x + 2}{\sin x + 2\cos x + 3}dx$

Sol. $I = \int \frac{3\cos x + 2}{\sin x + 2\cos x + 3}dx$

Let 3 cos x + 2 = λ (sinx + 2cox + 3) + μ (cosx – 2 sin x) + ν

Comparing the coefficients of sinx, cos x and constant term on both sides, we get

$\lambda - 2\mu = 0,\ 2\lambda + \mu = 3,\ 3\lambda + \nu = 2$

$$\Rightarrow \quad \lambda = \frac{6}{5},\ \mu = \frac{3}{5} \text{ and } \nu = -\frac{8}{5}$$

$$\therefore \quad I = \int \frac{\lambda(\sin x + 2\cos x + 3) + \mu(\cos x - 2\sin x) + \nu}{\sin x + 2\cos x + 3}dx$$

$\Rightarrow$ I

$$= \lambda \int dx + \mu \int \frac{\cos x - 2\sin x}{\sin x + 2\cos x + 3}dx + \nu \int \frac{1}{\sin x + 2\cos x + 3}dx$$

$\Rightarrow$ I = λ x + μ log | sinx + 2 cos x + 3 | + ν I_1, where

$$I_1 = \int \frac{1}{\sin x + 2\cos x + 3}dx$$

Putting $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}, \cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$, we get

$$I_1 = \int \frac{1}{\frac{2\tan x/2}{1+\tan^2 x/2} + \frac{2(1-\tan^2 x/2)}{1+\tan^2 x/2} + 3}dx$$

$$= \int \frac{1+\tan^2 x/2}{2\tan x/2 + 2 - 2\tan^2 x/2 + 3(1+\tan^2 x/2)}dx$$

$$= \int \frac{\sec^2 x/2}{\tan^2 x/2 + 2\tan x/2 + 5}dx$$

Putting $\tan\frac{x}{2} = t$ and $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

or, $\sec^2\frac{x}{2}\,dx = 2\,dt$, we get $I_1 = \int \frac{2dt}{t^2 + 2t + 5}$

$$= 2\int \frac{dt}{(t+1)^2 + 2^2} = \frac{2}{2}\tan^{-1}\left(\frac{t+1}{2}\right) = \tan^{-1}\left(\frac{\tan\frac{x}{2} + 1}{2}\right)$$

$$I = \lambda x + \mu \log|\sin x + 2\cos x + 3| + \nu \tan^{-1}\left(\frac{\tan\frac{x}{2} + 1}{2}\right) + C$$

where $\lambda = \frac{6}{5}, \mu = \frac{3}{5}$ and $\nu = -\frac{8}{5}$

Integration by parts:

It u and v are two functions of x, then

$$\int (u.v)\,dx = \left(u\int v\,dx\right) - \int\left(\frac{du}{dx}\right).\left(\int v\,dx\right)dx$$

i.e., integral of the product of two functions

= first x integral of second $-\int$ (derivative of first) x (integral of second)

Selection of first function: For applying this method we take x^n as the first function (if it is there) provided we know integral of the second. If logarithmic function, or inverse trigonometrical function is one of the products, then that should be taken as the first function.

Note: We can also choose the first function as the function which comes first in the word **ILATE**, where

I – stands for the inverse trigonometric function ($\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ etc.)

L – stands for the logarithmic function

A – stands for the algebraic functions

T – stands for the trigonometric functions

E – stands for the exponential functions

Short method: When one function is x^n, $n \in N$

$$\int x^n\, v dx = x^n \int v\,dx \;- F(x)$$

where F(x) is obtained by applying the rule (derivative of I) x (Integral of II) on every pervious obtained and writing them in alternate sign. By this method.

$$\int x^2 \cos x\,dx = x^2(\sin x) - (2x)(-\cos x) + 2(-\sin x)$$

$$= x^2\sin x + 2x\cos x - 2\sin x + c$$

Example 41 :

Evaluate : $\int \sqrt{x}e^{\sqrt{x}}dx$

Sol. Put $\sqrt{x} = t$ $\therefore$ $\frac{1}{2\sqrt{x}}dx = dt$ $\therefore I = 2\int t^2e^t dt$

$$= 2[t^2e^t - (2t)e^t + 2e^t] = (2x - 4\sqrt{x} + 4)\,e^{\sqrt{x}}$$

Example 42 :

Evaluate : $\int \tan^{-1}\sqrt{\left(\frac{1-x}{1+x}\right)}dx$

Sol. Put $x = \cos\theta$

$\therefore$ $\frac{1-\cos\theta}{1+\cos\theta} = \tan^2\frac{\theta}{2}$ and $dx = -\sin\theta\,d\theta$

$$I = \int \tan^{-1}[\tan(\theta/2)](-\sin\theta\,d\theta)]$$

$$= -\frac{1}{2}\int \theta\sin\theta d\theta = -\frac{1}{2}\int -\theta\cos\theta + \sin\theta]$$

$$= \frac{1}{2}[x\cos^{-1}x - \sqrt{(1-x^2)}]$$

Example 43 :

Evaluate $\int \sin\sqrt{x}\,dx$

Sol. Let $I = \int \sin\sqrt{x}\,dx$

Let $\sqrt{x} = t$. Then, $d(\sqrt{x}) = dt \Rightarrow \frac{1}{2\sqrt{x}}dx = dt$

$\Rightarrow dx = 2\sqrt{x}\,dt$

$$\therefore \int \sin\sqrt{x}\,dx = \int (\sin t)\,2t\,dt = 2\int \underset{I}{t}\,\underset{II}{\sin t}dt$$

$$= 2\left\{t(-\cos t) - \int 1.(-\cos t).dt\right\} = 2\left\{-t\cos t + \int \cos t\,dt\right\}$$

$$= 2\left[-t\cos t + \sin t\right] + C = 2[-\sqrt{x}\cos\sqrt{x} + \sin\sqrt{x}] + C$$

Example 44 :

Evaluate $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}dx$

Sol. $I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}dx$

$$= \int \frac{\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}}dx \quad \left[\because \sin^{-1}\theta + \cos^{-1}\theta = \frac{\pi}{2}\right]$$

$$= \frac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x} - \frac{\pi}{2}\right)dx = \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\,dx - \int 1.dx$$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x} - x + C \qquad ....(1)$$

Putting $x = \sin^2\theta$ and $dx = 2\sin\theta\cos\theta\,d\theta = \sin 2\theta\,d\theta$, we get

$$\int \sin^{-1}\sqrt{x}\,dx$$

$$= \int \underset{I}{\theta}\,\underset{II}{\sin 2\theta}\,d\theta = -\theta\frac{\cos 2\theta}{2} + \int\frac{1}{2}\cos 2\theta\,d\theta$$

$$= -\frac{\theta}{2}\cos 2\theta + \frac{1}{4}\sin 2\theta$$

$$= -\frac{1}{2}\theta(1 - 2\sin^2\theta) + \frac{1}{2}\sin\theta\sqrt{1-\sin^2\theta}$$

$$= -\frac{1}{2}\sin^{-1}\sqrt{x}(1-2x) + \frac{1}{2}\sqrt{x}\sqrt{1-x} \qquad ....(2)$$

From (1) and (2), we have

$$I = -\frac{4}{\pi}\left\{-\frac{1}{2}(1-2x)\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x-x^2}\right\} - x + C$$

$$\Rightarrow I = \frac{2}{\pi}\left\{\sqrt{x-x^2} - (1-2x)\sin^{-1}\sqrt{x}\right\} - x + C$$

Two Important integrals :

(i) $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

$= \frac{e^{ax}}{a^2+b^2}\sin(bx - \tan^{-1} b/a) + c$

(ii) $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos(bx - \tan^{-1} b/a) + c$

Example 45 :

Evaluate $\int e^x \cos^2 x\, dx$

Sol. $I = \int e^x \cos^2 x\, dx$

$\Rightarrow I = \int e^x\left(\frac{1+\cos 2x}{2}\right)dx = \frac{1}{2}\int e^x(1+\cos 2x)\,dx$

$\Rightarrow I = \frac{1}{2}\int e^x dx + \frac{1}{2}\int e^x \cos 2x\, dx$

$I = \frac{1}{2}e^x + \frac{1}{2}I_1 + C$ ....(1)

Where $I_1 = \int e^x \cos 2x\, dx$. Now, $I_1 = \int \underset{I}{e^x}\, \underset{II}{\cos 2x}\, dx$

$\Rightarrow I_1 = e^x \frac{\sin 2x}{2} - \int e^x \frac{\sin 2x}{2} dx$ [Integrating by parts]

$\Rightarrow I_1 = \frac{1}{2}e^x \sin 2x - \frac{1}{2}\int \underset{I}{e^x}\, \underset{II}{\sin 2x}\, dx$

$\Rightarrow I_1 = \frac{1}{2}e^x \sin 2x - \frac{1}{2}\left[-e^x\frac{\cos 2x}{2} - \int e^x\left(\frac{-\cos 2x}{2}\right)dx\right]$

$\Rightarrow I_1 = \frac{1}{2}e^x \sin 2x + \frac{1}{4}e^x \cos 2x - \frac{1}{4}\int e^x \cos 2x\, dx$

$\Rightarrow I_1 = \frac{1}{2}e^x \sin 2x + \frac{1}{4}e^x \cos 2x - \frac{1}{4}I_1$

$\Rightarrow I_1 + \frac{1}{4}I_1 = \frac{1}{4}e^x(\cos 2x + 2\sin 2x)$

$\Rightarrow \frac{5}{4}I_1 = \frac{1}{4}e^x(\cos 2x + 2\sin 2x)$

$\Rightarrow I_1 = \frac{e^x}{5}(\cos 2x + 2\sin 2x)$ ...(2)

$\therefore I = \frac{1}{2}e^x + \frac{e^x}{10}(\cos 2x + 2\sin 2x) + C$ [Using (1) & (2)]

Two standard forms of an Integral :

(i) $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$

$\Rightarrow \int e^x [f(x) + f'(x)]\, dx = \int e^x f(x)dx + \int e^x f'(x)\, dx$

$= e^x f(x) - \int e^x f'(x)\, dx + \int e^x f'(x)$ (on integrating by parts)

$= e^x f(x) + c$

(ii) $\int [xf'(x) + f(x)]dx = x f(x) + c$

$\int [x f'(x) + f(x)]\, dx = \int x f'(x)\, dx + \int f(x)\, dx$

$= xf(x) - \int f(x)\, dx + \int f(x)\, dx = xf(x) + c$

Example 46 :

Evaluate $\int [\log\log x + (\log x)^{-2}]dx$

Sol. Put $\log x = t$

$\therefore$ $x = e^t$ and $dx = e^t\, dt$

$\therefore \; I = \int e^t\left[\log t + \frac{1}{t^2}\right]dt$

$= e^t\left(\log t + \frac{1}{t}\right) - e^t\left(\frac{1}{t} + \frac{-1}{t^2}\right)$

We have added and subtracted $e^t \cdot \frac{1}{t}$ to give the form

$e^x [f(x) + f'(x)]$

$\therefore I = e^t \log t - e^t \cdot \frac{1}{t} = \left[x\log(\log x) - \frac{x}{\log x}\right]$

Example 47 :

Evaluate $\int \frac{e^x(1+\sin x)}{1+\cos x}dx$

Sol. $I = \int e^x\left[\tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2}\right] = e^x \tan\frac{x}{2}$

$\because$ $1 + \cos x = 2\cos^2\frac{x}{2}$ and $\sin x = 2\sin\frac{x}{2}\frac{\cos x}{2}$

Integrations of rational algebraic functions by using partial fraction:

If f (x) and g (x) are two polynomials, then $\frac{f(x)}{g(x)}$ defines a rational algebraic function or a rational function of x.

If degree of f (x) < degree of g (x), then $\frac{f(x)}{g(x)}$ is called a proper rational function.

If degree of f (x) ≥ degree of g (x) then $\frac{f(x)}{g(x)}$ is called an improper rational function.

If $\frac{f(x)}{g(x)}$ is an improper rational function, we divide f (x) by g (x) so that the rational function $\frac{f(x)}{g(x)}$ is expressed in the form f (x) + $\frac{\psi(x)}{g(x)}$ where ϕ (x) and $\psi(x)$ are polynomials such that the degree of $\psi(x)$ is less than that of g (x). Thus $\frac{f(x)}{g(x)}$ is expressible as the sum of a polynomial and a proper rational function.

1. When denominator is expressible as the product of non-repeating linear factors.
Let $g(x) = (x-a_1)(x-a_2)....(x-a_n)$. Then, we assume that

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + ... + \frac{A_n}{x-a_n}$$

where A_1, A_2........, A_n are constants and can be determined by equating the numerator on RHS to the numerator on LHS and then substituting $x = a_1, a_2,........a_n$

2. When the denominator g (x) is expressible as the product of the linear factors such that some of them are repeating
Let $g(x) = (x-a)^k (x-a_1)(x-a_2).......(x-a_r)$.
Then we assume that

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3}$$

$$+ ... + \frac{A_k}{(x-a)^k} + \frac{B_1}{x-a_1} + \frac{B_2}{x-a_2} + ... + \frac{B_r}{(x-a_r)}$$

i.e. corresponding to non-repeating factors we assume as in Case I and for each repeating factor $(x-a)^k$, we assume partial fractions.

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + ... + \frac{A_k}{(x-a)^{k'}}$$

where A_1, A_2 ,........, A_k are constants.
Now to determine constants we equate numerators on both sides. Some of the constants are determined by substitution as in case I and remaining are obtained by comparing coefficients of equal powers of x on both sides.

3. When some of the factors of denominator g (x) are quadratic but non-repeating. Corresponding to each quadratic factor $ax^2 + bx + c$, we assume partial fraction of the type

$\frac{Ax+B}{ax^2+bx+c}$ where A and B are constants to be determined by comparing coefficient of similar powers of x in the numerator of both sides. In practice it is advisable to assume Partial fractions of the type $\frac{A(2ax+b)}{ax^2+bx+c} + \frac{B}{ax^2+bx+c}$

When some of the factors of the denominator g (x) are quadratic and repeating.
For every quadratic repeating factor of the type $(ax^2 + bx + c)^k$, we assume 2k partial fractions of the form

$$\left\{\frac{A_0(2ax+b)}{ax^2+bx+c} + \frac{A_1}{ax^2+bx+c}\right\}$$

$$+\left\{\frac{A_1(2ax+b)}{(ax^2+bx+c)^2} + \frac{A_2}{(ax^2+bx+c^2)}\right\}$$

$$+ ... + \left\{\frac{A_{2k-1}(2ax+b)}{(ax^2+bx+c)^k} + \frac{A_{2k}}{(ax^2+bx+c)^k}\right\}$$

Example 48 :

Evaluate $\int \frac{x-1}{(x+1)(x-2)} dx$

Sol. Let $\frac{x-1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$ ..(i)

$\Rightarrow x - 1 = A(x-2) + B(x+1)$ ..(ii)

Putting $x - 2 = 0$ or, $x = 2$ in (ii), we get
$1 = 3B \Rightarrow B = 1/3$
Putting $x + 1 = 0$ or $x = -1$ in (ii) we get
$-2 = -3A \Rightarrow A = 2/3$
Substituting the values of A and B in (i), we get

$$\frac{x-1}{(x+1)(x-2)} = \frac{2}{3}.\frac{1}{x+1} + \frac{1}{3}.\frac{1}{x-2}$$

$$\therefore \int \frac{x-1}{(x+1)(x-2)} dx$$

$$= \frac{2}{3}\int \frac{1}{x+1} dx + \frac{1}{3}\int \frac{1}{x-2} dx$$

$$= \frac{2}{3}\log|x+1| + \frac{1}{3}\log|x-2| + C$$

Example 49 :

Evaluate $\int \frac{x^2}{x^2-1} dx$

Sol. Given integral

$$I = \int\left(1 + \frac{1}{x^2-1}\right) dx = \int dx + \int \frac{dx}{(x-1)(x+1)}$$

$$= x + \frac{1}{2}\int\left(\frac{1}{x-1} - \frac{x}{x+1}\right) dx$$

$$= x + \frac{1}{2}\log\left(\frac{x-1}{x+1}\right) + c$$

Example 50 :

Evaluate $\int \frac{8}{(x+2)(x^2+4)}dx$

Sol. Let $\frac{8}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$ ...(i)

Then, $8 = A(x^2+4) + (Bx+C)(x+2)$ ...(ii)

Putting $x + 2 = 0$ i.e. $x = -2$ in (ii), we get

$8 = 8A \Rightarrow A = 1$

Putting $x = 0$ and 1 respectively in (ii), we get

$8 = 4A + 2C$ and $8 = 5A + 3B + 3C$

Solving these equation, we obtain

$A = 1, C = 2, B = -1$

Substituting the values of A, B and C in (i) , we obtain

$$\frac{8}{(x+2)(x^2+4)} = \frac{1}{x+2} + \frac{-x+2}{x^2+4}$$

$$\therefore \int \frac{8}{(x+2)(x^2+4)}dx = \int \frac{1}{(x+2)}dx + \int \frac{-x+2}{x^2+4}dx$$

$$= \int \frac{1}{(x+2)}dx - \int \frac{x}{x^2+4}dx + 2\int \frac{1}{x^2+4}dx$$

$$= \log|x+2| - \frac{1}{2}\int \frac{1}{t}dt + 2.\frac{1}{2}\tan^{-1}\frac{x}{2} + C,$$

where $t = x^2 + 4$

$$= \log|x+2| - \frac{1}{2}\log t + \tan^{-1}\frac{x}{2} + C$$

$$= \log|x+2| - \frac{1}{2}\log(x^2+4) + \tan^{-1}\frac{x}{2} + C$$

Integrals of the form

$$\int \frac{x^2+1}{x^4+\lambda x^2+1}dx, \int \frac{x^2-1}{x^4+\lambda x^2+1}dx, \int \frac{1}{x^4+\lambda x^2+1}dx,$$

where ª:Č:R.

To evaluate this type of integrals, we use the following algorithm

(i) Divide numerator and denominator by x^2

(ii) Express the denominator of integrand in the form $\left(x+\frac{1}{x}\right)^2 \pm k^2$

(iii) Introduce $d\left(x+\frac{1}{x}\right)$ or $d\left(x-\frac{1}{x}\right)$ or both in the numerator.

(iv) Substitute $x+\frac{1}{x} = t$ or $x-\frac{1}{x} = t$ as the case may be

This substitution reduces the integral in one of the following forms $\int \frac{1}{x^2+a^2}dx, \int \frac{1}{x^2-a^2}dx$

(v) Use the appropriate formula.

Example 51 :

Evaluate $\int \frac{1}{x^4+1}dx$

Sol. $I = \int \frac{1}{x^4+1}dx \Rightarrow I = \int \frac{\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$

$$\Rightarrow I = \frac{1}{2}\int \frac{\frac{2}{x^2}}{x^2+\frac{1}{x^2}}dx \Rightarrow I = \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} - \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$$

$$\Rightarrow I = \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$$

$$\Rightarrow I = \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}dx$$

Putting $x - \frac{1}{x} = u$ in 1st integral and $x + \frac{1}{x} = v$ in 2nd integral, we get $I = \frac{1}{2}\int \frac{du}{u^2+(\sqrt{2})^2} - \frac{1}{2}\int \frac{dv}{t^2-(\sqrt{2})^2}$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2}.\frac{1}{2\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right) + \frac{1}{2}.\frac{1}{2\sqrt{2}}\log\left|\frac{x+1/x-\sqrt{2}}{x+1/x+\sqrt{2}}\right| + C$$

Example 52 :

Evaluate $\int\left\{\sqrt{\tan\theta}+\sqrt{\cot\theta}\right\}d\theta$

Sol. $I = \int\left\{\sqrt{\tan\theta}+\sqrt{\cot\theta}\right\}d\theta = \int\left\{\sqrt{\tan\theta}+\frac{1}{\sqrt{\tan\theta}}\right\}d\theta$

$$= \int \frac{\tan\theta+1}{\sqrt{\tan\theta}}d\theta$$

Let $\tan\theta = x^2$. Then, $d(\tan\theta) = d(x^2) \Rightarrow \sec^2\theta\, d\theta = 2x\,dx$

$$\Rightarrow d\theta = \frac{2xdx}{\sec^2\theta} = \frac{2xdx}{1+\tan^2\theta} = \frac{2xdx}{1+x^4}$$

$$= \int \frac{x^2+1}{\sqrt{x^2}}.\frac{2xdx}{1+x^4} = 2\int \frac{x^2+1}{x^4+1} = 2\int \frac{1+1/x^2}{x^2+1/x^2}dx$$

$$\Rightarrow 2\int \frac{1+1/x^2}{(x-1/x)^2+2}dx = 2\int \frac{1+1/x^2}{(x-1/x)^2+(\sqrt{2})^2}dx$$

$$\Rightarrow 2\int \frac{du}{u^2+(\sqrt{2})^2} = \frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+C, \text{ where } x-\frac{1}{x}=u.$$

$$\Rightarrow \quad I=\sqrt{2}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right)+C \Rightarrow I = \sqrt{2}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right)+C$$

$$\Rightarrow I=\sqrt{2}\tan^{-1}\left(\frac{\tan\theta-1}{\sqrt{2}\tan\theta}\right)+C$$

Integration of some special irrational algebraic functions

In this section, we shall discuss four integrals of the form

$\int \frac{\phi(x)}{p\sqrt{Q}}dx$, where P and Q are polynomial functions of x.

Integrals of the form $\int \frac{\phi(x)}{p\sqrt{Q}}dx$, where p and Q both are linear. Functions of x

To evaluate this type of integrals we put $Q = t^2$ i.e., to evaluate integrals of the form $\int \frac{1}{(ax+b)\sqrt{cx+d}}dx$,

put $cx + d = t^2$.

Integrals of the form $\int \frac{\phi(x)}{p\sqrt{Q}}dx$ where p is a Quadratic expression and Q is a linear expression.

To evaluate this type of integrals we put $Q = t^2$ i.e., to evaluate integrals of the form $\int \frac{1}{(ax^2+bx+c)\sqrt{px+q}}dx$,

put $px + q = t^2$.

Example 53 :

Evaluate $\int \frac{1}{(x-3)\sqrt{x+1}}dx$

Sol. Let $I = \int \frac{1}{(x-3)\sqrt{x+1}}dx$

Here, P and Q both are linear, so we put $Q = t^2$, i.e. $x+1 = t^2$ and $dx = 2t\,dt$.

$$\therefore \quad I = \int \frac{1}{(t^2-1-3)}\frac{2t}{\sqrt{t^2}}dt$$

$$\Rightarrow \quad I = 2\int \frac{dt}{t^2-2^2} = 2.\frac{1}{2(2)}\log\left|\frac{t-2}{t+2}\right|+C$$

$$\Rightarrow \quad I=\frac{1}{2}\log\left|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right|+C$$

Integrals of the form $\int \frac{\phi(x)}{p\sqrt{Q}}dx$ where p is a linear expression. and Q is a quadratic expression

To evaluate this type of integrals we put P = 1/t i.e., to evaluate integrals of the form $\int \frac{1}{(ax+b)\sqrt{px^2+qx+r}}dx$,

put $ax + b = \frac{1}{t}$

Example 54 :

Evaluate $\int \frac{1}{(x+1)\sqrt{(x^2-1)}}dx$

Sol. Let $I = \int \frac{1}{(x+1)\sqrt{(x^2-1)}}dx$

Putting $x + 1 = \frac{1}{t}$ and $dx = -\frac{1}{t^2}dt$, we get

$$\therefore \quad I = \int \frac{1}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2-1}}.\left(-\frac{1}{t^2}\right)dt$$

$$= -\int \frac{dt}{\sqrt{1-2t}} = -\int (1-2t)^{-1/2}\,dt$$

$$= -\frac{(1-2t)^{1/2}}{(-2)\left(\frac{1}{2}\right)}+C=\sqrt{1-2t} = \sqrt{1-\frac{2}{x+1}}+C=\sqrt{\frac{x-1}{x+1}}+C$$

Integrals of the form $\int \frac{\phi(x)}{p\sqrt{Q}}dx$ where p and Q both are pure quadratic expression in x i.e. $p = ax^2 + b$ and $Q = cx^2 + d$:

To evaluate this type of integrals we put x = 1/t and then $c + dt^2 = u^2$ i.e., to evaluate integrals of the form

$\int \frac{1}{(ax^2+b)\sqrt{cx^2+d}}dx$, we put $x = \frac{1}{t}$ to

obtain $\int \frac{-tdt}{(a+bt^2)\sqrt{c+dt^2}}$ and then $c + dt^2 = u^2$.

SOME INTEGRATES OF DIFFERENT EXPRESSIONS OF e^x:

(i) $\int \frac{ae^x}{b+ce^x}dx$ [Put $e^x = t$]

(ii) $\int \frac{1}{1+e^x}\,dx$ [multiply and divide by e^{-x} and put $e^{-x} = t$]

(iii) $\int \frac{1}{1-e^x}\,dx$ [multiply and divide by e^{-x} and put $e^{-x} = t$]

(iv) $\int \frac{1}{e^x - e^{-x}}\,dx$ [multiply and divide by e^x]

(v) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$ $\left[\frac{f'(x)}{f(x)}\text{form}\right]$

(vi) $\int \frac{e^x + 1}{e^x - 1}\,dx$ [multiply and divide by $e^{-x/2}$]

(vii) $\int \frac{1}{(1+e^x)(1-e^{-x})\,dx}$ [multiply and divide by e^x & put $e^x = t$]

(viii) $\int \frac{1}{\sqrt{1-e^x}}\,dx$ [multiply and divide by $e^{-x/2}$]

(ix) $\int \frac{1}{\sqrt{1+e^x}}\,dx$ [multiply and divide by $e^{-x/2}$]

(x) $\int \frac{1}{\sqrt{e^x - 1}}\,dx$ [multiply and divide by $e^{-x/2}$]

(xi) $\int \frac{1}{\sqrt{2e^x - 1}}\,dx$ [multiply and divide by $\sqrt{2}\, e^{-x/2}$]

(xii) $\int \sqrt{1-e^x}\,dx$ [integrand = $(1 - e^x)/\sqrt{1-e^x}$]

(xiii) $\int \sqrt{1+e^x}\,dx$ [integrand $= (1 + e^x)/\sqrt{1+e^x}$]

(xiv) $\int \sqrt{e^x - 1}\,dx$ [integrand $= (e^x - 1)/\sqrt{e^x - 1}$]

(xv) $\int \sqrt{\frac{e^x + a}{e^x - a}}\,dx$ [integrand $= (e^x + a)/\sqrt{e^{2x} - a^2}$

Example 55 :

Evaluate $\int \frac{1}{e^x - 1}\,dx$

Sol. Here $I = \int \frac{1}{e^x - 1}\,dx \Rightarrow \int \frac{e^{-x}}{1 - e^{-x}}\,dx = \log(1 - e^{-x}) + C$

TRY IT YOURSELF-1

Evaluate :

Q.1 $\int \frac{(x^2 + \sin^2 x)\sec^2 x}{1+x^2}\,dx$ **Q.2** $\int \frac{x^2 \tan^{-1} x^3}{1+x^6}\,dx$

Q.3 $\int \frac{e^x(1+x)}{\cos(xe^x)}\,dx$ **Q.4** $\int \frac{e^x dx}{\sqrt{e^{2x} - 1}}$

Q.5 $\int \frac{5x+4}{\sqrt{x^2 + 2x + 5}}\,dx$ **Q.6** $\int \sin(\ln x)\,dx$

Q.7 $\int \sin(\ln x) + \cos(\ln x)\,dx$ **Q.8** $\int \left(\ln(\ln x) + \frac{1}{\ln^2 x}\right) dx$

Q.9 $\int \frac{x^7}{(1-x^2)^5}\,dx$ **Q.10** $\int \frac{dx}{4 - 5\sin^2 x}$

Q.11 $\int \frac{dx}{(3\sin x - 4\cos x)^2}$ **Q.12** $\int \frac{dx}{(x+2)\sqrt{x+1}}$

Q.13 (i) $\int \frac{\sqrt{x}}{x+1}\,dx$ (ii) $\int \frac{dx}{e^x + e^{-x}}$ **Q.14** $\int \frac{\sin 2x}{a^2 + b^2 \sin^2 x}\,dx$

ANSWERS

(1) $\tan x - \tan^{-1} x + c$ **(2)** $\frac{1}{6}(\tan^{-1} x^3)^2 + c$

(3) $\ln(\sec(xe^x) + \tan(xe^x)) + c$ **(4)** $\ln(e^x + \sqrt{e^{2x} - 1}) + c$

(5) $5\sqrt{x^2 + 2x + 5} - \ln|x + 1 + \sqrt{x^2 + 2x + 5}| + C$

(6) $\frac{1}{2}x(\sin(\ln x) - \cos(\ln x)) + c$

(7) $x\sin(\ln x) + c$ **(8)** $x\left[\ln(\ln x) - \frac{1}{\ln x}\right] + c$

(9) $\frac{1}{8}\frac{1}{(x^{-2} - 1)^4} + c$ **(10)** $-\frac{1}{4}\ln\left(\frac{2\cot x - 1}{2\cot x + 1}\right) + c$

(11) $-\frac{1}{3}\frac{1}{(3\tan x - 4)} + c$ **(12)** $2\tan^{-1}(\sqrt{x+1}) + c$

(13) (i) $2\sqrt{x} - 2\tan^{-1}\sqrt{x} + C$ (ii) $\tan^{-1}(e^x) + C$

(14) $\frac{1}{ab}\tan^{-1}\left(\frac{b\sin^2 x}{a}\right) + c$

DEFINITE INTEGRAL

If $\frac{d}{dx}[f(x)] = f(x)$ and a and b, are two values independent of variable x, then $\int_a^b \phi(x)dx = [f(x)]_a^b = f(b) - f(a)$

is called Definite Integral of f (x) within limits a and b. Here a is called the lower limit and b is called the upper limit of the integral. The interval [a,b] is known as range of integration. It should be noted that every definite integral has a unique value.

Example 56 :

Evaluate $\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}}\, dx$

Sol. $\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}}\, dx = \int_0^1 \frac{\sqrt{1+x}-\sqrt{x}}{(\sqrt{1+x}+\sqrt{x})(\sqrt{1+x}-\sqrt{x})}\, dx$

$= \int_0^1 (\sqrt{1+x}-\sqrt{x})\, dx = \left[\frac{2}{3}(1+x)^{3/2} - \frac{2}{3}x^{3/2}\right]_0^1$

$= \left[\frac{2}{3}(1+1)^{3/2} - \frac{2}{3}(1)^{3/2}\right] - \left[\frac{2}{3}(1+0)^{3/2} - \frac{2}{3}(0)^{3/2}\right]$

$= \frac{2}{3}[2^{3/2}-1] - \frac{2}{3}[1-0] = \frac{2}{3}[2\sqrt{2}-2] = \frac{4}{3}[\sqrt{2}-1]$

Example 57 :

Evaluate $\int_0^{\pi/4} \sin 3x \sin 2x\, dx$

Sol. We have $\int_0^{\pi/4} \sin 3x \sin 2x\, dx$

$= \frac{1}{2}\int_0^{\pi/4} (2\sin 3x \sin 2x)\, dx = \frac{1}{2}\int_0^{\pi/4} (\cos x - \cos 5x)\, dx$

$= \frac{1}{2}\left[\sin x - \frac{\sin 5x}{5}\right]_0^{\pi/4}$

$= \frac{1}{2}\left[\left(\sin\frac{\pi}{4} - \frac{\sin\frac{5\pi}{4}}{5}\right) - \left(\sin 0 - \frac{\sin 0}{5}\right)\right]$

$= \frac{1}{2}\left[\frac{1}{\sqrt{2}} + \frac{1}{(\sqrt{2})5}\right] = \frac{1}{2(5\sqrt{2})} = \frac{3\sqrt{2}}{10}$

Example 58 :

Evaluate $\int_0^{\pi} \sin^3 x\, dx$

Sol. We have $\int_0^{\pi} \sin^3 x\, dx = \int_0^{\pi} \frac{3\sin x - \sin 3x}{4} dx$

$[\because \sin 3x = 3\sin x - 4\sin^3 x]$

$= \frac{1}{4}\int_0^{\pi} (3\sin x - \sin 3x)\, dx = \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right]_0^{\pi}$

$= \frac{1}{4}\left[\left(-3\cos\pi + \frac{\cos 3\pi}{3}\right) - \left(-3\cos 0 + \frac{\cos 0}{3}\right)\right]$

$= \frac{1}{4}\left[\left(3 - \frac{1}{3}\right) - \left(-3 + \frac{1}{3}\right)\right] = \frac{4}{3}$

Example 59 :

Evaluate $\int_0^4 \frac{1}{\sqrt{x^2+2x+3}} dx$

Sol. We have

$\int_0^4 \frac{1}{\sqrt{x^2+2x+3}} dx = \int_0^4 \frac{1}{\sqrt{(x^2+1)^2 + (\sqrt{2})^2}} dx$

$= \left[\log\left|x+1+\sqrt{(x+1)^2+(\sqrt{2})^2}\right|\right]_0^4$

$= \left[\log\left|x+1+\sqrt{x^2+2x+3}\right|\right]_0^4$

$= \log(5+\sqrt{16+8+3}) - \log(1+\sqrt{3})$

$= \log(5+3\sqrt{3}) - \log(1+\sqrt{3}) = \log\left(\frac{5+3\sqrt{3}}{1+\sqrt{3}}\right)$

Example 60 :

Evaluate $\int_0^{2\pi} e^x \sin\left(\frac{\pi}{4}+\frac{x}{2}\right) dx$

Sol. Let $I = \int_0^{2\pi} \underset{II}{e^x} \underset{I}{\sin}\left(\frac{\pi}{4}+\frac{x}{2}\right) dx$

On integrating by parts, we get

$I = \left[\sin\left(\frac{\pi}{4}+\frac{x}{2}\right).e^x\right]_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} \underset{I}{e^x} \underset{II}{\cos}\left(\frac{\pi}{4}+\frac{x}{2}\right) dx$

$$\Rightarrow I = \left[\sin\frac{5\pi}{4}e^{2\pi} - \sin\frac{\pi}{4}\right]$$

$$-\frac{1}{2}\left[\left\{e^x\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} e^x \sin\left(\frac{\pi}{4}+\frac{x}{2}\right)dx\right]$$

$$\Rightarrow \; I = \left(-\frac{e^{2\pi}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \frac{1}{2}\left[\left(-\frac{e^{2\pi}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) + \frac{1}{2}I\right]$$

$$\Rightarrow \; I = -\left(\frac{e^{2\pi}+1}{\sqrt{2}}\right) + \left(\frac{e^{2\pi}+1}{2\sqrt{2}}\right) - \frac{1}{4}I$$

$$I = \; I + \frac{1}{4}I = \frac{e^{2\pi}+1}{2\sqrt{2}}(1-2)$$

$$\Rightarrow \; \frac{5I}{4} = -\frac{e^{2\pi}+1}{2\sqrt{2}} \quad \Rightarrow I = -\frac{\sqrt{2}}{5}\left(e^{2\pi}+1\right)$$

Example 61 :

Evaluate $\displaystyle\int_0^{1/\sqrt{2}} \frac{\sin^{-1}x}{(1-x^2)^{3/2}}dx$

Sol. Let $\sin^{-1} x = \theta$ or, $x = \sin\theta$. Then, $dx = d(\sin\theta) = \cos\theta \, d\theta$.
Now, $x = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

$$\& \; x = \frac{1}{\sqrt{2}} \Rightarrow \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4} \; \therefore \; \int_0^{1/\sqrt{2}} \frac{\sin^{-1}x}{(1-x^2)^{3/2}}dx$$

$$= \int_0^{\pi/4} \frac{\theta}{\cos^3\theta}\cos\theta \, d\theta = \int_0^{\pi/4} \underset{\text{I}}{\theta}\,\underset{\text{II}}{\sec^2\theta}\, d\theta$$

$$= \left[\theta\tan\theta\right]_0^{\pi/4} - \int_0^{\pi/4} 1.\tan\theta \, d\theta = \left[\theta\tan\theta\right]_0^{\pi/4} + \left[\log\cos\theta\right]_0^{\pi/4}$$

$$= \left(\frac{\pi}{4} - 0\right) + \left(\log\left(\frac{1}{\sqrt{2}}\right) - \log 1\right) = \frac{\pi}{4} - \frac{1}{2}\log 2$$

Example 62 :

Evaluate $\displaystyle\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}dx$

Sol. Let $I = \displaystyle\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}dx$

Let $\cos x = t$ and $-\sin x \, dx = dt$.

Now, $x = 0 \Rightarrow t = \cos 0 = 1$ and $x = \dfrac{\pi}{2} \Rightarrow t = \cos\dfrac{\pi}{2} = 0$

$$\therefore \; I = \int_1^0 \frac{\sin x}{1+t^2}\left(\frac{-dt}{\sin x}\right) = -\int_1^0 \frac{dt}{1+t^2}$$

$$= -\left[\tan^{-1} t\right]_1^0 = -[\tan^{-1}0 - \tan^{-1}] = -\left[0 - \frac{\pi}{4}\right] = \frac{\pi}{4}$$

PROPERTIES OF DEFINITE INTEGRAL

[P- 1] $\displaystyle\int_a^b f(x)dx = \int_a^b f(t)dt$

i.e. the value of a definite integral remains unchanged if its variable is placed by any other symbol.

[P- 2] $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx$

i.e. the interchange of limits of a definite integral changes only its sign.

[P- 3] $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where $a < c < b$.

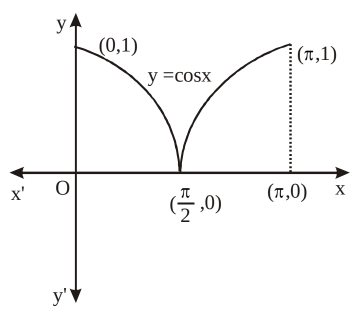
or $\displaystyle\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \ldots\ldots \int_{c_n}^{b} f(x)dx$

where $a < c_1 < c_2 < \ldots\ldots c_n < b$
Generally this property is used when the integrand has two or more rules in the integration interval.

Example 63 :

Evaluate $\displaystyle\int_0^{\pi} |\cos x| \, dx$

Sol. We have $|\cos x| = \begin{cases} \cos x \text{ when} & 0 \le x \le \dfrac{\pi}{2} \\ -\cos x \text{ when} & \dfrac{\pi}{2} \le x \le \pi \end{cases}$



$$\therefore \; \int_0^{\pi} |\cos x| \, dx = \int_0^{\pi/2} |\cos x| \, dx + \int_{\pi/2}^{\pi} |\cos x| \, dx$$

$$= \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} (-\cos x)\,dx$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi} = 1 + 1 = 2$$

Example 64 :

Evaluate $\int_{1/e}^{e} |\log_e x|\,dx$

Sol. We have

$$|\log_e x| = \begin{cases} -\log_e x & , \text{ if } 1/e < x < 1 \\ \log_e x & , \text{ if } 1 < x < e \end{cases}$$

$$\therefore\ I = \int_{1/e}^{e} |\log_e x|\,dx = \int_{1/e}^{1} -\log_e x + \int_1^e \log_e x\,dx$$

$$= -\int_{1/e}^{e} \log_e x\,dx + \int_1^e \log_e x\,dx$$

$$= -\left[x\left(\log_e x - 1\right)\right]_{1/e}^{1} + \left[x(\log_e x - 1)\right]_1^e$$

$$\left[\because \int \log_e x\,dx = x\left(\log_e x - 1\right)\right]$$

$$= -\left[1(0-1) - \frac{1}{e}(-1-1)\right] + \left[e(1-1) - 1(0-1)\right]$$

$$= -\left[-1 + \frac{2}{e}\right] + [0+1] = 2 - \frac{2}{e}$$

Example 65 :

Evaluate $\int_0^3 [x]\,dx$

Sol. $\int_0^3 [x]\,dx = \int_0^1 [x]\,dx + \int_1^2 [x]\,dx + \int_2^3 [x]\,dx = \int_0^1 0\,dx + \int_1^2 1.dx + \int_2^3 2.dx$

$$= 0 + [x]_1^2 + [2x]_2^3 = (2-1) + (6-4) = 3$$

Example 66 :

Evaluate $\int_0^{\sqrt{3}} \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$

Sol.

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \begin{cases} 2\tan^{-1} x & , \text{ if } -1 < x < 1 \\ -\pi + 2\tan^{-1} x & , \text{ if } x > 1 \\ \pi + 2\tan^{-1} x & , \text{ if } x < -1 \end{cases}$$

$$\therefore \int_0^{\sqrt{3}} \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx + \int_1^{\sqrt{3}} \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$$

$$= \int_0^1 2\tan^{-1} x\,dx + \int_1^{\sqrt{3}} (-\pi + 2\tan^{-1} x)\,dx$$

$$= \int_0^1 2\tan^{-1} x\,dx + \int_1^{\sqrt{3}} -\pi\,dx + \int_1^{\sqrt{3}} 2\tan^{-1} x\,dx$$

$$= \left\{\int_0^1 2\tan^{-1} x\,dx + \int_1^{\sqrt{3}} 2\tan^{-1} x\,dx\right\} - \pi \int_1^{\sqrt{3}} 1.dx$$

$$= 2\int_0^{\sqrt{3}} \tan^{-1} x\,dx - \pi \int_1^{\sqrt{3}} 1.dx$$

$$= 2\left[\{x\tan^{-1} x\}_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x}{1+x^2}\,dx\right] - \pi[x]_1^{\sqrt{3}}$$

$$= 2\left[\left\{\sqrt{3}\tan^{-1}\sqrt{3} - 0\right\} - \frac{1}{2}\left\{\log(1+x^2)\right\}_0^{\sqrt{3}}\right] - \pi(\sqrt{3}-1)$$

$$= 2\left[\frac{\pi}{3}\sqrt{3} - \frac{1}{2}(\log 4 - \log 1)\right] - \pi(\sqrt{3}-1)$$

$$= \frac{2\pi}{3}\sqrt{3} - \log 4 - \pi(\sqrt{3}-1) = \pi\left(1 - \frac{1}{\sqrt{3}}\right) - \log 4$$

Example 67 :

Evaluate $\int_{-1}^{1} (x - [x])\,dx$

Sol. $\int_{-1}^{1} (x-[x])\,dx = \int_{-1}^{1} x\,dx - \int_{-1}^{1} [x]\,dx$

$$= \left[\frac{x^2}{2}\right]_{-1}^{1} - \left[\int_{-1}^{0} [x]\,dx + \int_0^1 [x]\,dx\right] = 0 - \left[\int_{-1}^{0} -1\,dx + \int_0^1 0\,dx\right] = 1$$

[P- 4] $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Note : This property can be used only when lower limit is zero. It is generally used for those complicated integrals whose denominators are unchanged when x is replaced by a – x.

(i) $\int_0^{\pi/2} f(\sin x)dx = \int_0^{\pi/2} f(\cos x)dx$

(ii) $\int_0^{\pi/2} f(\tan x)dx = \int_0^{\pi/2} f(\cot x)dx$

(iii) $\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \int_0^{\pi/2} f(\sin 2x)\cos x\,dx$

(iv) $\int_0^1 f(\log x)dx = \int_0^1 f[\log(1-x)]dx$

(v) $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}dx$

$= \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x}dx = \pi/4$

(vi) $\int_0^{\pi/2} \frac{\tan^n x}{1+\tan^n x}dx = \int_0^{\pi/2} \frac{\cot^n x}{1+\cot^n x}dx = \pi/4$

(vii) $\int_0^{\pi/2} \frac{1}{1+\tan^n x}dx = \int_0^{\pi/2} \frac{1}{1+\cot^n x}dx = \pi/4$

(viii) $\int_0^{\pi/2} \frac{\sec^n x}{\sec^n x + \operatorname{cosec}^n x}dx$

$= \int_0^{\pi/2} \frac{\operatorname{cosec}^n x}{\operatorname{cosec}^n x + \sec^n x}dx = \pi/4$

(ix) $\int_0^{\pi/4} \log(1+\tan x)\,dx = (\pi/8)\log 2$

(x) $\int_0^{\pi/2} \log\cot x\,dx = \int_0^{\pi/2} \log\tan x\,dx = 0$

Example 68 :

Evaluate $\int_0^{\pi/4} \log(1+\tan x)\,dx$

Sol. Let $I = \int_0^{\pi/4} \log(1+\tan x)\,dx$ .... (i)

Then $I = \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$

$\Rightarrow I = \int_0^{\pi/4} \log\left(1+\frac{\tan\pi/4-\tan x}{1+\tan\pi/4\tan x}\right)dx = \int_0^{\pi/4} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$

$= \int_0^{\pi/4} \log\left(\frac{1+\tan x+1-\tan x}{1+\tan x}\right)dx$

$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx = \int_0^{\pi/4} \{\log 2-\log(1+\tan x)\}dx$

$= \int_0^{\pi/4} \log 2dx - \int_0^{\pi/4} \log(1+\tan x)dx$

$\Rightarrow I = (\log 2)\,[x]_0^{\pi/4} - I \Rightarrow I = \frac{\pi}{4}\log 2 \Rightarrow I = \frac{\pi}{8}\log 2$

Example 69 :

Evaluate $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

Sol. Let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$ .... (i)

Then, $I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2-x)}}{\sqrt{\sin(\pi/2-x)}+\sqrt{\cos(\pi/2-x)}}dx$

$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$ .... (ii)

Adding (i) and (ii), we get

$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

$= \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx = \int_0^{\pi/2} 1.dx = [x]_0^{\pi/2} = \frac{\pi}{2}-0$

$\Rightarrow I = \frac{\pi}{4} \Rightarrow \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx = \frac{\pi}{4}$

Example 70 :

Evaluate $\int_0^{\pi} \frac{x}{1+\sin x}dx$

Sol. We have $I = \int_0^{\pi} \frac{x}{1+\sin x}dx$ .... (i)

$$\Rightarrow \; I = \int_0^{\pi} \frac{\pi - x}{1 + \sin(\pi - x)} dx \quad \left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

$$\Rightarrow \; I = \int_0^{\pi} \frac{\pi - x}{1 + \sin x} dx \qquad \text{....... (ii)}$$

Adding (i) and (ii), we get $2I = \int_0^{\pi} \frac{x + \pi - x}{1 + \sin x} dx$

$$= \pi \int_0^{\pi} \frac{1}{1 + \sin x} dx = \pi \int_0^{\pi} \frac{1 - \sin x}{1 - \sin^2 x} dx \qquad \text{....... (iii)}$$

$$\Rightarrow 2I = \pi \int_0^{\pi} (\sec^2 x - \tan x \sec x) dx = \pi [\tan x - \sec x]_0^{\pi}$$

$\Rightarrow 2I = \pi$ [(tanπ – secπ) (tan0 – sec0)]

$= \pi [(0 - (-1) - (0 - 1)] = 2\pi$

$\Rightarrow I = \pi$

Example 71 :

Evaluate $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx$

Sol. Let $I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}} dx$ ......(i)

Then, $I = \int_0^{\pi} \frac{e^{\cos(\pi - x)}}{e^{\cos x} + e^{-\cos x(\pi - x)}} dx$

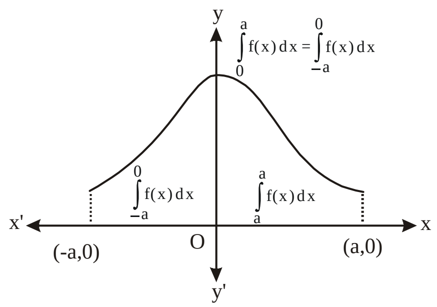
$$\Rightarrow \; I = \int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}} dx \qquad \text{......(ii)}$$

Adding (i) and (ii), we get $2I = \int_0^{\pi} 1 dx = \pi \Rightarrow I = \frac{\pi}{2}$

[P- 5]

$$\int_{-a}^{a} f(x) dx = \begin{cases} 0, \text{ if } f(-x) = -f(x) \;\; \text{i.e. if } f(x) \text{ is odd} \\ 2\int_0^a f(x) dx, \text{ if } f(-x) = f(x) \;\; \text{i.e. if } f(x) \text{ is even} \end{cases}$$

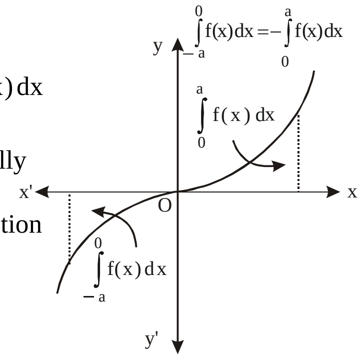
Note : The graph of an even function is symmetric about y-axis that is the curve on left side of y-axis is exactly identical to curve on its right side. So $\int_0^a f(x) dx = \int_{-a}^0 f(x) dx$



In case of an odd function the curve is symmetric in opposite quadrants, so

$$\int_{-a}^{0} f(x) dx = -\int_0^a f(x) dx$$

This property is generally used when integrand is either even or odd function of x.



Example 72 :

Evaluate $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x \, dx$

Sol. Let $f(x) = x^3 \sin^4 x$. Then,

$f(-x) = (-x)^3 \sin^4 (-x) = -x^3 \{\sin(-x)\}^4$

$\Rightarrow f(-x) = -x^3 (-\sin x)^4 = -x^3 \sin^4 x = -f(x)$.

So, f (x) is an odd function

Hence, $\int_{-\pi/4}^{\pi/4} f(x) dx = 0 \Rightarrow \int_{-\pi/4}^{\pi/4} x^3 \sin^4 x \, dx = 0$

Example 73 :

Evaluate $\int_{-\pi/2}^{\pi/2} |\sin x| dx$

Sol. Let $f(x) = |\sin x|$.

Then $f(-x) = |\sin(-x)| = |-\sin x| = |\sin| x = f(x)$

So, f (x) is an even function.

$$\therefore \int_{-\pi/2}^{\pi/2} |\sin x| dx = 2 \int_0^{\pi/2} |\sin x| dx = 2 \int_0^{\pi/2} \sin x \, dx$$

$$\left[\because \sin x \geq 0 \text{ for } 0 \leq x \leq \frac{\pi}{2} \right]$$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} |\sin x| dx = 2[-\cos x]_0^{\pi/2} = 2\left(-\cos\frac{\pi}{2} + \cos 0\right) = 2$$

Example 74 :

Evaluate $\int_{\log 1/2}^{\log 2} \sin\left\{\frac{e^x-1}{e^x+1}\right\}dx$

Sol. $I = \int_{\log 1/2}^{\log 2} \sin\left\{\frac{e^x-1}{e^x+1}\right\}dx$; If $f(x) = \sin\left\{\frac{e^x-1}{e^x+1}\right\}$

$$f(-x) = \sin\left\{\frac{1-e^x}{1+e^x}\right\} = -\sin\left\{\frac{e^x-1}{e^x+1}\right\} = -f(x)$$

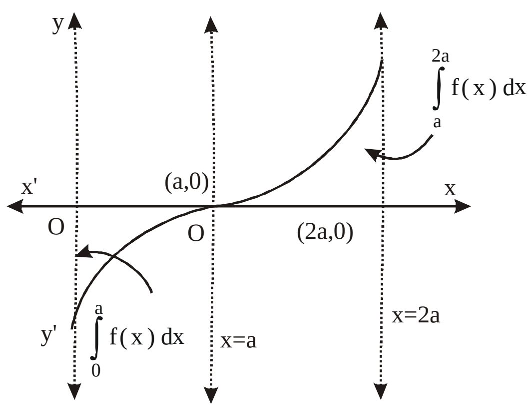
Hence f (x) is an odd function of x

$\therefore$ I = 0 by Prop. V.

[P-6] $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \text{if } f(2a-x) = f(x) \\ 0, \quad \text{if } f(2a-x) = -f(x) \end{cases}$

It is generally used to make half the upper limit

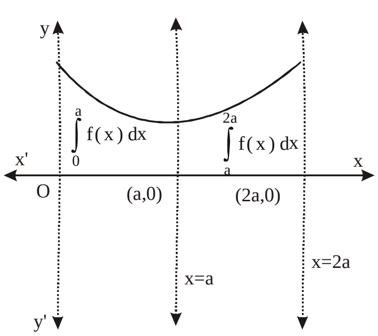
Note: If f (2a – x) = f (x), then the graph of f (x) is symmetrical about x = a as shown in fig.



$$\therefore \int_a^{2a} f(x)dx = \int_0^a f(x)dx$$

If (2a – x) = – f (x),then the graph of f (x) is shown

$$\therefore f\int_0^a f(x)dx = -\int_a^{2a} f(x)dx$$



Example 75 :

Prove that $\int_0^{\pi/2} \log\sin x\, dx = \int_0^{\pi/2} \log\cos x dx = -\frac{\pi}{2}\log 2$

Sol. Let $I = \int_0^{\pi/2} \log\sin x\, dx$ ...... (i)

$\int_0^{\pi/2} \log\sin\left(\frac{\pi}{2}-x\right)dx \Rightarrow I = \int_0^{\pi/2} \log\cos x\, dx$ ..... (ii)

Adding (i) and (ii), we get

$$\Rightarrow 2I = \int_0^{\pi/2} \log\sin x\, dx + \int_0^{\pi/2} \log\cos x\, dx = \int_0^{\pi/2} (\log\sin x\cos x)dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log\left(\frac{2\sin x\cos x}{2}\right)dx = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log\sin 2x\, dx - \int_0^{\pi/2} \log 2dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log\sin 2x\, dx - \frac{\pi}{2}(\log 2)$$

$\Rightarrow 2I = \int_0^{\pi/2} \log\sin 2x\, dx - \frac{\pi}{2}\log 2$ .....(iii)

Let $I_1 = \int_0^{\pi/2} \log\sin 2x\, dx$

Putting 2x = t, we get

$$I_1 = \int_0^{\pi} \log\sin t\frac{dt}{2} \Rightarrow I_1 = \frac{1}{2}.2\int_0^{\pi/2} \log\sin t\, dt$$

$$\Rightarrow I_1 = \int_0^{\pi/2} \log\sin x\, dx = I$$

So, from (iii), we get $2I = I - \frac{\pi}{2}\log 2 \Rightarrow I = -\frac{\pi}{2}\log 2$

Hence, $\int_0^{\pi/2} \log\sin x\, dx = \int_0^{\pi/2} \log\cos x dx = -\frac{\pi}{2}\log 2$

[P- 7] $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Example 76 :

Prove that $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx = \frac{b-a}{2}$

Sol. Let $I = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx$ .....(i)

Then, $I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(a+b)-(a+b-x)}dx$

$\Rightarrow I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}dx$ .....(ii)

Adding (i) and (ii), we get

$$2I = \int_a^b \frac{f(x)+f(a+b-x)}{f(x)+f(a+b-x)}dx$$

$$\Rightarrow 2I = \int_a^b 1.dx = (b-a) \Rightarrow I = \left(\frac{b-a}{2}\right)$$

Example 77 :

Evaluate $\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}dx$

Sol. $I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}dx = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$ ....(i)

Then, $I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx$

$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$ .....(ii)

Adding (i) and (ii), we get $2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$

$$\Rightarrow 2I = \int_{\pi/6}^{\pi/3} 1.dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6} \Rightarrow I = \pi/12$$

[P- 8] If f (x) is a periodic function of period a, i.e. $f(a+x) = f(x)$, then

(a) $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$

(b) $\int_{na}^{b+na} f(x)\,dx = \int_0^b f(x)dx$, where $b \in R^+$

(c) $\int_b^{b+a} f(x)\,dx$ is independent of b

(d) $\int_b^{b+na} f(x)\,dx = n\int_0^a f(x)dx$, where $n \in I$. In particular ,

(i) If b = 0, $\int_0^{na} f(x)\,dx = n\int_0^a f(x)dx$

(ii) If n = 1, $\int_b^{b+a} f(x)\,dx = \int_0^a f(x)dx$

Example 78 :

Evaluate $\int_0^{10} (x-[x])dx$

Sol. Since x – [x] is a periodic function with period one unit. Therefore

$$\int_0^{10}(x-[x])dx = 10\int_0^1(x-[x])dx = 10\left[\int_0^1 x\,dx - \int_0^1 [x]dx\right]$$

$$= 10\left[\left[\frac{x^2}{2}\right]_0^1 - 0\right] = \frac{10}{2} = 5$$

[P- 9] Differentiation of Integration (Leibnithz's Rule) :

$$\frac{d}{dt}\left[\int_{\varphi(t)}^{\Psi(t)} f(x)dx\right] = f\{\psi(t)\}\,\psi'(t) - f\{\phi(t)\}\,\phi'(t)$$

Example 79 :

Let $f(x) = \int_1^x \sqrt{2-t^2}$ dt then find the real roots of the equation $x^2 - f'(x) = 0$

Sol. $f(x) = \int_1^x \sqrt{2-t^2}\,dt$; $f'(x) = \sqrt{2-x^2}\,.\,1 - \sqrt{2-1}\,.\,0$

$= \sqrt{2-x^2}$

$\therefore$ $x^2 = f'(x) = \sqrt{2-x^2}$ or $x^4 + x^2 - 2 = 0$
or $(x^2+2)(x^2-1) = 0$ $\therefore$ $x = \pm 1$ (only real)

[P- 10] If $f(x) \geq 0$ on the interval [a, b] , then $\int_a^b f(x)dx \geq 0$

[P- 11] If $f(x) \leq g(x)$ on [a, b] , then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$

[P- 12] If m and M are the smallest and greatest values of a function f (x) defined on an interval [a, b], then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

Example 80 :

Prove that $1 \le \int_0^1 e^{x^2} dx \le e$

Sol. For $0 \le x \le 1$, we have $e^0 \le e^{x^2} \le e^1$

$$\therefore\ e^0(1-0) \le \int_0^1 e^{x^2} dx \le e(1-0) \Rightarrow 1 \le \int_0^1 e^{x^2} dx \le e$$

[P- 13] If f (x) is defined on [a, b], then

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx$$

SOME IMPORTANT FORMULAE

Gamma function: If n is a positive rational number, then the improper integral $\int_0^\infty e^{-x}x^{n-1}dx$ is defined as Gamma function and is denoted by Γn

i.e. $\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx$, where $x \in Q^+$

For Ex. $\Gamma 1 = \int_0^\infty e^{-x}x^0 dx = \lim_{b\to\infty}\int_0^b e^{-x}dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b$

$$= \lim_{b\to\infty}(-e^{-b}+e^0) = 0+1 = 1$$

PROPERTIES OF GAMMA FUNCTION

(i) $\Gamma 1 = 1$, $\Gamma 0 = \infty$ and $\Gamma n+1 = n\Gamma n$

(ii) If $n \in N$, then $\Gamma n+1 = n!$ (iii) $\Gamma 1/2 = \sqrt{\pi}$

(iv) $\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \dfrac{\Gamma\frac{m+1}{2}\Gamma\frac{n+1}{2}}{2\Gamma\frac{m+n+2}{2}}$

(v) In place of gamma function, we can also use the following formula : $\int_0^{\pi/2} \sin^m \times \cos^n x\,dx$

$$= \frac{(m-1)(m-3).....(2\text{ or }1)(n-1)(n-3).....(2\text{ or }1)}{(m+n)(m+n-2).....(2\text{ or }1)} \times (1 \text{ or } \pi/2)$$

It is important to note that we multiply by $(\pi/2)$ when both m and n are even.

Example 81 :

Evaluate $\int_0^{\pi/2} \sin^4 x \cos^5 x\,dx$

Sol. Using gamma function formula

$$I = \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{2\Gamma\left(\frac{4+5+2}{2}\right)} = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma(3)}{2\Gamma\left(\frac{11}{2}\right)}$$

$$= \frac{\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}\,2.1}{2.\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}} = \frac{8}{315}$$

Walli's formula : $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$

$$= \begin{cases} \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.....\frac{2}{3}, & \text{when } n \text{ is odd} \\ \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.....\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}, & \text{when } n \text{ is even} \end{cases}$$

Example 82 :

Evaluate $\int_0^{\pi/2} \sin^6 x\,dx$

Sol. $I = \frac{5}{6}.\frac{3}{2}.\frac{1}{2} \times \frac{\pi}{2} = \frac{5}{32}\pi$

SUMMATION OF SERIES BY INTEGRATION

For finding sum of an infinite series with the help of definite integration, following formula is used

$$\lim_{n\to\infty}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)\frac{1}{n} = \int_0^1 f(x)\,dx$$

The following method is used to solve the questions on summation of series.

(i) After writing $(r-1)^{th}$ or r^{th} term of the series, express it in the form $\frac{1}{n}f\left(\frac{r}{n}\right)$. Therefore the given series will take the form. $\lim_{n\to\infty}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)\frac{1}{n}$

(ii) Now writing $\int$ in place of $\left(\lim_{n\to\infty}\Sigma\right)$, x in place of $\left(\frac{r}{n}\right)$ and dx in place of $\frac{1}{n}$, we get the integral $\int f(x)\,dx$ in place of above series.

(iii) The lower limit of this integral = $\lim_{n\to\infty}\left(\frac{r}{n}\right)_{r=0}$

where r = 0 is taken corresponding to first term of the series

and upper limit = $\lim_{n\to\infty}\left(\frac{r}{n}\right)_{r=n-1}$

where r = n – 1 is taken corresponding to the last term.

Example 83 :

Evaluate $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+.....+\frac{1}{2n}\right]$

Sol. Limit $= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r} = \lim_{n\to\infty}\sum\frac{1}{1+\frac{r}{n}}.\frac{1}{n}$

$$= \int_0^1 \frac{1}{1+x}dx = \left[\log(1+x)\right]_0^1 = \log 2$$

Example 84 :

Evaluate $\lim_{n\to\infty}\prod_{r=1}^{n}\left(\frac{n+r}{n}\right)^{1/n}$.

Sol. $S = \lim_{n\to\infty}\prod_{r=1}^{n}\left(\frac{n+r}{n}\right)^{1/n} = \text{Lim}\left[\frac{n+1}{n}\cdot\frac{n+2}{n}.......\frac{n+n}{n}\right]^{1/n}$

Taking in both sides, we have

$$\ln S = \lim_{n\to\infty}\frac{1}{n}\left[\ln\left(\frac{n+1}{n}\right)+\ln\left(\frac{n+2}{n}\right)+....\ln\left(\frac{n+n}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{n=1}^{n}\ln\left(1+\frac{r}{n}\right) = \int_0^1 \ln(1+x)\,dx$$

$= \ln 2 – (1 – \ln 2) = \ln 4 – 1 = \ln (4/e)$

$\therefore$ S = 4 / e.

Example 85 :

If $n\to\infty$, then find the limit of $\frac{1}{n}\sum_{r=1}^{n}\sin^{2k}\left(\frac{r\pi}{2n}\right)$.

Sol. Let $P = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\sin^{2k}\left(\frac{r\pi}{2n}\right) = \int_0^1 \sin^{2k}\left(\frac{\pi}{2}x\right)dx$

Put $\frac{\pi}{2}x = t \quad \therefore\ dx = \frac{2}{\pi}dt = \frac{2}{\pi}\int_0^{\pi/2}\sin^{2k}t\,dt$

$$= \frac{2}{\pi}\frac{(2k-1)(2k-3)(2k-5)......\ 3.1}{2k(2k-2)(2k-4)......\ 4.2}\cdot\frac{\pi}{2}$$

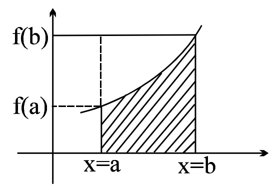
$$= \frac{2k(2k-1)(2k-2)(2k-3)......3.2.1}{[2k(2k-2)(2k-4)......4.2]^2}$$

Hence $P = \frac{2k!}{(2^k k!)^2} = \frac{2k!}{2^{2k}(k!)^2}$

INEQUALITIES INVOLVED IN DEFINITE INTEGRAL

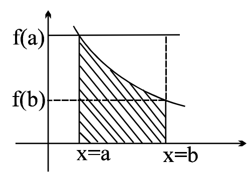
(i) For a monotonic increasing function in (a, b)

$$(b-a)f(a) < \int_a^b f(x)dx < (b-a)f(b)$$



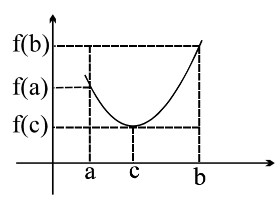
(ii) Foa a monotonic decreasing function in (a, b)

$$f(b).(b-a) < \int_a^b f(x)dx < (b-a)f(a)$$



(iii) For a non monotonic function in (a, b)

$$f(c)(b-a) < \int_a^b f(x)dx < (b-a)f(b)$$



(iv) $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$ equality holds when f (x) lies completely above the x-axis

(v) If $h(x) \le f(x) \le g(x)\ \forall\ x \in [a, b]$ then

$$\int h(x)dx < \int f(x)dx < \int g(x)dx$$

Example 86 :

Show that $1 < \int_0^{\pi/2} \frac{\sin x}{x} dx < \frac{\pi}{2}$

Sol. $f(x) = \frac{\sin x}{x}$ or $f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x}{x^2}[x - \tan x]$

$\Rightarrow f'(x) < 0$ hence $f(x)_{min} = \frac{2}{\pi}$ $f(x)_{max} = 1.$

$\Rightarrow \frac{2}{\pi}\left(\frac{\pi}{2} - 0\right) < I < 1\left(\frac{\pi}{2} - 0\right)$ or $1 < I < \frac{\pi}{2}$.

Example 87 :

Show that $\frac{1}{4} \le \int_0^1 \frac{dx}{1 + x^2 + 2x^5} \le 1$.

Sol. Consider the following function

$f(x) = \frac{1}{1 + x^2 + 2x^5}, x \in [0, 1]$

In the interval [0, 1], f (x) is strictly decreasing, therefore

we have $f(1) \le f(x) \le f(0)$ i.e. $\frac{1}{4} \le f(x) \le 1$

Hence, we have $(1 - 0)\ \frac{1}{4} \le \int_0^1 f(x)\,dx \le (1 - 0)\,1$

i.e. $\frac{1}{4} \le \int_0^1 f(x)\,dx \le 1$ which is the desired result.

Example 88 :

Prove that $\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \frac{\pi\sqrt{2}}{8}$

Sol. $4 - 2x^2 \le 4 - x^2 - x^3 \le 4 - x^2$

$\Rightarrow \frac{1}{\sqrt{4 - 2x^2}} \ge \frac{1}{\sqrt{4 - x^2 - x^3}} \ge \frac{1}{\sqrt{4 - x^2}}$

$\Rightarrow \int_0^1 \frac{dx}{\sqrt{4 - x^2}} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \int_0^1 \frac{dx}{\sqrt{4 - 2x^2}}$

$\Rightarrow \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1 < I < \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^1$

or $\frac{\pi}{6} < I < \frac{\sqrt{2}}{8}\pi$

TRY IT YOURSELF-2

Q.1 If $\lim_{n \to \infty} \int_{-\sqrt[3]{a}}^{\sqrt[3]{a}} \left(1 - \frac{t^3}{n}\right)^n t^2 dt = \frac{2\sqrt{2}}{3}\ (n \in N)$, then find the value of a.

Q.2 Evaluate : $\int_3^8 \frac{\sin\sqrt{x+1}}{\sqrt{x+1}} dx$

Q.3 Evaluate : $\int_2^4 \frac{\sqrt{x^2 - 4}}{x^4} dx$

Q.4 Evaluate : $\int_0^{\pi} \sqrt{\frac{1 + \cos 2x}{2}} dx$

Q.5 Evaluate : $\int_0^3 |5x - 9|\,dx$

Q.6 Evaluate : $\int_1^4 \ln[x]\,dx$, $[\cdot]$ is the greatest integer function.

Q.7 The value of $\int_{-1}^{3} \left(\tan^{-1}\frac{x}{x^2 + 1} + \tan^{-1}\frac{x^2 + 1}{x}\right) dx$ is –

(A) π
(B) 2π
(C) 3π
(D) $5\pi/2$

Q.8 Evaluate : $\int_{50}^{100} \frac{\ln x}{\ln x + \ln(150 - x)} dx$

Q.9 Evaluate : $\int_0^{\pi} \frac{dx}{1 + 2^{\tan x}}$

Q.10 Evaluate : $\int_0^{\pi/2} \frac{dx}{1 + \sin x}$

Q.11 $I = \int_0^{2\pi} \sin^4 x\,dx = k\int_0^{\pi/2} \cos^4 dx$ find value of k

Q.12 Evaluate : $I = \int_0^{2\pi} x \cdot \cos^5 x\,dx$

Q.13 Evaluate : $\int_0^{10} \sqrt{1 - \cos \pi x}\,dx$

Q.14 Evaluate : $\lim_{x \to 0} \frac{\int_0^x (1 - \cos 2x)\,dx}{x\int_0^x \tan x\,dx}$

Q.15 Find the value of $\lim_{n \to \infty} \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots \frac{1}{4n}$

ANSWERS

(1) $\ln(\sqrt{2}+\sqrt{3})$ **(2)** 2 (cos 2 – cos 3) **(3)** $\frac{\sqrt{3}}{32}$

(4) 2 **(5)** 15/2 **(6)** ln 6

(7) (A) **(8)** 25 **(9)** $\pi/2$

(10) 1 **(11)** 4 **(12)** 0

(13) $\frac{20\sqrt{2}}{\pi}$ **(14)** 4/3 **(15)** ln 4

IMPORTANT POINTS

* **Fundamental theorem of calculus :**

Part I : If f (x) is continuous on [a, b] then g (x) = $\int_a^x f(t)\,dt$ is also continuous on [a, b] and

$$g'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Part II : f (x) is continuous on [a, b], F (x) is an anti-derivative of f (x) i.e. $F(x) = \int f(x)\,dx$ then $\int_a^b f(x)\,dx = F(b) - F(a)$

* **Integration by Substitution :** The substitution u = g (x) will convert $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$,

using du = g ' (x) dx.

* Integration by parts : $\int u\,dv = uv - \int v\,du$ and

$\int_a^b u\,dv = uv\big|_a^b - \int_a^b v\,du$. Choose u and dv from integral and compute du by differentiating u and compute v using $v = \int dv$

* Integration by Partial fraction : If integrating $\int \frac{P(x)}{Q(x)}dx$ where the degree of P (x) is smaller than the degree of Q (x). Factor denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.).

* **Some properties of definite integral :**

(a) If an interval [a, b] (a < b), the function f (x) and ϕ (x) satisfy the condition $f(x) \le \phi(x)$, then

$$\int_b^a f(x)\,dx \le \int_a^b \phi(x)\,dx$$

(b) If m and M are the smallest and greatest values of a function f (x) on an interval [a, b] and a ≤ b, then

$$m(b-a) \le \int_b^a f(x)\,dx \le M(b-a)$$

(c) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)$$

(d) $$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

(e) $$\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$$

(f) $$\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{If } f(x) \text{ is even function, i.e. } f(-x) = f(x) \\ 0, & \text{If } f(x) \text{ is odd function, i.e. } f(x) = -f(x) \end{cases}$$

(g) $$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{If } f(2a-x) = f(x) \\ 0, & \text{If } f(2a-x) = -f(x) \end{cases}$$

* **Leibnitz rule :**

$$\frac{d}{dx}\int_{f(x)}^{g(x)} F(t)\,dt = g'(x)F(g(x)) - f'(x)F(f(x))$$

* If a series can be put in the form $\frac{1}{n}\sum_{r=0}^{r=n-1} f\left(\frac{r}{n}\right)$ or $\frac{1}{n}\sum_{r=1}^{r=n} f\left(\frac{r}{n}\right)$, then its limit as $n \to \infty$ is $\int_0^1 f(x)\,dx$

* If $f(x) \le g(x)$ on [a, b], then $\int_a^b f(x)dx \le \int_a^b g(x)dx$

* **(i)** For a monotonic decreasing function in

$$(a, b)\,;\ f(b).(b-a) < \int_a^b f(x)\,dx < f(a).(b-a)$$

(ii) For a monotonic increasing function in

$$(a, b)\,;\ f(a).(b-a) < \int_a^b f(x)\,dx < f(b).(b-a)$$

* If $\int_a^b f(x)\,dx = 0$, then the equation $f(x) = 0$ has atleast one root in (a, b) provided f is continuous in (a, b).

Note that the converse is not true.

* $\lim_{n\to\infty}\left(\int_a^b f_n(x)dx\right) = \int_a^b\left(\lim_{n\to\infty} f_n(x)\right)dx$

* $\int_a^b f(x)\cdot d(g(x)) = \int_{g^{-1}(a)}^{g^{-1}(b)} f(x)\cdot g'(x)\,dx$.

* If g (x) is the inverse of f (x) and f (x) has domain $x \in [a, b]$ where f (a) = c and f (b) = d then the value of

$$\int_a^b f(x)\,dx + \int_c^d g(y)\,dy = (bd - ac)$$

* If f(x) is continuous in (a, b), Then $\int_a^b \frac{d}{dx}(f(x)) = [f(x)]_a^b$

and if f(x) is discontinuous in (a, b) at $x = c \in (a, b)$, then

$$\int_a^b \frac{d}{dx}(f(x)) = [f(x)]_a^{c^-} + [f(x)]_{c^+}^b$$

ADDITIONAL EXAMPLES

Example 1 :

Evaluate $\int \frac{\cos x + x\sin x}{x(x+\cos x)}\,dx$

Sol. $I = \int \frac{(x+\cos x) - x + \sin x}{x(x+\cos x)}\,dx = \int \frac{1}{x}dx - \int \frac{1-\sin x}{x+\cos x}dx$

$= \log x - \log(x + \cos x) + c = \log\left(\frac{x}{x+\cos x}\right) + c$

Example 2 :

If $\int \frac{2^{1/x}}{x^2}\,dx = K.2^{1/x}$, then find the value of K.

Sol. We have, $\int \frac{2^{1/x}}{x^2}\,dx = K.\,2^{1/x}$

Differentiating both sides w.r.t. x, we get

$$\frac{2^{1/x}}{x^2} = K.2^{1/x}.\left(\frac{-1}{x^2}\right).\log 2 \Rightarrow K = \frac{-1}{\log 2}$$

Example 3 :

Evaluate $\int [\sin(\log x) + \cos(\log x)]\,dx$.

Sol. Put $\log x = z \Rightarrow x = e^z \Rightarrow dx = e^z\,dz$.

$\therefore \int [\sin(\log x) + \cos(\log x)]\,dx$

$= \int e^z(\sin z + \cos z)\,dz = e^z \sin z + C = x\sin(\log x) + C$

Example 4 :

Find the value of $\int \frac{(x - x^3)^{1/3}}{x^4}\,dx$.

Sol. $\int \frac{(x-x^3)^{1/3}}{x^4}\,dx = \int \frac{1}{x^3}\left(\frac{1}{x^2} - 1\right)^{1/3} dx$

$= \frac{-1}{2}\int t^{1/3}\,dt \quad \left[\text{Putting } \frac{1}{x^2} - 1 = t \Rightarrow \frac{-2}{x^3}dx = dt\right]$

$= \frac{-1}{2}.\frac{t^{4/3}}{4/3} + C = \frac{-3}{8}\left(\frac{1}{x^2} - 1\right)^{4/3} + C$

Example 5 :

Evaluate $\int \frac{x^2+1}{(x-1)(x-2)}\,dx$

Sol. Here since the highest powers of x in Num[r] and Den[r] are equal and coefficients of x^2 are also equal, therefore

$$\int \frac{x^2+1}{(x-1)(x-2)} \equiv 1 + \frac{A}{x-1} + \frac{B}{x-2}$$

On solving we get A = – 2, B = 5

Thus $\int \frac{x^2+1}{(x-1)(x-2)} \equiv 1 - \frac{2}{x-1} + \frac{5}{x-2}$

The above method is used to obtain the value of constant corresponding to non repeated linear factor in the Den[r].

Now, $I = \int\left(1 - \frac{2}{x-1} + \frac{5}{x-2}\right)dx$

$= x - 2\log(x-1) + 5\log(x-2) + C$

$= x + \log\left[\frac{(x-2)^5}{(x-1)^2}\right] + C$

Example 6 :

Evaluate $\lim_{n\to\infty}\frac{1}{n}\left[\tan\frac{\pi}{4n} + \tan\frac{2\pi}{4n} + ... + \tan\frac{n\pi}{4n}\right]$

Sol. $I = \lim_{n\to\infty}\Sigma\frac{1}{n}\tan\frac{\pi}{4}\frac{r}{n} = \int_0^1 \tan\frac{\pi}{4}x\,dx$

$= \frac{4}{\pi}\int_0^{\pi/4}\tan t\,dt = \frac{4}{\pi}\,[\log\sec t]_0^{\pi/4} = \frac{4}{\pi}\log\sqrt{2} = \frac{2}{\pi}\log 2$

Example 7 :

Evaluate $\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}\ (\beta > \alpha)$.

Sol. Put $x - \alpha = t^2 \Rightarrow dx = 2t\,dt$

$$\therefore \quad I = 2\int \frac{t\,dt}{\sqrt{t^2(\beta-\alpha-t^2)}} = 2\int \frac{dt}{\sqrt{(\beta-\alpha)-t^2}}$$

$$= 2\sin^{-1}\frac{t}{\sqrt{\beta-\alpha}} + c = 2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} + c$$

Example 8 :

Evaluate $\int \sqrt{e^{2x}-1}\,dx$

Sol. $= \int \frac{e^{2x}-1}{\sqrt{e^{2x}-1}}\,dx = \frac{1}{2}\int \frac{2e^{2x}}{\sqrt{e^{2x}-1}}\,dx - \int \frac{e^x}{e^x\sqrt{e^{2x}-1}}\,dx$

$= \sqrt{e^{2x}-1} - \sec^{-1} e^x + c$

Example 9 :

Evaluate $\int \frac{\log(x+1)-\log x}{x(x+1)}\,dx$

Sol. $I = -\int (\log(x+1)-\log x).\frac{1}{-x(x+1)}\,dx = \frac{-1}{2}\left[\log\left(\frac{x+1}{x}\right)\right]^2 + C$

$$\left[\because \frac{d}{dx}[\log(x+1)-\log x] = \frac{1}{x+1} - \frac{1}{x} = -\frac{1}{(x+1)x}\right]$$

Example 10 :

Evaluate $\int \frac{dx}{4\sin^2 x + 4\sin x\cos x + 5\cos^2 x}$

Sol. After dividing by $\cos^2 x$ to numerator and denominator of integration $I = \int \frac{\sec^2 x\,dx}{4\tan^2 x + 4\tan x + 5}$

$$= \int \frac{\sec^2 x\,dx}{(2\tan x+1)^2+4} = \frac{1}{22}\tan^{-1}\left(\frac{2\tan x+1}{2}\right) + c$$

Example 11 :

For the function f (x) = 1 + 3^x log3, the antiderivative F assumes the value 7 for x = 2. At what value of x does the curve y = F(x) cut the abscissa ?

Sol. $f(x) = 1 + 3^x \log 3 \;\therefore\; F(x) = \int (1+3^x\log 3)\,dx = x + 3^x + c$

Since $F(2) = 2 + 9 + c = 7 \Rightarrow c = -4$

$\therefore\; F(x) = x + 3^x - 4 = 0 \Rightarrow x = 1$

Example 12 :

Evaluate $\int \frac{x^2+1}{x^4+x^2+1}\,dx$

Sol. $I = \int \frac{1+1/x^2}{x^2+1+1/x^2}\,dx = \int \frac{d\,(x-1/x)}{(x-1/x)^2+3}$

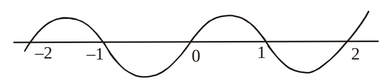
$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{(x-1/x)}{\sqrt{3}} + c = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{(x^2-1)}{\sqrt{3}x}\right) + c$$

Example 13 :

Find the points of maxima/minima of $\int_0^{x^2} \frac{t^2-5t+4}{2+e^t}\,dt$.

Sol. f (x) =

$\int_0^{x^2} \frac{t^2-5t+4}{2+e^t}\,dt$



$$\Rightarrow f'(x) = \frac{x^4-5x^2+4}{2+e^{x^2}}2x - 0 = \frac{(x-1)(x+1)(x-2)(x+2)2x}{2+e^{x^2}}$$

From the wavy curve, it is clear that f ' (x) changes its sign at x = ± 2, ± 1, 0 and hence the points of maxima are –1, 1 and that of the minima are –2, 0, 2.

Example 14 :

Find $\lim_{n\to\infty} n^{-\frac{1}{2}\left(1+\frac{1}{n}\right)} \cdot (1^1 \cdot 2^2 \cdot 3^3 \,.......\, n^n)^{\frac{1}{n^2}}$.

Sol. Let $L = \lim_{n\to\infty} n^{-\frac{1}{2}\left(1+\frac{1}{n}\right)} \cdot (1^1 \cdot 2^2 \cdot 3^3 \,.......\, n^n)^{\frac{1}{n^2}}$

$$\ln L = \lim_{n\to\infty} -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n + \frac{1}{n^2}\sum_{k=1}^{n} k\ln k$$

$$= \lim_{n\to\infty} -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n + \frac{1}{n^2}\sum_{k=1}^{n}(k\ln k - k\ln n + k\ln n)$$

$$= \lim_{n\to\infty} -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n + \frac{1}{n^2}\sum_{k=1}^{n} k\ln\frac{k}{n} + \frac{\ln n}{n^2}\sum_{k=1}^{n} k$$

$$= \lim_{n\to\infty} -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n + \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\ln\frac{k}{n} + \frac{\ln n}{n^2}\cdot\frac{n(n+1)}{2}$$

$$= -\frac{1}{2}\left(\frac{n+1}{n}\right)\ln n + \int_0^1 x\ln x\,dx + \frac{1}{2}\left(\frac{n+1}{n}\right)\ln n$$

$$= \int_0^1 \underset{II}{x}\,\underbrace{\ln x}_{I}\,dx = -\frac{1}{4} \qquad \therefore L = e^{-\frac{1}{4}}$$

Example 15 :

Find the values of L that satisfy the following equation

$$\frac{\int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)\,dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)\,dt} = L$$

Sol. $I_1 = \int_0^{\pi} e^t(\sin^6 at + \cos^4 at)\,dt + \int_{\pi}^{2\pi} e^t(\sin^6 at + \cos^4 at)\,dt$

$$+\int_{2\pi}^{3\pi} e^t(\sin^6 at + \cos^4 at)\,dt + \int_{3\pi}^{4\pi} (\sin^6 at + \cos^4 at)\,dt$$

$$= (1 + e^{\pi} + e^{2\pi} + e^{3\pi}\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)\,dt$$

$$\Rightarrow \frac{I_1}{I_2} = 1 + e^{\pi} + e^{2\pi} + e^{3\pi} = \frac{e^{4\pi}-1}{e^{\pi}-1}$$

Example 16 :

Evaluate : $\int_0^{\pi/2} \cos^7 x\,dx$

Sol. $I = \int_0^{\pi/2} \cos^7 x\,dx = \frac{6\cdot4\cdot2}{7\cdot5\cdot3\cdot1}$

Example 17 :

Evaluate :

$$\lim_{n\to\infty}\left[\frac{\sqrt{n}}{\sqrt{n^3}} + \frac{\sqrt{n}}{\sqrt{(n+2)^3}} + \frac{\sqrt{n}}{\sqrt{(n+4)^3}} + \frac{\sqrt{n}}{\sqrt{(n+8)^3}} + ...\right]$$

Sol. We have

$$\lim_{n\to\infty}\left[\frac{\sqrt{n}}{\sqrt{n^3}} + \frac{\sqrt{n}}{\sqrt{(n+2)^3}} + \frac{\sqrt{n}}{\sqrt{(n+4)^3}} + \frac{\sqrt{n}}{\sqrt{(n+8)^3}} + ...\right]$$

$$= \lim_{n\to\infty}\sum_{r=0}^{n}\frac{\sqrt{n}}{\sqrt{(n+2r)^3}} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{(n+2r)^3}}$$

[omitting one term will not affect the limit]

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{n^{3/2}}{(n+2r)^{3/2}} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\left\{1+2\left(\frac{r}{n}\right)\right\}^{3/2}}$$

$$= \int_0^1 \frac{1}{(1+2x)^{3/2}}dx = \left[\frac{(1+2x)^{-1/2}}{-1/2}\cdot\frac{1}{2}\right]_0^1$$

$$= \left[\frac{1}{\sqrt{1+2x}}\right]_1^0 = 1 - \frac{1}{\sqrt{3}}$$

Example 18 :

Prove that $\int_{1/2}^{2} (\ln x)^2\,dx < \int_{1/2}^{2} |\ln x|\,dx$

Sol. In the interval $\left[\frac{1}{4}, \frac{1}{2}\right]$, $|\ln x|$ is a fraction. hence, we have $(\ln x)^2 < |\ln x|$

i.e. $\int_{1/2}^{2} (\ln x)^2\,dx < \int_{1/2}^{2} |\ln x|\,dx$

Example 19 :

Evaluate the definite integrals $\int_0^{\pi/2} \sin x \ln(\cos x)\,dx$

Sol. $I = \int_0^{\pi/2} \sin x \ln(\cos x)\,dx$

$$= [-\cos x \ln(\cos x)]_0^{\pi/2} + \int_0^{\pi/2} \cos x . \frac{-\sin x}{\cos x}dx$$

$$= \lim_{x\to\pi/2}\frac{-\ln(\cos x)}{\sec x} + [\cos x]_0^{\pi/2} = \lim_{x\to\pi/2}\frac{\tan x}{\sec x\tan x} - 1$$

$$= \lim_{x\to\pi/2} \cos x - 1 = -1.$$

QUESTION BANK

CHAPTER 7 : INTEGRATION

EXERCISE - 1 [LEVEL-1]

PART - 1 - INDEFINITE INTEGRATION

Q.1 $\int\left(\sqrt{\frac{\cos x}{x}} - \sqrt{\frac{x}{\cos x}}\sin x\right)dx =$

(A) $-\sqrt{x\cos x}+C$ (B) $\sqrt{x\sin x}+C$

(C) $2\sqrt{x\cos x}+C$ (D) $C-2\sqrt{x\cos x}$

Q.2 $\int\frac{2^x+3^x}{5^x}dx$ is equal to –

(A) $\frac{(2/5)^x}{\log_3(3/5)}+\frac{(3/5)^x}{\log_3(2/5)}+C$ (B) $\frac{(2/5)^x}{\log_3(2/5)}+\frac{(3/5)^x}{\log_3(3/5)}+C$

(C) $\frac{(3/5)^x}{\log_3(2/5)}+\frac{(2/5)^x}{\log_3(3/5)}+C$ (D) None of the above

Q.3 $\int \tan 2x\,\tan 3x\,\tan 5x\,dx$ is equal to –

(A) $\frac{1}{2}\log|\sec 2x| - \frac{1}{3}\log|\sec 3x| - \frac{1}{5}\log|\sec 5x|+C$

(B) $\frac{1}{2}\log|\sec 2x| + \frac{1}{3}\log|\sec 3x| + \frac{1}{5}\log|\sec 5x|+C$

(C) $\frac{1}{5}\log|\sec 5x| - \frac{1}{2}\log|\sec 2x| - \frac{1}{3}\log|\sec 3x|+C$

(D) None of these

Q.4 $\int\frac{\sin^4 x}{\cos^8 x}dx$ is equal to –

(A) $\frac{(1+\tan^5 x)}{5}+\frac{\tan^5 x}{7}+C$ (B) $\frac{\tan^5 x}{5}+\frac{\tan^7 x}{7}+C$

(C) $\frac{\tan^7 x}{5}+\frac{\tan^5 x}{7}+C$ (D) None of the above

Q.5 $\int e^x\left\{\frac{1+\sin x\cos x}{\cos^2 x}\right\}dx$ is equal to –

(A) $e^x\cos x + c$ (B) $e^x\sec x\,.\,\tan x + c$

(C) $e^x\tan x + c$ (D) $e^x\cos^2 x - 1 + c$

Q.6 $\int\frac{\cos^2 x}{\cos^2 x+9\sin^2 x}dx$ is equal to –

(A) $\frac{1}{8}\{3\tan^{-1}(3\tan x)-x\}+c$ (B) $\frac{1}{8}\{3\tan^{-1}(3\tan x)+x\}+c$

(C) $\frac{1}{27}\{\tan^{-1}(3\tan x)-x\}+c$ (D) $8\left\{\frac{1}{27}\tan^{-1}(3\tan x)-x\right\}+c$

Q.7 Find $\int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$

(A) $\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}+C$

(B) $-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}+C$

(C) $-2\sqrt{1-x}-\cos^{-1}x^3+\sqrt{x}\sqrt{1-x}+C$

(D) $-5\sqrt{1-x}+\cos^{-1}\sqrt{x}-x\sqrt{1-x}+C$

Q.8 $\int\frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}dx$ equals

(A) $\frac{3}{2}x^{2/3}+6\tan^{-1}\sqrt[6]{x}+c$ (B) $\frac{3}{2}x^{2/3}+6\tan^{-1}\sqrt{x}+c$

(C) $\frac{3}{2}x^{2/3}+\tan^{-1}x+c$ (D) $\frac{3}{2}x^{2/3}+6\tan^{-1}x^{1/3}+c$

Q.9 The value of the integral $\int\frac{\cos^3 x+\cos^5 x}{\sin^2 x+\sin^4 x}dx$ is –

(A) $\sin x - 6\tan^{-1}(\sin x) + c$

(B) $\sin x - 2(\sin x)^{-1} + c$

(C) $\sin x - 2(\sin x)^{-1} - 6\tan^{-1}(\sin x) + c$

(D) $\sin x - 2(\sin x)^{-1} + 5\tan^{-1}(\sin x) + c$

Q.10 Evaluate: $\int\sqrt{\frac{x}{4-x^3}}dx$.

(A) $\frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{2}\right)+c$ (B) $\frac{2}{3}\sin^{-1}\left(x^{3/2}\right)+c$

(C) $2\sin^{-1}\left(\frac{x^{3/2}}{2}\right)+c$ (D) $\frac{1}{3}\sin^{-1}\left(\frac{x^{3/2}}{2}\right)+c$

Q.11 $\int\frac{(ax^2-b)dx}{x\sqrt{c^2x^2-(ax^2+b)^2}}$ is equal to

(A) $\sin^{-1}\left(\frac{ax+bx^2}{c}\right)+C$ (B) $\sin^{-1}\left(\frac{ax^2+b}{cx}\right)+C$

(C) $\tan^{-1}\left(\frac{a+bx^2}{cx}\right)+C$ (D) $\tan^{-1}(ax^2+bx+c)+C$

Q.12 Evaluate: $\int x\tan^{-1}x\,dx$

(A) $\frac{1}{2}(x^2+1)\tan^{-1}x-\frac{1}{2}x+c$ (B) $\frac{1}{2}(x^2+1)\tan^{-1}x+\frac{1}{2}x+c$

(C) $\frac{1}{2}(x^2-1)\tan^{-1}x-\frac{1}{2}x+c$ (D) None of these

Q.13 If $\int f(x)\sin x.\cos x\,dx = \frac{1}{2(b^2-a^2)}\log f(x) + c$, where c is the constant of integration, then f (x) =

(A) $\frac{2}{ab\cos 2x}$ (B) $\frac{2}{(b^2-a^2)\cos 2x}$

(C) $\frac{2}{ab\sin 2x}$ (D) $\frac{2}{(b^2-a^2)\sin 2x}$

Q.14 If $\int \frac{\sqrt{x}}{x(x+1)}dx = k\tan^{-1} m$, then (k, m) is –

(A) (2, x) (B) (1, x)

(C) $(1, \sqrt{x})$ (D) $(2, \sqrt{x})$

Q.15 When x > 0, then $\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx$ is

(A) $2x\tan^{-1} x - \log(1+x^2) + c$
(B) $2x\tan^{-1} x + \log(1+x^2) + c$
(C) $2[x\tan^{-1} x + \log(1+x^2)] + c$
(D) $2[x\tan^{-1} x - \log(1+x^2)] + c$

Q.16 If $n \in N$ and $I_n = \int (\log x)^n dx$, then $I_n + nI_{n-1} =$

(A) $\frac{(\log x)^n n}{n}$ (B) $\frac{(\log x)^{n+1}}{n+1}$

(C) $x(\log x)^n + c$ (D) $(\log x)^{n-1}$

Q.17 $\int \frac{\cos^{n-1} x}{\sin^{n+1} x}dx, n \neq 0$ is –

(A) $\frac{\cot^n x}{n}$ (B) $\frac{-\cot^{n-1} x}{n-1}$

(C) $\frac{-\cot^n x}{n}$ (D) $\frac{\cot^{n-1} x}{n-1}$

Q.18 $\int \frac{(x-1)e^x}{(x+1)^3}dx =$

(A) $\frac{e^x}{x+1}$ (B) $\frac{e^x}{(x+1)^2}$

(C) $\frac{e^x}{(x+1)^3}$ (D) $\frac{x\cdot e^x}{(x+1)}$

Q.19 If linear function f (x) and g (x) satisfy
$\int [(3x-1)\cos x + (1-2x)\sin x]\,dx$
$= f(x)\cos x + g(x)\sin x + C$, then –

(A) $f(x) = 3(x-1)$ (B) $f(x) = 3x - 5$
(C) $g(x) = 3(x-1)$ (D) $g(x) = 3 + x$

Q.20 $\int \frac{\sin 2x}{\sin^2 x + 2\cos^2 x}dx =$

(A) $-\log(1+\sin^2 x) + C$ (B) $\log(1+\cos^2 x) + C$
(C) $-\log(1+\cos^2 x) + C$ (D) $\log(1+\tan^2 x) + C$

Q.21 $\int \frac{1}{x^2(x^4+1)^{3/4}}dx$ is equal to –

(A) $\frac{-(1+x^4)^{1/4}}{x} + C$ (B) $\frac{-(1+x^4)^{1/4}}{x^2} + C$

(C) $\frac{-(1+x^4)^{1/4}}{2x} + C$ (D) $\frac{-(1+x^4)^{3/4}}{x} + C$

Q.22 $\int \frac{\sin^2 x}{1+\cos x}dx$

(A) $x + \sin x + C$ (B) $x - \sin x + C$
(C) $\sin x + C$ (D) $\cos x + C$

Q.23 $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right) dx$ is –

(A) $e^x \tan\left(\frac{x}{2}\right) + C$ (B) $\tan\left(\frac{x}{2}\right) + C$

(C) $e^x + C$ (D) $e^x \sin x + C$

PART - 2 - DEFINITE INTEGRATION

Q.24 If $I_1 = \int_x^1 \frac{1}{1+t^2}dt$ and $I_2 = \int_1^{1/x} \frac{1}{1+t^2}dt$ for x > 0, then

(A) $I_1 = I_2$ (B) $I_1 > I_2$
(C) $I_2 > I_1$ (D) None of these

Q.25 If $g(x) = \int_0^x \cos^4 t\,dt$, then $g(x+\pi)$ equals

(A) $g(x) + g(\pi)$ (B) $g(x) - g(\pi)$
(C) $f(x)\,g(\pi)$ (D) $g(x)/g(\pi)$

Q.26 $\int_0^{\pi^2/4} \left(\frac{2}{\operatorname{cosec}\sqrt{x}} + \frac{\sqrt{x}}{\sec\sqrt{x}}\right) dx =$

(A) $\pi^2/4$ (B) $\pi^2/2$
(C) π^2 (D) $\pi/8$

Q.27 $\int_{-\pi/4}^{\pi/4} \ln\sqrt{1+\sin 2x}\,dx =$

(A) $-2\pi\ln 2$ (B) $-\frac{\pi}{4}\ln 2$

(C) $-\pi\ln 2$ (D) $-\frac{\pi}{2}\ln 2$

Q.28 $\int_{0}^{2\pi} \ln(1+\sin x)\,dx =$

(A) $-2\pi \ln 2$ (B) $-\frac{\pi}{4}\ln 2$

(C) $-\pi \ln 2$ (D) $-\frac{\pi}{2}\ln 2$

Q.29 $\int_{1/2010}^{2010} \frac{1}{x}\sin\left(x-\frac{1}{x}\right) dx$ is equal to –

(A) 2010 (B) –2010
(C) –1 (D) 0

Q.30 If $g(x) = \int_{0}^{x} \cos^4 t\,dt$, then g (x + π) equals –

(A) g (x) + g (π) (B) g (x) – g (π)

(C) g (x) g (π) (D) $\frac{g(x)}{g(\pi)}$

Q.31 If $A = \int_{0}^{\pi/2} \sin^2 x\,dx$ and $B = \int_{0}^{\pi/2} \cos^2 x\,dx$ and

$C = \int_{0}^{\pi/2} \sin^3 x\,dx$, then –

(A) A + B = 0 (B) A – B = 0
(C) A + C = 0 (D) A – C = 0

Q.32 Evaluate $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\,dx$

(A) 1 (B) 0
(C) 2 (D) 4

Q.33 If $\int_{0}^{1} \frac{e^t dt}{t+1} = a$, then $\int_{b-1}^{b} \frac{e^{-t}dt}{t-b-1} = ?$

(A) ae^{-b} (B) $-ae^{-b}$
(C) $-be^{-a}$ (D) ae^{b}

Q.34 The value of the integral

$\int_{0}^{2008} \left(3x^2 - 8028x + (2007)^2 + \frac{1}{2008}\right) dx$ equals –

(A) $(2008)^2$ (B) $(2009)^2$
(C) 2009 (D) 1

Q.35 Evaluate $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{(1+e^x)(1+x^2)}$

(A) π/3 (B) π/2
(C) π/4 (D) π/6

Q.36 The value of $\int_{-2\pi}^{5\pi} \cot^{-1}(\tan x)\,dx$ is equal to –

(A) $5\pi^2/2$ (B) $7\pi^2/2$
(C) $9\pi^2/2$ (D) None of these

Q.37 $\int_{0}^{\pi/2} \frac{2^{\sin x}}{2^{\sin x} + 2^{\cos x}} dx$ equals

(A) 2 (B) π
(C) π/4 (D) π/2

Q.38 Evaluate $\int_{0}^{\pi/4} (\sqrt{\tan x} + \sqrt{\cot x})\,dx$

(A) $\pi/\sqrt{2}$ (B) π/2

(C) π/6 (D) $\pi/\sqrt{3}$

Q.39 $\int_{\pi/4}^{3\pi/4} \frac{\phi}{1+\sin\phi} d\phi$ equals

(A) $\pi(\sqrt{2}-1)$ (B) $\pi(\sqrt{2}+1)$

(C) $\pi(2-\sqrt{2})$ (D) None of these

Q.40 Evaluate $\int_{0}^{1} \log\left(\frac{1}{x}-1\right) dx$

(A) 1 (B) 2
(C) 3 (D) 0

Q.41 Value of definite integral $\int_{0}^{1} (1+e^{-x^2})\,dx$ can be –

(A) –1 (B) 2
(C) $1 + e^{-1}$ (D) None of these

Q.42 Evaluate $\int_{0}^{\infty} \frac{dx}{1+e^x}$

(A) log 2 (B) log 3
(C) log 7 (D) log 5

Q.43 Let f (x) = x – [x], fore every real number x, where [x] is integral part of x. Then find the value of $\int_{-1}^{1} f(x)\,dx$.

(A) 1 (B) 2
(C) 0 (D) 3

Q.44 Evaluate $\int_{-1}^{2} \frac{|x|}{x} dx$

(A) 3 (B) 2
(C) 0 (D) 1

Q.45 $\int_0^{\pi/2n} \frac{1}{1+\cot^n nx}dx$ =

(A) 0 (B) $\pi/4n$
(C) $\pi/2n$ (D) $\pi/2$

Q.46 $\int_0^{\pi/4} \frac{\sin x + \cos x}{3+\sin 2x}dx$ =

(A) (1/4) log 3 (B) log 3
(C) (1/2) log 3 (D) 2 Log 3

Q.47 $\int_0^1 x\,(1-x)^{3/2}dx$ =

(A) –2/35 (B) 4/35
(C) 24/35 (D) –8/35

Q.48 The value of $\int_0^4 |x-1|\,dx$ is

(A) 1 (B) 4
(C) 5 (D) 5/2

Q.49 If $I_n = \int_0^4 \tan^n x\,dx$, where n is a positive integer, then $I_{10} + I_8$ is –

(A) 9 (B) 1/7
(C) 1/8 (D) 1/9

Q.50 $\int_{\pi/6}^{\pi/3} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}dx$ =

(A) $\pi/6$ (B) $\pi/2$
(C) $\pi/3$ (D) $\pi/12$

Q.51 If [x] is the greatest integer function not greater than x, then $\int_0^{11} [x]\,dx$ =

(A) 55 (B) 45
(C) 66 (D) 35

Q.52 If $I_1 = \int_0^{\pi/2} x \cdot \sin x\,dx$ and $I_2 = \int_0^{\pi/2} x\cdot \cos x\,dx$, then which one of the following is true?

(A) $I_1 = I_2$ (B) $I_1 + I_2 = 0$
(C) $I_1 = \frac{\pi}{2} I_2$ (D) $I_1 + I_2 = \frac{\pi}{2}$

Q.53 The value of the definite integral $\int_0^{\pi/2} \sin|2x-\alpha|\,dx$ where $\alpha \in [0, \pi]$

(A) 1 (B) $\cos\alpha$
(C) $\frac{1+\cos\alpha}{2}$ (D) $\frac{1-\cos\alpha}{2}$

Q.54 $\int_0^{\pi} \frac{\cos^4 x}{\cos^4 x + \sin^4 x}dx$ =

(A) $\pi/4$ (B) $\pi/2$
(C) $\pi/8$ (D) π

Q.55 If $f(x) = f(\pi + e - x)$ & $\int_e^{\pi} f(x)\,dx = \frac{2}{e+\pi}$, then $\int_e^{\pi} x\,f(x)\,dx$ =

(A) $\pi - e$ (B) $(\pi + e)/2$
(C) 1 (D) $(\pi - e)/2$

Q.56 The value of the integral $\int_{-\pi/4}^{\pi/4} \log(\sec\theta - \tan\theta)\,d\theta$ is –

(A) 0 (B) $\pi/4$
(C) π (D) $\pi/2$

Q.57 $\int_0^{\pi/4} \log\left(\frac{\sin x + \cos x}{\cos x}\right)dx$ =

(A) $\frac{\pi}{4}\log 2$ (B) $\frac{\pi}{2}\log 2$
(C) $\frac{\pi}{8}\log 2$ (D) log 2

Q.58 $\int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x}$ is equal to –

(A) 2 (B) 1
(C) 4 (D) 0

PART - 3 - SERIES SUMMATION

Q.59 If $S_n = \left[\frac{1}{1+\sqrt{n}} + \frac{1}{2+\sqrt{2n}} + ... + \frac{1}{n+\sqrt{n^2}}\right]$, then find $\lim_{n\to\infty} S_n$.

(A) log 4 (B) 5/2
(C) log 3 (D) None of these

Q.60 $\lim_{n\to\infty} \frac{2^k + 4^k + 6^k + ... + (2n)^k}{n^{k+1}}$; $(k \ne 1)$ is equal to –

(A) 2^k (B) $\frac{2^k}{k+1}$
(C) $\frac{1}{k+1}$ (D) 0

Q.61 $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\sqrt{\left(\frac{n+r}{n-r}\right)}$ equals

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{2}+1$

(C) π (D) None of these

Q.62 $\lim_{n\to\infty}\frac{1}{n}(e^{1/n}+e^{2/n}+e^{3/n}+....+e^{n/n})$ is equal to

(A) e (B) $e-1$

(C) $1-e$ (D) None of these

Q.63 Let $\lambda=\int_0^1\frac{dx}{1+x^3}$, $p=\lim_{n\to\infty}\left[\frac{\prod_{r=1}^{n}(n^3+r^2)}{n^{3n}}\right]^{1/n}$

then ln p is equal to –

(A) $\ln 2-1+\lambda$ (B) $\ln 2-3+3\lambda$

(C) $2\ln 2-\lambda$ (D) $\ln 4-3+3\lambda$

Q.64 If $f(x)=\lim_{n\to\infty}\{2x+4x^3+.........+2nx^{2n-1}\}$ $(0<x<1)$;

then $\int f(x)\,dx$ is equal to

(A) $-(1-x^2)+C$ (B) $-(1-x^2)^{-1}+C$

(C) $(1-x^2)^{-1}+C$ (D) $(1-x)^{-1}+C$

PART - 4 - MISCELLANEOUS

Q.65 The value of the definite integral

$$I=\int_0^{\pi/2}e^x\left\{\cos(\sin x)\cos^2\frac{x}{2}+\sin(\sin x)\sin^2\frac{x}{2}\right\}dx\text{, is}$$

(A) $\frac{1}{2}[e^{\pi/2}(\cos 1+\sin 1)-1]$ (B) $\frac{e^{\pi/2}}{2}(\cos 1+\sin 1)$

(C) $\frac{1}{2}(e^{\pi/2}\cos 1-1)$ (D) $\frac{e^{\pi/2}}{2}(\cos 1+\sin 1-1)$

Q.66 $I=\int\frac{1}{1-\cos^4 x}dx$ is equal to

(A) $2\sqrt{2}(\cot x+\sqrt{2}\tan^{-1}\sqrt{2}\cot x)$

(B) $-\frac{1}{2\sqrt{2}}[\sqrt{2}\cot x+\tan^{-1}(\sqrt{2}\cot x)]+c$

(C) $2\sqrt{2}\{\cot x+\tan^{-1}(\cos x)\}$

(D) none of these

Q.67 The value of the definite integral

$$\int_0^{3\pi/4}(1+x)\sin x+(1-x)\cos x)\,dx\text{ is –}$$

(A) $2(\sqrt{2}+1)$ (B) $2\sqrt{2}$

(C) $2(\sqrt{2}-1)$ (D) 0

Q.68 The value of $\int e^x\frac{1+n\,x^{n-1}-x^{2n}}{(1-x^n)\sqrt{1-x^{2n}}}dx$,

(where n is a nonzero constant) is

(A) $\frac{e^x\sqrt{1-x^n}}{1-x^n}+C$ (B) $e^x\frac{\sqrt{1+x^{2n}}}{1-x^{2n}}+C$

(C) $e^x\frac{\sqrt{1-x^{2n}}}{1-x^n}+C$ (D) none of these

Q.69 $\int 4\cos\left(x+\frac{\pi}{6}\right)\cos 2x.\cos\left(\frac{5\pi}{6}+x\right)dx$

(A) $-\left(x+\frac{\sin 4x}{4}+\frac{\sin 2x}{2}\right)+c$

(B) $-\left(x+\frac{\sin 4x}{4}-\frac{\sin 2x}{2}\right)+c$

(C) $-\left(x-\frac{\sin 4x}{4}+\frac{\sin 2x}{2}\right)+c$

(D) $-\left(x-\frac{\sin 4x}{4}+\frac{\cos 2x}{2}\right)+c$

Q.70 $\int\sqrt{x}e^{\sqrt{x}}dx$ equals

(A) $2\sqrt{x}\,e^{\sqrt{x}}-4\sqrt{x}e^{\sqrt{x}}$ (B) $(2x-4\sqrt{x}+4)\,e^{\sqrt{x}}$

(C) $(1-4\sqrt{x})\,e^{\sqrt{x}}$ (D) None of these

Q.71 Evaluate: $\int(\cos x-\sin x)(3+4\sin 2x)dx$.

(A) $\left(\frac{\sin x-\cos x}{3}\right)(1+4\sin 2x)+c$

(B) $\left(\frac{\sin x-\cos x}{3}\right)(1-4\sin 2x)+c$

(C) $\left(\frac{\sin x+\cos x}{3}\right)(1+4\sin 2x)+c$

(D) None of these

Q.72 $\int\frac{(x^2-1)}{(x^2+1)\sqrt{x^4+1}}dx$ is equal to

(A) $\sec^{-1}\left(\frac{x^2+1}{\sqrt{2}\,x}\right)+c$ (B) $\frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{x^2+1}{\sqrt{2}\,x}\right)+c$

(C) $\frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{x^2+1}{\sqrt{2}}\right)+c$ (D) None of these

Q.73 $\int_{0}^{\pi/2} x\left|\sin^2 x - \frac{1}{2}\right| dx$ is equal to

(A) $\pi/2$ (B) $\pi/4$
(C) $\pi/8$ (D) $3\pi/4$

Q.74 $\int_{-1}^{1} (1+x)^{1/2}(1-x)^{3/2} dx$ equals

(A) π (B) $\pi/2$
(C) $\pi/8$ (D) $3\pi/4$

Q.75 $\int \frac{x+\sin x}{1+\cos x} dx =$

(A) $\cot\frac{x}{2} + c$ (B) $x\tan\frac{x}{2} + c$

(C) $\log(1+\cos x) + c$ (D) $x\tan\frac{x}{2} + \tan x + c$

Q.76 $\int \frac{1}{a^2\cos^2 x + b^2\sin^2 x} dx$ equals

(A) $\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan x\right) + c$ (B) $\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\cot x\right) + c$

(C) $\frac{1}{ab}\cot^{-1}\left(\frac{b}{a}\tan x\right) + c$ (D) $-\frac{1}{ab}\cot^{-1}\left(\frac{a}{b}\cot x\right) + c$

Q.77 $\int \frac{x^3+x}{x^4-9} dx =$

(A) $\frac{1}{4}\log_e|x^2-9| + \frac{1}{12}\left|\frac{x^2-3}{x^2+3}\right| + C$

(B) $\frac{1}{8}\log_e|x^4-9| + \frac{1}{12}\log_e\left|\frac{x^2-3}{x^2+3}\right| + C$

(C) $\frac{1}{4}\log_e|x^4-9| + \frac{1}{12}\log_e\left|\frac{x^4-3}{x^4+3}\right| + C$

(D) $\frac{1}{4}\log_e|x^4-9| + \frac{1}{12}\log_e\left|\frac{x^2-3}{x^2+3}\right| + C$

Q.78 $\int \frac{e^{x-1}}{(x^2-5x+4)}.2x\,dx = A\,F(x-1) + B\,(x-4) + c$,

where $F(x) = \int \frac{e^x}{x} dx$ then A and B ordered set is –

(A) $\left(-\frac{2}{3}, \frac{8}{3}\right)$ (B) $\left(-\frac{2}{3}, \frac{8e^3}{3}\right)$

(C) $\left(\frac{8}{3}, \frac{2}{3}\right)$ (D) $\left(-\frac{2}{3}, -\frac{8e^3}{3}\right)$

Q.79 Evaluate $\int \frac{a^x}{\sqrt{1-a^{2x}}} dx$

(A) $\frac{1}{\log a}\sin^{-1}(a^x) + C$ (B) $\frac{1}{\log a}\cos^{-1}(a^x) + C$

(C) $\frac{1}{\log a}\sec^{-1}(a^x) + C$ (D) None of these

Q.80 Given $f'(x) = \frac{\cos x}{x}$, $f\left(\frac{\pi}{2}\right) = a$, $f\left(\frac{3\pi}{2}\right) = b$. The value of the definite integral $\int_{\pi/2}^{3\pi/2} f(x)\,dx$ equals –

(A) $2 - \frac{\pi}{2}(a-3b)$ (B) $2 - \frac{\pi}{2}(a+3b)$

(C) $2 + \frac{\pi}{2}(3a+b)$ (D) $2 - \frac{\pi}{2}(3a-b)$

Q.81 $\int \tan^{-1}\sqrt{\left(\frac{1-x}{1+x}\right)} dx$ equals

(A) $x\cos^{-1}x - \sqrt{(1-x^2)}$ (B) $x\cos^{-1}x + \sqrt{(1-x^2)}$

(C) $\frac{1}{2}[x\cos^{-1}x - \sqrt{(1-x^2)}]$ (D) None of these

Q.82 Evaluate: $\int x^2 \sin x\,dx$

(A) $-x^2\cos x + 2x\sin x - 2\cos x + c$
(B) $x^2\cos x + 2x\sin x - 2\cos x + c$
(C) $x^2\cos x + 2x\sin x + 2\cos x + c$
(D) None of these

Q.83 $\int \left(x+\frac{1}{x}\right)^{n+5}\left(\frac{x^2-1}{x^2}\right) dx$ is equal to

(A) $\frac{\left(x+\frac{1}{x}\right)^{n+6}}{n+6} + c$ (B) $\left(\frac{x^2+1}{x^2}\right)^{n+6}(n+6) + c$

(C) $\left(\frac{x}{x^2+1}\right)^{n+6}(n+6) + c$ (D) none of these

Q.84 Evaluate $\int \frac{x^2}{x^2-1} dx$

(A) $x - \frac{1}{2}\log\left(\frac{x-1}{x+1}\right) + c$ (B) $x + \frac{1}{2}\log\left(\frac{x+1}{x-1}\right) + c$

(C) $x + \frac{1}{2}\log\left(\frac{x-1}{x+1}\right) + c$ (D) None of these

Q.85 $\int \frac{1}{(x+1)\sqrt{x^2-1}}dx =$

(A) $\sqrt{\frac{x+1}{x-1}}+C$ (B) $\sqrt{\frac{x^2-1}{x+1}}+C$

(C) $\sqrt{\frac{x-1}{x+1}}+C$ (D) $\sqrt{\frac{x-1}{x^2+1}}+C$

Q.86 $\int \frac{1}{x(x^n+1)}dx$, (where n is a non-zero constant) is equal to

(A) $\frac{1}{n}\ell n\left|\frac{x^n}{x^n+1}\right|+C$ (B) $\frac{1}{n}\ell n\left|\frac{x^n+1}{x^n}\right|+C$

(C) $\ell n\left|\frac{x^n}{x^n+1}\right|+C$ (D) none of these

Q.87 $\int \frac{a^{\sqrt{x}}}{\sqrt{x}}dx$ is equal to

(A) $\frac{a^{\sqrt{x}}}{\ell n\, a}+C$ (B) $\frac{2a^{\sqrt{x}}}{\ell n\, a}+C$

(C) $2a^{\sqrt{x}}.\ell n\, a+C$ (D) none of these

Q.88 Evaluate $\int \frac{1}{9x^2-4}dx$

(A) $\frac{1}{12}\log\left|\frac{3x-2}{3x+2}\right|+C$ (B) $\frac{1}{12}\log\left|\frac{x-2}{3x+2}\right|+C$

(C) $\frac{1}{12}\log\left|\frac{x-2}{x+2}\right|+C$ (D) None of these

Q.89 If $\int_{-2}^{2} x^4.\sqrt{x^4-x^2}\,dx$ has the value equal to $k\pi$ then the value of k equals –

(A) 0 (B) 2

(C) 8 (D) 4

Q.90 Let A be a 3 × 3 diagonal matrix with $a_{11} = e_x$, $a_{22} = e^{e^x}$ and $a_{33} = e^{e^{e^x}}$. The value of the integral $\int_0^1 (\det.A)\,dx$ equals –

(A) $e^{e^e} - e^e$ (B) $e^{e^e} - 1$

(C) $e^e - e$ (D) e^{e^e}

Q.91 Evaluate $\int \frac{x^2+1}{x^4+1}dx$

(A) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right)+C$ (B) $\frac{1}{2}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}}\right)+C$

(C) $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}}\right)+C$ (D) None of these

Q.92 Let $\int \frac{dx}{x^{2008}+x} = \frac{1}{p}\ln\left(\frac{x^q}{1+x^r}\right)+C$, where p, q, r $\in$ N and need not be distinct, then the value of (p+q+ r) equals

(A) 6024 (B) 6022

(C) 6021 (D) 6020

Q.93 The value of $\lim_{x\to 0}\frac{\int_0^{x^2}\cos t^2 dt}{x\sin x}$ is –

(A) 3/2 (B) 1

(C) – 1 (D) None of these

Q.94 $\int_0^{\pi/2} (\sin x)^x (\ln(\sin x) + x\cot x)\,dx$ is –

(A) –1 (B) 1

(C) 0 (D) indeterminant

Q.95 $\int \frac{3+2\cos x}{(2+3\cos x)^2}dx$ is equal to

(A) $\frac{\sin x}{(2+3\cos x)}+c$ (B) $\left(\frac{2\cos x}{3\sin x+2}\right)+c$

(C) $\left(\frac{2\cos x}{3\cos x+2}\right)+c$ (D) $\left(\frac{2\sin x}{3\sin x+2}\right)+c$

Q.96 Evaluate: $\int \frac{dx}{\sin x(2\cos^2 x-1)}$.

(A) $\frac{1}{2}\ell n\left|\frac{\cos x-1}{\cos x+1}\right|+\frac{1}{\sqrt{2}}\ell n\left|\frac{\sqrt{2}\cos x-1}{\sqrt{2}\cos x+1}\right|+C$

(B) $\frac{1}{2}\ell n\left|\frac{\cos x-1}{\cos x+1}\right|-\frac{1}{\sqrt{2}}\ell n\left|\frac{\sqrt{2}\cos x-1}{\sqrt{2}\cos x+1}\right|+C$

(C) $\frac{1}{2}\ell n\left|\frac{\cos x-1}{\cos x+1}\right|-\frac{1}{2}\ell n\left|\frac{\sqrt{2}\cos x-1}{\sqrt{2}\cos x+1}\right|+C$

(D) None of these

Q.97 $\int_0^2 \frac{2x^3-6x^2+9x-5}{x^2-2x+5}dx$ is equal to

(A) 0 (B) 3

(C) 3/2 (D) 4

Q.98 Evaluate: $\int \sin^{-7/5} x \cos^{-3/5} x \, dx$.

(A) $-\frac{5}{2}(\cot x)^{2/5} + C$ (B) $-\frac{5}{2}(\cot x)^{3/5} + C$

(C) $-\frac{1}{2}(\cot x)^{2/5} + C$ (D) $-\frac{5}{2}(\cot x)^{1/5} + C$

Q.99 Evaluate $\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}} dx$

(A) $\frac{1}{3}[\sqrt{2}-1]$ (B) $\frac{4}{3}[\sqrt{2}+1]$

(C) $\frac{4}{3}[\sqrt{2}-1]$ (D) $\frac{2}{3}[\sqrt{2}-1]$

Q.100 Evaluate $\int_0^{\pi/4} \sqrt{1-\sin 2x}\, dx$

(A) $\sqrt{2}$ (B) $\sqrt{2}-1$

(C) $\sqrt{2}+1$ (D) 1

Q.101 Evaluate $\int_0^4 \frac{1}{\sqrt{x^2+2x+3}} dx$

(A) $\log\left(\frac{5+3\sqrt{3}}{1+\sqrt{3}}\right)$ (B) $\log\left(\frac{5+3\sqrt{3}}{1-\sqrt{3}}\right)$

(C) $\log\left(\frac{5-2\sqrt{3}}{1+\sqrt{3}}\right)$ (D) None of these

Q.102 If [x] stands for the greatest integer function, then

$\int_1^3 [x]^x \, dx$

(A) Cannot be evaluated
(B) Has the value $1 + 4 (\log 2)^{-1}$
(C) Has the value $1 + 4 \log 2$
(D) Has the value $1 - 4 \log 2$

Q.103 $\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx = \pi \frac{\cos\alpha}{1+\sin^2\alpha}$

(A) for no value of α
(B) for exactly two values of α in $(0, \pi)$.
(C) for at least one α in $(\pi/2, \pi)$.
(D) for exactly one α in $(0, \pi/2)$.

Q.104 Find $\int_{-1}^{1} x^3 . e^{x^4} dx$

(A) 0 (B) 1
(C) 2 (D) 4

Q.105 $\int_{\pi/4}^{3\pi/4} \frac{\cos x}{1-\cos x} dx$ is equal to-

(A) $2-\frac{\pi}{2}$ (B) $2+\frac{\pi}{2}$

(C) $\frac{1}{2}-2\pi$ (D) $\frac{1}{2}+2\pi$

Q.106 $\lim_{n\to\infty}\left[\frac{1}{na}+\frac{1}{na+1}+\frac{1}{na+2}+...+\frac{1}{nb}\right]$ is equal to

(A) ln (b/a) (B) ln (a/b)
(C) ln a (D) ln b

Q.107 Evaluate $\int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}} dx$

(A) 1 (B) 0
(C) 1/2 (D) 2

Q.108 Evaluate $\int_0^{\pi/4} \sin 3x \sin 2x \, dx$

(A) $\frac{3\sqrt{2}}{10}$ (B) $\frac{2\sqrt{2}}{10}$

(C) 3 (D) $\sqrt{2}$

Q.109 If $f(t) = \begin{cases} at-1, & t<1 \\ t^2+b, & t\geq 1 \end{cases}$ then possible set of value of (a, b) so that $\int_0^x f(x)\, dx$ is differentiable for all $x \geq 0$ is –

(A) (5, 1) (B) (1, 3)
(C) (4, 2) (D) None of these

Q.110 Evaluate $\int_0^{\pi} \sin^3 x \, dx$

(A) 1/3 (B) 2/3
(C) 4/3 (D) 3

Q.111 $\int_{\log 1/2}^{\log 2} \sin\left\{\frac{e^x-1}{e^x+1}\right\} dx$ equals

(A) $\cos\frac{1}{3}$ (B) $\sin\frac{1}{2}$

(C) 2 cos 2 (D) 0

Q.112 $\int_0^{2\pi} \frac{e^{|\sin x|}\cos x}{1+e^{\tan x}} dx =$

(A) e^{π} (B) 1
(C) $e^{\pi}-1$ (D) 0

Q.113 Evaluate $\int_{-3/2}^{10} \{2x\}dx$, where $\{.\}$ denotes the fractional part of x.

(A) 1/3 (B) 23/4
(C) 5/3 (D) 4/23

Q.114 Evaluate $\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}}dx$

(A) $\frac{\pi}{2}-\frac{1}{2}\log 2$ (B) $\frac{\pi}{4}+\frac{1}{2}\log 2$
(C) $\frac{\pi}{4}-\frac{1}{2}\log 2$ (D) $\frac{\pi}{4}-\log 2$

Q.115 Evaluate $\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}dx$

(A) π/2 (B) π/4
(C) π/3 (D) π

Q.116 Evaluate $\int_{-1}^{1} f(x)dx$, where $f(x) = \begin{cases} 1-2x, & x \le 0 \\ 1+2x, & x \ge 0 \end{cases}$.

(A) 2 (B) 1
(C) 3 (D) 4

Q.117 Evaluate : $\int_0^1 \frac{dx}{\sqrt{2-x^2}}$.

(A) π/4 (B) π
(C) π/2 (D) π/3

Q.118 $\int_0^4 \{\sin\{x\}\}\,dx = \lambda$, where $\{ x \}$ represents fractional part of x. Then λ/4 is –

(A) – cos 1 (B) 1 – cos 1
(C) 1 – sin 1 (D) None of these

Q.119 The value of integral $\int_0^{\pi/2} \frac{\phi(x)}{\phi(x)+\phi(\pi/2-x)}dx$ is

(A) π/4 (B) π/2
(C) π (D) none of these

Q.120 $\int_0^{\pi} x \sin x \cos^4 x\, dx =$

(A) π/10 (B) π/5
(C) –π/5 (D) none of these

Q.121 Evaluate $\int_{-\pi/2}^{\pi/2} |\sin x|\,dx$

(A) 2 (B) 1
(C) 3 (D) 0

Q.122 Evaluate $\int_0^{\pi/2} \log \tan x\, dx$

(A) 1 (B) 0
(C) 2 (D) 3

Q.123 Evaluate $\int_0^3 [x]dx$

(A) 1 (B) 0
(C) 2 (D) 3

Q.124 Find the value of $\int_0^{4\pi} |\sin x|\; dx$

(A) 8 (B) 6
(C) 4 (D) 2

Q.125 A function f (x) which satisfies the relation

$f(x) = e^x + \int_0^1 e^x f(x)\,dt$, then f (x) is –

(A) $\frac{e^x}{2-e}$ (B) $(e-2)\,e^x$
(C) $2e^x$ (D) $e^x/2$

Q.126 Let $I_1 = \int_0^1 e^{-x}\cos^2 x\,dx$ and $I_2 = \int_0^1 e^{-x^2}\cos^2 x\,dx$, then–

(A) $I_1 > I_2$ (B) $I_1 = I_2$
(C) $I_1 < I_2$ (D) None of these

Q.127 If $I_1 = \int_0^{3\pi} f(\cos^2 x)\,dx$ and $I_2 = \int_0^{\pi} f(\cos^2 x)\,dx$ then

(A) $I_1 = I_2$ (B) $I_1 = 2I_2$
(C) $I_1 = 5I_2$ (D) $I_1 = 3I_2$

Q.128 If $I_m = \int_1^e (\ln x)^m dx$, where m ∈ N, then $I_{10} + 10\, I_9$ is equal to –

(A) e^{10} (B) $\frac{e^{10}}{10}$
(C) e (D) e – 1

Q.129 $\int_0^{\pi/2} \frac{\sin 4\theta}{\sin\theta}\,d\theta$ is equal to-

(A) 4/3 (B) 8/3
(C) 3/4 (D) None of these

Q.130 Let $f(x) = \int_1^x \sqrt{2-t^2}\,dt$. Then the real roots of the equation $x^2 - f'(x) = 0$ are

(A) ± 1 (B) $\pm 1/\sqrt{2}$
(C) ± 1/2 (D) 0 and 1

Q.131 If $I = \int_0^{\pi/2} e^{-\alpha \sin x} dx$, where $\alpha \in (0,\infty)$, then –

(A) $I > \frac{\pi}{2}$ (B) $I < \frac{\pi}{2}(e^{-\alpha}+1)$

(C) $I < \frac{\pi}{2}e^{-\alpha}$ (D) $I > 0$

Q.132 Evaluate $\int_1^4 (\{x\})^{[x]} dx$, where {.} and [.] denote the fractional part and the greatest integer functions respectively.

(A) 12/13 (B) 13/12
(C) 7/12 (D) 12/7

Q.133 The value of the definite integral $\int_{-1}^{1} \frac{dx}{(1+e^x)(1+x^2)}$ is –

(A) $\pi/2$ (B) $\pi/4$
(C) $\pi/8$ (D) $\pi/16$

EXERCISE - 2 [LEVEL-2]

Q.1 Find the value of $\lim_{x\to 0} \frac{\int_0^{x^2} \sin\sqrt{t}\,dt}{x^3}$

(A) 2/3 (B) 1/3
(C) 1/4 (D) 1/2

Q.2 If $\int \frac{dx}{\sqrt{2ax - x^2}} = fog(x) + C$, then

(A) $f(x) = \sin^{-1}x, g(x) = \frac{x+a}{a}$

(B) $f(x) = \sin^{-1}x, g(x) = \frac{x-a}{a}$

(C) $f(x) = \cos^{-1}x, g(x) = \frac{x-a}{a}$

(D) $f(x) = \tan^{-1}x, g(x) = \frac{x-a}{a}$

Q.3 Let $f(x) = \int_x^2 \frac{dy}{\sqrt{1+y^3}}$. The value of the integral $\int_0^2 x\,f(x)\,dx$ is equal to –

(A) 1 (B) 1/3
(C) 4/3 (D) 2/3

Q.4 $\int \frac{\tan(\ln x)\tan\left(\ln\frac{x}{2}\right)\tan(\ln 2)}{x} dx =$

(A) $\ln\left(\frac{\sec(\ln x)}{\sec\left(\ln\frac{x}{2}\right)}\right) + C$ (B) $\ln(\sec\ln x) + C$

(C) $\ln\left(\sec\ln\left(\frac{x}{2}\right)x^{\tan(\ln x)}\right) + C$

(D) $\ln\left(\frac{\sec(\ln x)}{\sec\left(\ln\frac{x}{2}\right)x^{\tan(\ln 2)}}\right) + C$

Q.5 The value of the definite integral $\int_0^{\pi/3} \ln(1+\sqrt{3}\tan x)\,dx$ equals –

(A) $\frac{\pi}{3}\ln 2$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi^2}{6}\ln 2$ (D) $\frac{\pi^2}{2}\ln 2$

Q.6 The value of the definite integral $\int_0^{2(2+\sqrt{3})} \frac{16}{(4+x^2)^2} dx$ is equal to –

(A) $\frac{5+3\pi}{12}$ (B) $\frac{3+5\pi}{6}$ (C) $\frac{5+3\pi}{6}$ (D) $\frac{3+5\pi}{12}$

Q.7 $\int_{-\pi/2}^{\pi/2} \frac{\ln(\cos x)}{1+e^x.e^{\sin x}} dx =$

(A) $-2\pi\ln 2$ (B) $-\frac{\pi}{4}\ln 2$

(C) $-\pi\ln 2$ (D) $-\frac{\pi}{2}\ln 2$

Q.8 The equation of a curve is y = f (x). The tangents at (1, f (A),), (2, f (B)) and (3, f (C)) make angles $\pi/6$, $\pi/3$ and $\pi/4$ respectively with the positive direction of the x-axis.

Then the value of $\int_2^3 f'(x)\,f''(x)\,dx + \int_1^3 f''(x)\,dx =$

(A) $-1/\sqrt{3}$ (B) $1/\sqrt{3}$
(C) 0 (D) None of these

Q.9 Let $I = \int_0^{\pi/2} \frac{\cos x + 4}{3\sin x + 4\cos x + 25} dx$ and

$J = \int_0^{\pi/2} \frac{\sin x + 3}{3\sin x + 4\cos x + 25} dx$.

If $25I = a\pi + b \ln \frac{c}{d}$ where a, b, c and d $\in$ N and $\frac{c}{d}$ is not a perfect square of a rational then find the value of (a + b + c + d).

(A) 29 (B) 62
(C) 58 (D) 60

Q.10 If $a_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin x} dx$ then $a_2 - a_1, a_3 - a_2, a_4 - a_3$, ... are in –

(A) AP (B) GP
(C) HP (D) None of these

Q.11 The value of $\int_1^2 [f\{g(x)\}]^{-1} f'\{g(x).g'(x) dx$, where g (A) = g (B) is equal to –

(A) 0 (B) 1
(C) 2 (D) None of these

Q.12 Evaluate : $\int_0^{\pi} \frac{x\,dx}{a^2\cos^2 x + b^2 \sin^2 x}$

(A) π^2/ab (B) $\pi^2/2ab$
(C) $\pi^2/3ab$ (D) $\pi^2/5ab$

Q.13 If m any natural number, then the value of the integral $\int (x^{3m} + x^{2m} + x^m)(2x^{2m} + 3x^m + 6)^{1/m} dx$ is –

(A) $\frac{1}{6(m+1)} \{2x^{3m} + 3x^{2m} + 6x^m\}^{(1/m)+1} + C$

(B) $\frac{1}{6m} \{2x^{3m} + 3x^{2m} + 6x^m\}^{(1/m)+1} + C$

(C) $\frac{1}{6m} \{2x^{3m} + 3x^{2m} + 6x^m\}^{1/m} + C$

(D) None of these

Q.14 $\int \tan^{-1}\sqrt{x}\, dx$ is equal to –

(A) $x\tan^{-1}\sqrt{x} - \sqrt{x} + (\tan^{-1}\sqrt{x}) + c$

(B) $x\tan^{-1}\sqrt{x} + \sqrt{x} - \frac{1}{2}(\tan^{-1}\sqrt{x}) + c$

(C) $\tan^{-1}\sqrt{x} - \sqrt{x} + \frac{1}{2}(\tan^{-1}\sqrt{x}) + c$

(D) $x\tan^{-1}\sqrt{x} + \sqrt{x} - \frac{1}{2}(\tan^{-1}\sqrt{x}) + c$

Q.15 $\int_{2-\log 3}^{3+\log 3} \frac{\log(4+x)}{\log(4+x) + \log(9-x)} dx =$

(A) (1/2) + log 3 (B) 5/2
(C) 1 + 2 log 3 (D) None of these

Q.16 The greatest value of $f(x) = \int_1^x |t|\,dt$ on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ is –

(A) 3/8 (B) 1/2
(C) –3/8 (D) –1/2

Q.17 Find the value of $\int_{-1}^{1} \frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right) dx$

(A) –π/6 (B) –π/4
(C) – π/2 (D) – π/3

Q.18 Evaluate $\int_0^1 \frac{\log(1+x)}{1+x^2} dx$

(A) $\frac{\pi}{8}\log 4$ (B) $\frac{\pi}{8}\log 2$

(C) $\frac{\pi}{4}\log 2$ (D) None of these

Q.19 Evaluate $\int_{-2}^{2} [x^2]\, dx$

(A) $10 - 2\sqrt{3} - 2\sqrt{2}$ (B) $10 + 2\sqrt{3} - 2\sqrt{2}$
(C) $10 - \sqrt{3} - 3\sqrt{2}$ (D) $10 - 2\sqrt{3} + 2\sqrt{2}$

Q.20 Find the value of $\int_{\pi}^{2\pi} [2\sin x]\, dx$, where [ ] represents the greatest integer function.

(A) –π/6 (B) –π/4
(C) – π/2 (D) –5π/3

Q.21 $\int_{-1}^{1} [x[1 + \sin \pi x] + 1]\, dx$ is equal to –
(where [.] is G.I.F.)

(A) 0 (B) 1
(C) 2 (D) –2

Q.22 $\int_0^{5\pi/12} [\tan x]\, dx$, (where [.] dentotes G.I.F.) is equal to –

(A) π/2 (B) 5π/4
(C) –π/2 (D) –π/4

Q.23 If $f\left(\frac{3x-4}{3x+4}\right) = x+2$, then $\int f(x)\,dx$ is equal to –

(A) $\frac{2}{3}x + \log|1-x| + C$ (B) $\frac{2}{3}x - \frac{8}{3}\ln|1-x| + C$

(C) $\frac{2}{3}x + \frac{8}{3}\log|1-x| + C$ (D) None of these

Q.24 $\int_1^{2.5} x\,[2x]\,\text{sgn}(x-2)\,dx$ is equal to –

(where [.] is G.I.F. & Sgn → Signum function)

(A) –31/8 (B) 67/8

(C) 5/8 (D) 13/2

Q.25 Let $I_1 = \int_0^{\pi/2} \cos\theta.f(\sin\theta + \cos^2\theta)\,d\theta$ and

$I_2 = \int_0^{\pi/2} \sin 2\theta.f(\sin\theta + \cos^2\theta)\,d\theta$ then $\frac{I_1}{I_2}$ is equal to

(A) 1 (B) 2

(C) 3 (D) 4

Q.26 Let $f(x) = \int_0^{g(x)} \frac{dt}{\sqrt{1+t^2}}$ where $g(x) = \int_0^{\cos x} (1+\sin^2 t)\,dt$.

Also $h(x) = e^{-|x|}$ and $\ell(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

then $f'(\pi/2)$ is equal to –

(A) $\ell'(0)$ (B) $h'(0^-)$

(C) $h'(0^+)$ (D) $\lim_{x\to 0} \frac{1-\cos x}{x\sin x}$

Q.27 Evaluate

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{4n^2-1}} + \frac{1}{\sqrt{4n^2-4}} + \frac{1}{\sqrt{4n^2-9}} + \ldots\ldots \frac{1}{\sqrt{3n^2}}\right].$$

(A) $\pi/6$ (B) $\pi/3$

(C) $\pi/2$ (D) $\pi/4$

Q.28 If $m = \int_{-2}^{0} \frac{|\sin x|}{\left[\frac{x}{\pi}\right] + \frac{1}{2}}\,dx$ and $n = \int_{0}^{2} \frac{|\sin x|}{\left[\frac{x}{\pi}\right] + \frac{1}{2}}\,dx$, where [.] represents greatest integer function, then –

(A) $m = n$ (B) $m = -n$

(C) $m = 2n$ (D) $m = -2n$

Q.29 $\int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt$ is equal to

(A) $\pi/2$ (B) 1

(C) $\pi/4$ (D) None of these

Q.30 $\int_0^2 \sqrt{x + \sqrt{x + \sqrt{x + \ldots\ldots\infty}}}\,dx$ is equal to $(x > 0)$

(A) 19/6 (B) 17/6

(C) 13/6 (D) Can't determine

Q.31 Evaluate $\int_0^{\infty} \frac{x\log x}{(1+x^2)^2}\,dx$

(A) 1/3 (B) 2

(C) 0 (D) 3

Q.32 Evaluate: $\int \frac{(x-1)\,dx}{(2x+1)(x-2)(x-3)}$

(A) $\frac{3}{35}\ln|2x+1| - \frac{1}{5}\ln|x-2| + \frac{2}{7}\ln|x-3| + C$

(B) $\frac{3}{35}\ln|2x+1| - \frac{1}{5}\ln|x+2| + \frac{2}{7}\ln|x-3| + C$

(C) $\frac{3}{35}\ln|2x+1| + \frac{1}{5}\ln|x+2| + \frac{2}{7}\ln|x-3| + C$

(D) None of these

Q.33 The value of the definite integral

$\int_1^e \{(1+x)\,e^x + (1-x)\,e^{-x}\}.\ln x\,dx$, is equal to –

(A) $e^{1+e} + e^{1-e} + e^{-e} - e^e + e + e^{-1}$

(B) $e^{1+e} - e^{1-e} + e^{-e} - e^e + e - e^{-1}$

(C) $e^{1+e} + e^{1-e} + e^{-e} - e^e - e - e^{-1}$

(D) $e^{1+e} - e^{1-e} + e^{-e} - e^e - e + e^{-1}$

Q.34 Let $I_n = \int_{-n}^{n} [\{x+1\}.\{x^2+2\} + \{x^2+2\}\{x^3+4\}]\,dx$

where { } denotes the fractional part of x, then I_1 has the value equal to –

(A) 1/3 (B) 2/3

(C) 0 (D) 4/3

Q.35 Evaluate: $\int \frac{(5\sin x + 6)\,dx}{\sin x + 2\cos x + 3}$.

(A) $x - 2\ell n|\sin x + 2\cos x + 3| + 3\tan^{-1}\left(\frac{1+\tan(x/2)}{2}\right) + C$

(B) $x - 2\ell n|\sin x + \cos x - 3| + 3\tan^{-1}\left(\frac{1+\tan(x/2)}{2}\right) + C$

(C) $x - 2\ell n|\sin x - \cos x + 2| + 2\tan^{-1}\left(\frac{1+\tan(x/2)}{2}\right) + C$

(D) None of these

Q.36 If $\int \frac{dx}{x^{22}(x^7-6)} = A\{\ln(p)^6 + 9p^2 - 2p^3 - 18p\} + c$, then

(A) $A=\frac{1}{9072}, p=\left(\frac{x^7-6}{x^7}\right)$ (B) $A=\frac{1}{54432}, p=\left(\frac{x^7-6}{x^7}\right)$

(C) $A=\frac{1}{54432}, p=\left(\frac{x^7}{x^7-6}\right)$ (D) $A=\frac{1}{9072}, p=\left(\frac{x^7-6}{x^7}\right)^{-1}$

Q.37 Evaluate $\int \frac{dx}{\sqrt[3]{x+1}+\sqrt{x+1}}$

(A) $6\left(\frac{(1+x)^{1/2}}{2}-\frac{(1+x)^{2/3}}{3}+(1+x)^{1/6}-\log\left((x+1)^{1/6}+1\right)\right)+C$

(B) $6\left(\frac{(1+x)^{1/2}}{3}-\frac{(1+x)^{1/3}}{2}+(1+x)^{1/6}+\log\left((x+1)^{1/3}+1\right)\right)+C$

(C) $6\left(\frac{(1+x)^{1/2}}{3}-\frac{(1+x)^{1/3}}{2}+(1+x)^{1/3}+\log\left((x+1)^{1/6}-1\right)\right)+C$

(D) $6\left(\frac{(1+x)^{1/2}}{3}-\frac{(1+x)^{1/3}}{2}+(1+x)^{1/6}-\log\left((x+1)^{1/6}+1\right)\right)+C$

Q.38 If $\int_{-\infty}^{\infty}\left(\frac{1}{e}\right)^{x^2} dx = \sqrt{\pi}$ then

$$\int_{-\infty}^{\infty}\frac{1}{(1+e^x)\sqrt[4]{e^{x^2}\sqrt{e^{2x^2}\sqrt{e^{3x^2}\,\ldots\ldots\infty}}}}dx =$$

(A) $\sqrt{\pi}$ (B) $\frac{\sqrt{\pi}}{2}$ (C) $\frac{\sqrt{\pi}}{4}$ (D) $\frac{\sqrt{\pi}}{8}$

Q.39 If $\int_0^{2\pi} \ln(\sec^2 x)\,dx = 2k^2 \sin^{-1}\left(\frac{k}{2}\right)\ln k$, then find the value of [k], (where [.] denotes greatest integer function).

(A) 2 (B) 3
(C) 4 (D) 1

Q.40 If the value of $\int_0^{100\pi}([\cot^{-1}x]+[\tan^{-1}x])\,dx$ is $100\pi + p\cot p$, then the value of p is (where [.] denotes greatest integer function).

(A) 2 (B) 3
(C) 4 (D) 1

Q.41 The absolute value of $\frac{\int_0^{\pi/2}(x\cos x+1)e^{\sin x}dx}{\int_0^{\pi/2}(x\sin x-1)e^{\cos x}dx} =$

(A) e (B) πe
(C) e/2 (D) π/e

Q.42 The value of definite integral

$\int_0^{\pi}\frac{x|\sin x|}{1+|\cos x|}dx$ is equal to $\frac{\pi}{A}\ln 2$. Find the value of A.

(A) 1 (B) 7
(C) 4 (D) 2

Q.43 If the value of the definite integral

$$\int_{-1}^{1}\cot^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)\cdot\left(\cot^{-1}\frac{x}{\sqrt{1-(x^2)^{|x|}}}\right)dx = \frac{\pi^2(\sqrt{a}-\sqrt{b})}{\sqrt{c}}$$

where a, b, c $\in$ N in their lowest form, then find the value of (a + b + c).

(A) 1 (B) 7
(C) 4 (D) 2

Q.44 If $y^2 = x^2 - x + 1$ and $I_n = \int \frac{x^n}{y}dx$ and $AI_3 + BI_2 + CI_1 = x^2y$ then ordered triplet A, B, C is –

(A) (1/2, –1/2, 1) (B) (3, 1, 0)
(C) (1, –1, 2) (D) (3, –5/2, 2)

Q.45 If f (x) is a function satisfying $f\left(\frac{1}{x}\right)+x^2 f(x)=0$ for all non-zero x, then find $\int_{\sin\theta}^{\operatorname{cosec}\theta} f(x)\,dx$.

(A) 1 (B) 2
(C) 0 (D) 3

Q.46 The value of $\int_{1/e}^{\tan x}\frac{t\,dt}{1+t^2}+\int_{1/e}^{\cot x}\frac{dt}{t(1+t^2)}$ is equal to

(A) 1 (B) 1/2
(C) π/4 (D) none of these

Q.47 If $\int\frac{x^3-6x^2+11x-6}{\sqrt{x^2+4x+3}}dx$

$= (Ax^2+Bx+C)\sqrt{x^2+4x+3}+\lambda\int\frac{dx}{\sqrt{x^2+4x+3}}$

then value of A is

(A) 1/3 (B) 1
(C) 3 (D) – 1/3

Q.48 In above question value of C is –

(A) – 37 (B) – 14/3
(C) 14/3 (D) 37

Q.49 In above question value of λ is –

(A) 66 (B) – 66
(C) 37/3 (D) – 37/3

Q.50 $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{{}^nC_k}{n^k(k+3)}$ is equal to –

(A) e (B) e + 2
(C) 0 (D) e – 2

Q.51 If f (x) be a real valued function ;
$f(x) + f(x+4) = f(x+2) + f(x+6)$;

$g(x) = \int_{x}^{x+8} f(t)\,dt$ then $g'(x)$ is equal to –

(A) f (x) (B) f (x + 8)
(C) 8 (D) 0

Q.52 If $f(x) = \int_0^x (\sin^4 t + \cos^4 t)\,dt$, then $f(x+\pi)$ will be equal to

(A) $f(x) + f(\pi/2)$ (B) $f(x) + 2f(\pi/2)$
(C) $f(x) + f(\pi)$ (D) (B) & (C) both

Q.53 $\int \sin 4x . e^{\tan^2 x} dx$ is equal to –

(A) $e^{\tan^2 x} + C$ (B) $-2\cos^4 x . e^{\tan^2 x} + C$

(C) $-2\cos^2 x . e^{\tan^2 x} + C$ (D) $-2\sec^4 x . e^{\tan^2 x} + C$

Directions : Assertion-Reason type questions.

Each questions contain Statement -1 (Assertion) and Statement -2 (Reason). Each question has 4 choices (A), (B), (C) and (D) out of which ONLY ONE is correct.

(A) Statement-1 is True,Statement-2 is True,Statement-2 is a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True;Statement-2 is NOT a correct explanation for Statement-1.
(C) Statement - 1 is True, Statement- 2 is False.
(D) Statement -1 is False, Statement -2 is True.

Q.54 Consider the function $F(x) = \int \frac{x}{(x-1)(x^2+1)} dx$

Statement 1 : F (x) is discontinuous at x = 1
Statement 2 : Integrand of F (x) is discontinuous at x = 1.

Q.55 **Statement -1 :** $\int_0^{\pi/2} \frac{\sin x}{x} dx < \frac{\pi}{2}$

Statement -2 : $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Q.56 **Statement -1 :** $\int 2^{\tan^{-1} x} d(\cot^{-1} x) = \frac{2^{\tan^{-1} x}}{\ln 2} + C$

Statement -2 : $\frac{d}{dx}(a^x + c) = a^x \ln a$

Q.57 **Statement–1:** $\int_0^{\pi/2} (\sin^6 x + \cos^6 x)\,dx$ lies in the interval $(\pi/8, \pi/2)$.

Statement–2 : $(\sin^6 x + \cos^6 x)$ is periodic with period $\pi/2$.

Q.58 **Statement 1 :** If $y = \sin^{-1} x$, then

$\int \sin^{-1} x\,dx = \int y \cos y\,dy + c$

Statement 2 : If $y = f^{-1} x$, then

$\int f^{-1} x\,dx = \int y\, f^{-1}(y)\,dy + c$

Q.59 **Statement 1 :**

$$\int e^{\sin^{-1} x}\left(1 - \frac{x}{\sqrt{1-x^2}}\right) dx = e^{\sin^{-1} x}\sqrt{1-x^2} + c$$

Statement 2 :

$$\int e^{g(x)}(g'(x) f(x) + f'(x))\,dx = e^{g(x)} f(x) + c$$

Q.60 **Statement 1 :** $\int_0^n \{x\}\,dx = \frac{n}{2}$, where {.} represents fractional part function and $n \in N$.

Statement 2 : $\int_0^n [x]\,dx = \frac{n(n-1)}{2}$, where [.] represents greatest integer function and $n \in N$.

Q.61 **Statement 1 :** $\int_0^{\pi} x \sin x \cos^2 x\,dx = \frac{\pi}{2}\int_0^{\pi} \sin x \cos^2 x\,dx$

Statement 2 : $\int_a^b x\, f(x)\,dx = \frac{a+b}{2}\int_a^b f(x)\,dx$

Passage (Q.62-Q.64)

Let $A = \int_0^1 \frac{e^x}{x+1} dx$ then answer the following questions in terms of A.

Q.62 $\int_0^1 \frac{x^2 e^x}{x+1} dx$ equals –

(A) A – e (B) e – 2 + A
(C) 2 + A (D) 2 – e + A

Q.63 $\int_0^1 \left(\frac{x}{x+1}\right)^2 e^x\,dx$ equals –

(A) $A - \frac{e}{2}$ (B) $\frac{e}{2} + 1 - A$ (C) $\frac{e}{2} - A$ (D) $A - 1 + \frac{e}{2}$

Q.64 $\int_0^1 \frac{x^3}{(x+1)^3} e^x\,dx$ equals –

(A) $\frac{e}{2} - 2 + A$ (B) $3A - \frac{3e}{2}$

(C) $3 - \frac{A}{2} - \frac{9e}{8}$ (D) None of these

Passage (Q.65-Q.67)

Consider the integral

$$I = \int_0^{10\pi} \frac{\cos 6x.\cos 7x.\cos 8x.\cos 9x}{1+e^{2\sin^3 4x}}\,dx$$

Q.65 If $I = k\int_0^{\pi/2} \cos 6x.\cos 7x.\cos 8x.\cos 9x\,dx$ then k =

(A) 1 (B) 5
(C) 10 (D) 20

Q.66 If $I = c\int_0^{\pi/4} \cos 6x.\cos 8x.\cos 2x\,dx$ then c equals –

(A) 5 (B) 10
(C) 20 (D) 5/2

Q.67 The value of I equals –

(A) $5\pi/4$ (B) $5\pi/16$
(C) $5\pi/32$ (D) $5\pi/8$

Q.68 Let $a_n = \int_0^{\pi/2} (1-\sin t)^n \sin 2t\,dt$ then $\lim_{n\to\infty}\sum_{n=1}^{n}\frac{a_n}{n}$ is equal to –

(A) 1/2 (B) 1
(C) 4/3 (D) 3/2

Q.69 Let a, b are real number such that a + b = 1 then the minimum value of the integral $\int_0^{\pi}(a\sin x + b\sin 2x)^2 dx$ is equal to

(A) $\pi/2$ (B) $\pi/4$
(C) $\pi/8$ (D) $3\pi/4$

Q.70 Evaluate: $\int \frac{dx}{\left(x+\sqrt{x^2-4}\right)^{5/3}}$.

(A) $\frac{1}{4}t^{-8/3}[1-t^2]+c$ (B) $\frac{3}{4}t^{-5/3}[1-t^2]+c$

(C) $\frac{3}{4}t^{-8/3}[1-t]+c$ (D) $\frac{3}{4}t^{-8/3}[1-t^2]+c$

Q.71 Evaluate $\int \frac{x-1}{(x+1)(x-2)}\,dx$

(A) $\frac{2}{3}\log|x+1|+\frac{1}{3}\log|x-2|+C$

(B) $\frac{1}{3}\log|x-1|+\frac{1}{3}\log|x-2|+C$

(C) $\frac{2}{3}\log|x^2+1|+\frac{1}{3}\log|x-2|+C$

(D) $\frac{2}{3}\log|x+1|+\frac{1}{3}\log|x^2+2|+C$

Q.72 Evaluate $\int_0^{\pi/4}\log(1+\tan x)dx$

(A) $\frac{\pi}{8}\log 2$ (B) $\log 2$

(C) $\frac{\pi}{8}\log 3$ (D) $\frac{\pi}{4}\log 2$

Q.73 Evaluate $\int_{\pi/6}^{\pi/3}\frac{1}{1+\sqrt{\cot x}}dx$

(A) $\pi/6$ (B) $\pi/8$
(C) $\pi/12$ (D) π

Q.74 Evaluate $\lim_{n\to\infty}\left[\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\dots\dots\dots+\frac{1}{64n}\right]$.

(A) 15/32 (B) 1/32
(C) 32/15 (D) 19/32

Q.75 Evaluate $\int_0^1 |5x-3|\,dx$

(A) 13/10 (B) 1/10
(C) 3/10 (D) 8/10

Q.76 $\int_{-\pi}^{\pi}(\cos px-\sin qx)^2\,dx$ where p, q are integers is equal to

(A) $-\pi$ (B) 0
(C) π (D) 2π

Q.77 If $S_n = \frac{1}{2n}+\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-4}}+\dots\dots+\frac{1}{\sqrt{3n^2+2n-1}}$, $n\in N$, then $\lim_{n\to\infty}S_n$ is equal to –

(A) $\pi/2$ (B) 2
(C) 1 (D) $\pi/6$

Q.78 Evaluate $\int_{-2}^{3}|x^2-1|\,dx$.

(A) 7/3 (B) 14/3
(C) 28/3 (D) 1/3

Q.79 The points of maxima of $f(x) = \int_0^{x^2-1} t\,(t-1)\,(t-3)^3 dt$ are

(A) $-\sqrt{2}, 0, \sqrt{2}$ (B) $-2, \sqrt{2}, -1$
(C) $-2, -1, 1, 2$ (D) 0, 1

Q.80 Evaluate : $\int \frac{dx}{x^3+1}$

(A) $\frac{1}{3}\log\left|\frac{x+1}{\sqrt{x^2-x+1}}\right|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$

(B) $\frac{1}{3}\log\left|\frac{x+1}{\sqrt{x^2+x+1}}\right|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$

(C) $\frac{1}{2}\log\left|\frac{x+1}{\sqrt{x^2-x+1}}\right|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right)+C$

(D) $\frac{1}{5}\log\left|\frac{x-1}{\sqrt{x^2-x+1}}\right|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right)+C$

Q.81 $\lim_{x\to 0}\left(\int_0^1 (by+a(1-y))^x\,dy\right)^{1/x}$ equals $(b \neq a)$ –

(A) $e^{-1}.\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$ (B) $e^{-1}.\left(\frac{a^b}{b^b}\right)^{\frac{1}{b-a}}$

(C) $e.\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$ (D) $e.\left(\frac{a^a}{b^b}\right)^{\frac{1}{b-a}}$

NOTE : The answer to each question is a NUMERICAL VALUE.

Q.82 If the value of the definite integral

$\int_0^1 {}^{207}C_7 x^{200}(1-x)^7\,dx$ is equal to $\frac{1}{k}$ where $k \in N$.

Find k.

Q.83 Consider a polynomial P (x) of the least degree that has a maximum equal to 6 at x = 1, and a minimum equal to 2 at x = 3. Compute the value of P (2) + P' (0).

Q.84 Let F (x) be a non-negative continuous function defined on R such that $F(x) + F\left(x+\frac{1}{2}\right) = 3$. Find the value of

$\int_0^{1500} F(x)\,dx$.

Q.85 If $\int_0^{\pi/2}\frac{1-\sin 2x}{(1+\sin 2x)^2}dx = \frac{a}{b}$ where a, b are relatively prime find a + b + ab.

Q.86 For $a \ge 2$, if the value of the definite integral

$\int_0^{\infty}\frac{dx}{a^2+\left(x-\frac{1}{x}\right)^2}$ equals $\frac{\pi}{5050}$. Find the value of a.

Q.87 If $\int_{-\pi/4}^{\pi/4}\frac{(\pi-4\theta)\tan\theta}{1-\tan\theta}d\theta = \pi\ln k - \frac{\pi^2}{w}$,

find the value of (kw), where $k, w \in N$.

Q.88 Given a function g, continuous everywhere such that

$g(1) = 5$ and $\int_0^1 g(t)\,dt = 2$. If $f(x) = \frac{1}{2}\int_0^x (x-t)^2\, g(t)\,dt$,

then compute the value of $f'''(1) - f''(1)$.

Q.89 If $f(x) = \begin{cases} e^{\cos x}.\sin x & \text{for} |x| \le 2 \\ 2 & \text{otherwise}\end{cases}$. Then $\int_{-2}^{3} f(x)dx$:

Q.90 The value of $\int_{-2}^{0}[x^3+3x^2+3x+3+(x+1)\cos(x+1)]dx$ is

Q.91 If $\int_{\sin x}^{1} t^2 f(t)dt = 1-\sin x\ \forall x \in [0, \pi/2)$, then $f(1/\sqrt{3})$ is

Q.92 If $\lim_{t\to a}\frac{\int_a^t f(x)dx - \frac{t-a}{2}(f(t)+f(a))}{(t-a)^3} = 0$,

then degree of polynomial function f(x) is

Q.93 Let $f : R \to R$ be a continuous function which satisfies

$f(x) = \int_0^x f(t)\,dt$. Then the value of f (ln 5) is

Q.94 For any real number, let [x] denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval [– 10, 10] by $f(x) = \begin{cases} x-[x], & \text{if } [x] \text{ is odd}, \\ 1+[x]-x, & \text{if } [x] \text{ is even}\end{cases}$.

Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x\,dx$ is

Q.95 The value of $\int_0^1 4x^3\left\{\frac{d^2}{dx^2}(1-x^2)^5\right\}dx$ is –

Q.96 Let $f : R \to R$ be a function defined by $f(x) = \begin{cases}[x], & x \le 2 \\ 0, & x > 2\end{cases}$

where [x] is the greatest integer less than or equal to x.

If $I = \int_{-1}^{2}\frac{xf(x^2)}{2+f(x+1)}dx$, then the value of (4I – 1) is –

Q.97 If $\alpha = \int_0^1 (e^{9x+3\tan^{-1}x})\left(\frac{12+9x^2}{1+x^2}\right)dx$, where $\tan^{-1}x$ takes only principal values, then the value of

$\left(\log_e|1+\alpha| - \frac{3\pi}{4}\right)$ is

EXERCISE - 3 [PREVIOUS YEARS JEE MAIN QUESTIONS]

Q.1 $\int \frac{\cos 2x - 1}{\cos 2x + 1} dx =$ **[AIEEE 2002]**

(A) $\tan x - x + c$ (B) $x + \tan x + c$
(C) $x - \tan x + c$ (D) $-x - \cot x + c$

Q.2 $\int \frac{(\log x)}{x^2} dx =$ **[AIEEE 2002]**

(A) $\frac{1}{2}(\log x + 1) + c$ (B) $-\frac{1}{x}(\log x + 1) + c$ (C) $\frac{1}{x}(\log x - 1) + c$ (D) $\log(x + 1) + c$

Q.3 If $I_n = \int_0^{\pi/4} \tan^n x \, dx$ then the value of $n(I_{n-1} + I_{n+1})$ is- **[AIEEE-2002]**

(A) 1 (B) $\pi/2$
(C) $\pi/4$ (D) n

Q.4 $\int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x} =$ **[AIEEE-2002]**

(A) π^2 (B) $\pi^2/4$
(C) $\pi/8$ (D) $\pi^2/8$

Q.5 $\int_{\pi}^{10\pi} |\sin x| \, dx =$ **[AIEEE-2002]**

(A) 9 (B) 10
(C) 18 (D) 20

Q.6 $\int_0^{\sqrt{2}} [x^2] dx =$ **[AIEEE-2002]**

(A) $\sqrt{2} - 1$ (B) $2(\sqrt{2} - 1)$
(C) $\sqrt{2}$ (D) None of these

Q.7 $\lim_{n\to\infty} \frac{1^P + 2^P + 3^P + ... + n^P}{n^{P+1}}$ equals- **[AIEEE 2002]**

(A) 1 (B) $\frac{1}{P+1}$ (C) $\frac{1}{P+2}$ (D) P^2

Q.8 Let $\frac{d}{dx} F(x) = \left(\frac{e^{\sin x}}{x}\right)$, $x > 0$. If $\int_1^4 \frac{3}{x} e^{\sin x^3} dx = F(k) - F(1)$, then one of the possible values of k, is- **[AIEEE 2003]**

(A) 64 (B) 15
(C) 16 (D) 63

Q.9 If $f(a + b - x) = f(x)$, then $\int_a^b x \, f(x) \, dx =$ **[AIEEE 2003]**

(A) $\frac{a+b}{2} \int_a^b f(a + b - x) \, dx$ (B) $\frac{a+b}{2} \int_a^b f(b - x) \, dx$

(C) $\frac{a+b}{2} \int_a^b f(x) \, dx$ (D) $\frac{b-a}{2} \int_a^b f(x) \, dx$

Q.10 The value of the integral $I = \int_0^1 x(1 - x)^n \, dx$ is **[AIEEE 2003]**

(A) $\frac{1}{n+1} + \frac{1}{n+2}$ (B) $\frac{1}{n+1}$

(C) $\frac{1}{n+2}$ (D) $\frac{1}{n+1} - \frac{1}{n+2}$

Q.11 The value of $\lim_{x\to 0} \frac{\int_0^{x^2} \sec^2 t \, dt}{x \sin x}$ is- **[AIEEE 2003]**

(A) 0 (B) 3
(C) 2 (D) 1

Q.12 $\lim_{n\to\infty} \frac{1 + 2^4 + 3^4 + ... + n^4}{n^5}$; $\lim_{n\to\infty} \frac{1 + 2^3 + 3^3 + ... + n^3}{n^5}$ is equal to- **[AIEEE 2003]**

(A) 1/5 (B) 1/30
(C) zero (D) 1/4

Q.13 If $f(y) = e^y$, $g(y) = y$; $y > 0$ and $F(t) = \int_0^t f(t - y) \, g(y) \, dy$, then **[AIEEE 2003]**

(A) $F(t) = t\,e^{-t}$ (B) $F(t) = 1 - e^{-1}(1 + t)$
(C) $F(t) = e^t - (1 + t)$ (D) $F(t) = t\,e^t$

Q.14 Let f(x) be a function satisfying $f'(x) = f(x)$ with $f(0) = 1$ and g(x) be a function that satisfies $f(x) + g(x) = x^2$. Then the value of the integral $\int_0^1 f(x) \, g(x) \, dx$, is **[AIEEE 2003]**

(A) $e + \frac{e^2}{2} + \frac{5}{2}$ (B) $e - \frac{e^2}{2} - \frac{5}{2}$

(C) $e + \frac{e^2}{2} - \frac{3}{2}$ (D) $e - \frac{e^2}{2} - \frac{3}{2}$

Q.15 If $\int \frac{\sin x}{\sin(x - \alpha)} dx = Ax + B \log \sin(x - \alpha) + C$, then value of (A,B) is- **[AIEEE 2004]**

(A) $(\sin\alpha, \cos\alpha)$ (B) $(\cos\alpha, \sin\alpha)$
(C) $(-\sin\alpha, \cos\alpha)$ (D) $(-\cos\alpha, \sin\alpha)$

Q.16 $\int \frac{dx}{\cos x - \sin x}$ is equal to- **[AIEEE 2004]**

(A) $\frac{1}{\sqrt{2}} \log \left|\tan\left(\frac{x}{2} - \frac{\pi}{8}\right)\right| + C$ (B) $\frac{1}{\sqrt{2}} \log \left|\cot\left(\frac{x}{2}\right)\right| + C$

(C) $\frac{1}{\sqrt{2}} \log \left|\tan\left(\frac{x}{2} - \frac{3\pi}{8}\right)\right| + C$ (D) $\frac{1}{\sqrt{2}} \log \left|\tan\left(\frac{x}{2} + \frac{3\pi}{8}\right)\right| + C$

Q.17 $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} e^{\frac{r}{n}}$ is- **[AIEEE 2004]**

(A) e (B) e – 1
(C) 1 – e (D) e + 1

Q.18 The value of $\int_{-2}^{3} |1 - x^2|\, dx$ is- **[AIEEE 2004]**

(A) 28/3 (B) 14/3
(C) 7/3 (D) 1/3

Q.19 The value of $I = \int_0^{\pi/2} \frac{(\sin x + \cos x)^2}{\sqrt{1 + \sin 2x}} dx$ is- **[AIEEE 2004]**

(A) 0 (B) 1
(C) 2 (D) 3

Q.20 If $\int_0^{\pi} x\, f(\sin x)\, dx = A \int_0^{\pi/2} f(\sin x)\, dx$, then A is-

(A) 0 (B) π **[AIEEE 2004]**
(C) $\pi/4$ (D) 2π

Q.21 If $f(x) = \frac{e^x}{1 + e^x}$, $I_1 = \int_{f(-a)}^{f(a)} x\, g\{x(1 - x)\}\, dx$ and

$I_2 = \int_{f(-a)}^{f(a)} g\{x(1 - x)\}\, dx$, then the value of $\frac{I_2}{I_1}$ is-

(A) 2 (B) – 3 **[AIEEE 2004]**
(C) – 1 (D) 1

Q.22 $\int \left\{\frac{(\log x - 1)}{1 + (\log x)^2}\right\}^2 dx$ is equal to - **[AIEEE-2005]**

(A) $\frac{\log x}{(\log x)^2 + 1} + C$ (B) $\frac{x}{x^2 + 1} + C$

(C) $\frac{xe^x}{1 + x^2} + C$ (D) $\frac{x}{(\log x)^2 + 1} + C$

Q.23 $\lim_{n\to\infty} \left[\frac{1}{n^2}\sec^2\frac{1}{n^2} + \frac{2}{n^2}\sec^2\frac{4}{n^2} + \ldots.. + \frac{1}{n}\sec^2 1\right]$ equals

[AIEEE-2005]

(A) $\frac{1}{2}\sec 1$ (B) $\frac{1}{2}\operatorname{cosec} 1$
(C) tan 1 (D) (1/2) tan 1

Q.24 If $I_1 = \int_0^1 2^{x^2} dx$, $I_2 = \int_0^1 2^{x^3} dx$, $I_3 = \int_1^2 2^{x^2} dx$

& $I_4 = \int_1^2 2^{x^3} dx$ then - **[AIEEE-2005]**

(A) $I_2 > I_1$ (B) $I_1 > I_2$
(C) $I_3 = I_4$ (D) $I_3 > I_4$

Q.25 Let $f : R \to R$ be a differentiable function having f(2) = 6,

$f'(2) = \left(\frac{1}{48}\right)$. Then $\underset{x\to 2}{\text{Lt}} \int_6^{f(x)} \frac{4t^3}{x - 2} dt =$ **[AIEEE-2005]**

(A) 24 (B) 36
(C) 12 (D) 18

Q.26 The value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1 + a^x} dx$, a > 0, is - **[AIEEE-2005]**

(A) $a\pi$ (B) $\pi/2$
(C) π/a (D) 2π

Q.27 The value of the integral, $\int_3^6 \frac{\sqrt{x}}{\sqrt{9 - x} + \sqrt{x}} dx$ is –

(A) 3/2 (B) 2 **[AIEEE 2006]**
(C) 1 (D) 1/2

Q.28 $\int_{-3\pi/2}^{-\pi/2} [(x + \pi)^3 + \cos^2(x + 3\pi)]\, dx$ is equal to –

(A) $(\pi^4/32) + (\pi/2)$ (B) $\pi/2$ **[AIEEE 2006]**
(C) $(\pi/4) - 1$ (D) $\pi^4/32$

Q.29 $\int_0^{\pi} x\, f(\sin x)\, dx$ is equal to– **[AIEEE 2006]**

(A) $\pi \int_0^{\pi} f(\sin x)\, dx$ (B) $\frac{\pi}{2} \int_0^{\pi/2} f(\sin x)\, dx$

(C) $\pi \int_0^{\pi/2} f(\cos x)\, dx$ (D) $\pi \int_0^{\pi} f(\cos x)\, dx$

Q.30 The value of $\int_1^a [x]\, f'(x)\, dx$, a > 1, where [x] denotes the greatest integer not exceeding x is – **[AIEEE 2006]**

(A) [a] f(a) – {f(1) + f(2) + ... + f([a])}
(B) [a] f([a]) – {f(1) + f(2) + ... + f(a)}
(C) a f([a]) – {f(1) + f(2) + ... + f(a)}
(D) a f(a) – {f(1) + f(2) + ... + f([a])}

Q.31 Let F(x) = f(x) + $f\left(\frac{1}{x}\right)$, where $f(x)=\int_1^x \frac{\log t}{1+t}dt$. Then F(e) equals **[AIEEE 2007]**
(A) 1/2 (B) 0
(C) 1 (D) 2

Q.32 The solution for x of the equation $\int_{\sqrt{2}}^{x}\frac{dt}{t\sqrt{t^2-1}}=\frac{\pi}{12}$ is **[AIEEE 2007]**
(A) 2 (B) π
(C) $\sqrt{3}/2$ (D) $2\sqrt{2}$

Q.33 $\int \frac{dx}{\cos x+\sqrt{3}\sin x}$ equals **[AIEEE 2007]**
(A) $\frac{1}{2}\log\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)+C$ (B) $\frac{1}{2}\log\tan\left(\frac{x}{2}-\frac{\pi}{12}\right)+C$
(C) $\log\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)+C$ (D) $\log\tan\left(\frac{x}{2}-\frac{\pi}{12}\right)+C$

Q.34 The value of $\sqrt{2}\int \frac{\sin x\,dx}{\sin\left(x-\frac{\pi}{4}\right)}$ is - **[AIEEE 2008]**
(A) $x-\log|\sin(x-\frac{\pi}{4})|+c$ (B) $x+\log|\sin(x-\frac{\pi}{4})|+c$
(C) $x-\log|\cos(x-\frac{\pi}{4})|+c$ (D) $x+\log|\cos(x-\frac{\pi}{4})|+c$

Q.35 Let I = $\int_0^1 \frac{\sin x}{\sqrt{x}}dx$ and J = $\int_0^1 \frac{\cos x}{\sqrt{x}}dx$. Then which one of the following is true ? **[AIEEE 2008]**
(A) I < $\frac{2}{3}$ and J < 2 (B) I < $\frac{2}{3}$ and J > 2
(C) I > 2/3 and J < 2 (D) I > 2/3 and J > 2

Q.36 $\int_0^{\pi}[\cot x]\,dx$ where [.] denotes the greatest integer function, is equal to **[AIEEE 2009]**
(A) $\pi/2$ (B) 1
(C) –1 (D) $-\pi/2$

Q.37 Let p (x) be a function defined on R such that $p'(x)=p'(1-x)$, for all $x\in[0,1]$, p(0) = 1 and p (1) = 41.
Then $\int_0^1 p(x)\,dx$ equals – **[AIEEE 2010]**
(A) 21 (B) 41
(C) 42 (D) 41

Q.38 The value of $\int_0^1 \frac{8\log(1+x)}{1+x^2}dx$ is – **[AIEEE 2011]**
(A) $\pi\log 2$ (B) $\frac{\pi}{8}\log 2$
(C) $\frac{\pi}{2}\log 2$ (D) log 2

Q.39 If $\frac{dy}{dx}=y+3>0$ and y (0) = 2, then y (ln 2) is equal to – **[AIEEE 2011]**
(A) 7 (B) 5
(C) 13 (D) –2

Q.40 If the integral $\int\frac{5\tan x}{\tan x-2}dx = x + a\ln|\sin x-2\cos x|+k$, then a is equal to – **[AIEEE 2012]**
(A) –1 (B) –2
(C) 1 (D) 2

Q.41 If $g(x)=\int_0^x \cos 4t\,dt$, then g (x + π) equals **[AIEEE 2012]**
(A) $\frac{g(x)}{g(\pi)}$ (B) $g(x)+g(\pi)$
(C) $g(x)-g(\pi)$ (D) $g(x)\,.\,g(\pi)$

Q.42 If $\int f(x)\,dx=\psi(x)$, then $\int x^5 f(x^3)\,dx$ is equal to – **[JEE MAIN 2013]**
(A) $\frac{1}{3}\left[x^3\psi(x^3)-\int x^2\psi(x^3)\,dx\right]+C$
(B) $\frac{1}{3}x^3\psi(x^3)-3\int x^3\psi(x^3)\,dx+C$
(C) $\frac{1}{3}x^3\psi(x^3)-\int x^2\psi(x^3)\,dx+C$
(D) $\frac{1}{3}\left[x^3\psi(x^3)-\int x^3\psi(x^3)\,dx\right]+C$

Q.43 **Statement-I :** The value of the integral $\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}$ is equal to $\pi/6$.
Statement-II : $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$ **[JEE MAIN 2013]**
(A) Statement-I is true; Statement-II is true; Statement-II is a correct explanation for Statement-I.
(B) Statement-I is true; Statement-II is true; Statement-II is not a correct explanation for Statement-I.
(C) Statement-I is true; Statement-II is false.
(D) Statement-I is false; Statement-II is true.

Q.44 The integral $\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ equals – **[JEE MAIN 2014]**

(A) $\pi-4$ (B) $\frac{2\pi}{3}-4-4\sqrt{3}$

(C) $4\sqrt{3}-4$ (D) $4\sqrt{3}-4-\frac{\pi}{3}$

Q.45 The integral $\int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx$ is equal to – **[JEE MAIN 2014]**

(A) $(x-1)\,e^{x+\frac{1}{x}}+c$ (B) $x\,e^{x+\frac{1}{x}}+c$

(C) $(x+1)\,e^{x+\frac{1}{x}}+c$ (D) $-xe^{x+\frac{1}{x}}+c$

Q.46 The integral $\int\frac{dx}{x^2(x^4+1)^{3/4}}=$ **[JEE MAIN 2015]**

(A) $(x^4+1)^{1/4}+c$ (B) $-(x^4+1)^{1/4}+c$

(C) $-\left(\frac{x^4+1}{x^4}\right)^{1/4}+c$ (D) $\left(\frac{x^4+1}{x^4}\right)^{1/4}+c$

Q.47 The integral $\int_2^4\frac{\log x^2}{\log x^2+\log(36-12x+x^2)}dx=$ **[JEE MAIN 2015]**

(A) 4 (B) 1
(C) 6 (D) 2

Q.48 The integral $\int\frac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx$ is equal to – **[JEE MAIN 2016]**

(A) $\frac{x^{10}}{2(x^5+x^3+1)^2}+C$ (B) $\frac{x^5}{2(x^5+x^3+1)^2}+C$

(C) $\frac{-x^{10}}{2(x^5+x^3+1)^2}+C$ (D) $\frac{-x^5}{(x^5+x^3+1)^2}+C$

(Where C is an arbitrary constant)

Q.49 $\lim_{n\to\infty}\left(\frac{(n+1)(n+2)....3n}{n^{2n}}\right)^{1/n}$ is equal to **[JEE MAIN 2016]**

(A) $27/e^2$ (B) $9/e^2$
(C) $3\log 3-2$ (D) $18/e^4$

Q.50 The integral $\int_{\pi/4}^{3\pi/4}\frac{dx}{1+\cos x}$ is equal to : **[JEE MAIN 2017]**

(A) 4 (B) –1 (C) –2 (D) 2

Q.51 Let $I_n=\int\tan^n x\,dx,\ (n>1)$. If $I_4+I_6=a\tan^5x+bx^5+C$, where C is a constant of integration, then the ordered pair (a, b) is equal to : **[JEE MAIN 2017]**

(A) (1/5, –1) (B) (–1/5, 0)
(C) (–1/5, 1) (D) (1/5, 0)

Q.52 The value of $\int_{-\pi/2}^{\pi/2}\frac{\sin^2 x}{1+2^x}dx$ is **[JEE MAIN 2018]**

(A) 4π (B) $\pi/4$
(C) $\pi/8$ (D) $\pi/2$

Q.53 The integral $\int\frac{\sin^2\cos^2 x}{(\sin^5 x+\cos^3 x\sin^2 x+\sin^3 x\cos^2 x+\cos^5 x)^2}dx$ is equal to : **[JEE MAIN 2018]**

(A) $\frac{1}{1+\cot^3 x}+C$ (B) $\frac{-1}{1+\cot^3 x}+C$

(C) $\frac{1}{3(1+\tan^3 x)}+C$ (D) $\frac{-1}{3(1+\tan^3 x)}+C$

(where C is a constant of integration)

Q.54 For $x^2\neq n\pi+1$, $n\in N$ (the set of natural numbers), the integral $\int x\sqrt{\frac{2\sin(x^2-1)-\sin 2(x^2-1)}{2\sin(x^2-1)+\sin 2(x^2-1)}}dx$ is equal to :

(where c is a constant of integration)
[JEE MAIN 2019 (JAN)]

(A) $\log_e\left|\sec\left(\frac{x^2-1}{2}\right)\right|+c$ (B) $\log_e\left|\frac{1}{2}\sec^2(x^2-1)\right|+c$

(C) $\frac{1}{2}\log_e\left|\sec^2\left(\frac{x^2-1}{2}\right)\right|+c$ (D) $\frac{1}{2}\log_e\left|\sec^2(x^2-1)\right|+c$

Q.55 The value of $\int_0^{\pi}|\cos x|^3\,dx$ **[JEE MAIN 2019 (JAN)]**

(A) 2/3 (B) 0
(C) – 4/3 (D) 4/3

Q.56 If $f(x)=\frac{2-x\cos x}{2+x\cos x}$ and $g(x)=\log_e x,\ (x>0)$ then the value of integral $\int_{-\pi/4}^{\pi/4}g(f(x))\,dx$ is **[JEE MAIN 2019 (APRIL)]**

(A) $\log_e 3$ (B) $\log_e 2$
(C) $\log_e e$ (D) $\log_e 1$

Q.57 $\int \frac{\sin\frac{5x}{2}}{\sin\frac{x}{2}}dx$ is equal to : (where c is a constant of integration) **[JEE MAIN 2019 (APRIL)]**

(A) $2x + \sin x + 2\sin 2x + c$ (B) $x + 2\sin x + 2\sin 2x + c$
(C) $x + 2\sin x + \sin 2x + c$ (D) $2x + \sin x + \sin 2x + c$

Q.58 Let $f(x) = \int_0^x g(t)\,dt$, where g is a non-zero even function.

If $f(x+5) = g(x)$, then $\int_0^x f(t)\,dt$ equals-

[JEE MAIN 2019 (APRIL)]

(A) $\int_{x+5}^{5} g(t)\,dt$ (B) $5\int_{x+5}^{5} g(t)\,dt$
(C) $\int_{5}^{x+5} g(t)\,dt$ (D) $2\int_{5}^{x+5} g(t)\,dt$

Q.59 If $\int \frac{dx}{x^3(1+x^6)^{2/3}} = xf(x)(1+x^6)^{1/3} + C$ where C is a constant of integration, then the function f (x) is equal to- **[JEE MAIN 2019 (APRIL)]**

(A) $-\frac{1}{6x^3}$ (B) $\frac{3}{x^2}$
(C) $-\frac{1}{2x^2}$ (D) $-\frac{1}{2x^3}$

Q.60 The value of $\int_0^{\pi/2} \frac{\sin^3 x}{\sin x + \cos x}dx$ is

[JEE MAIN 2019 (APRIL)]

(A) $\frac{\pi-2}{4}$ (B) $\frac{\pi-2}{8}$
(C) $\frac{\pi-1}{4}$ (D) $\frac{\pi-1}{2}$

Q.61 The integral $\int \sec^{2/3} x \operatorname{cosec}^{4/3} x\,dx$ is equal to (Hence C is a constant of integration)
[JEE MAIN 2019 (APRIL)]

(A) $3\tan^{-1/3}x + C$ (B) $-\frac{3}{4}\tan^{-4/3}x + C$
(C) $-3\cot^{-1/3}x + C$ (D) $-3\tan^{-1/3}x + C$

Q.62 The value of the integral $\int_0^1 x\cot^{-1}(1-x^2+x^4)\,dx$ is
[JEE MAIN 2019 (APRIL)]

(A) $\frac{\pi}{4} - \frac{1}{2}\log_e 2$ (B) $\frac{\pi}{2} - \log_e 2$
(C) $\frac{\pi}{2} - \frac{1}{2}\log_e 2$ (D) $\frac{\pi}{4} - \log_e 2$

Q.63 If $\int e^{\sec x}(\sec x \tan x\, f(x) + (\sec x \tan x + \sec^2 x))dx = e^{\sec x} f(x) + C$, then a possible choice of f (x) is
[JEE MAIN 2019 (APRIL)]

(A) $\sec x - \tan x - \frac{1}{2}$ (B) $x\sec x + \tan x + \frac{1}{2}$
(C) $\sec x + x\tan x - \frac{1}{2}$ (D) $\sec x + \tan x + \frac{1}{2}$

Q.64 The value of $\int_0^{2\pi} [\sin 2x(1+\cos 3x)]\,dx$, where [t] denotes the greatest integer function, is
[JEE MAIN 2019 (APRIL)]

(A) -2π (B) π
(C) $-\pi$ (D) 2π

Q.65 If $\int \frac{dx}{(x^2-2x+10)^2} = A\left(\tan^{-1}\left(\frac{x-1}{3}\right) + \frac{f(x)}{x^2-2x+10}\right) + C$ where C is a constant of integration, then :
[JEE MAIN 2019 (APRIL)]

(A) $A = 1/27$ and $f(x) = 9(x-1)$
(B) $A = 1/81$ and $f(x) = 3(x-1)$
(C) $A = 1/54$ and $f(x) = 9(x-1)^2$
(D) $A = 1/54$ and $f(x) = 3(x-1)$

Q.66 If $\int x^5 e^{-x^2}dx = g(x)e^{-x^2} + c$, where c is a constant of integration, then g (–1) is equal to :
[JEE MAIN 2019 (APRIL)]

(A) $-5/2$ (B) 1
(C) $-1/2$ (D) -1

Q.67 The integral $\int_{\pi/6}^{\pi/3} \sec^{2/3} x \operatorname{cosec}^{4/3} x\,dx$ equal to
[JEE MAIN 2019 (APRIL)]

(A) $3^{7/6} - 3^{5/6}$ (B) $3^{5/3} - 3^{1/3}$
(C) $3^{4/3} - 3^{1/3}$ (D) $3^{5/6} - 3^{2/3}$

Q.68 If $\int_0^{\pi/2} \frac{\cot x}{\cot x + \operatorname{cosec} x}dx = m(\pi + n)$, then m . n is equal to
[JEE MAIN 2019 (APRIL)]

(A) -1 (B) 1
(C) 1/2 (D) $-1/2$

Q.69 The integral $\int \frac{2x^3-1}{x^4+x}dx$ is equal to : (Here C is a constant of integration) **[JEE MAIN 2019 (APRIL)]**

(A) $\log_e\left|\frac{x^3+1}{x}\right|+C$ (B) $\frac{1}{2}\log_e\frac{(x^3+1)^2}{|x^3|}+C$

(C) $\frac{1}{2}\log_e\frac{|x^3+1|}{x^2}+C$ (D) $\log_e\frac{|x^3+1|}{x^2}+C$

Q.70 A value of α such that

$$\int_{\alpha}^{\alpha+1}\frac{dx}{(x+\alpha)(x+\alpha+1)}=\log_e\left(\frac{9}{8}\right) \text{ is}$$

[JEE MAIN 2019 (APRIL)]

(A) 1/2 (B) 2
(C) –1/2 (D) – 2

Q.71 Let $\alpha \in (0, \pi/2)$ be fixed. If the integral

$$\int\frac{\tan x+\tan\alpha}{\tan x-\tan\alpha}dx=$$

A (x) cos 2α + B (x) sin 2α + C, where C is a constant of integration, then the functions A(x) and B(x) are respectively : **[JEE MAIN 2019 (APRIL)]**

(A) $x-\alpha$ and $\log_e|\cos(x-\alpha)|$
(B) $x+\alpha$ and $\log_e|\sin(x-\alpha)|$
(C) $x-\alpha$ and $\log_e|\sin(x-\alpha)|$
(D) $x+\alpha$ and $\log_e|\sin(x+\alpha)|$

Q.72 Given $f(a+b+1-x)=f(x)\ \forall\ x\in R$ then the value of

$$\frac{1}{(a+b)}\int_a^b x\,[f(x)+f(x+1)]\,dx \text{ is equal to}$$

[JEE MAIN 2020 (JAN)]

(A) $\int_{a+1}^{b+1} f(x)\,dx$ (B) $\int_{a+1}^{b+1} f(x+1)\,dx$

(C) $\int_{a-1}^{b-1} f(x)\,dx$ (D) $\int_{a-1}^{b-1} f(x+1)\,dx$

Q.73 Let θ_1 and θ_2 (where $\theta_1<\theta_2$)are two solutions of

$$2\cot^2\theta-\frac{5}{\sin\theta}+4=0,\ \theta\in[0,2\pi] \text{ then } \int_{\theta_1}^{\theta_2}\cos^3 3\theta\,d\theta \text{ is}$$

equal to – **[JEE MAIN 2020 (JAN)]**

(A) π/3 (B) 2π/3

(C3) π/9 (D) $\frac{\pi}{3}+\frac{1}{6}$

Q.74 Let $4\alpha\int_{-1}^{2}e^{-\alpha|x|}dx=5$ then $\alpha=$ **[JEE MAIN 2020 (JAN)]**

(A) ln 2 (B) $\ln\sqrt{2}$
(C) ln (3/4) (D) ln (4/3)

Q.75 Let $\int\frac{\cos x\,dx}{\sin^3 x(1+\sin^6 x)^{2/3}}=f(x)(1+\sin^6 x)^{1/\lambda}+c$ then find value of $\lambda f\left(\frac{\pi}{3}\right)$. **[JEE MAIN 2020 (JAN)]**

(A) 4 (B) –2
(C) 8 (D) – 4

Q.76 Let $I=\int_1^2\frac{1\,dx}{\sqrt{2x^3-9x^2+12x+4}}$ then

[JEE MAIN 2020 (JAN)]

(A) $\frac{1}{9}<I^2<\frac{1}{8}$ (B) $\frac{1}{3}<I^2<\frac{1}{2}$

(C) $\frac{1}{9}<I<\frac{1}{8}$ (D) $\frac{1}{3}<I<\frac{1}{2}$

Q.77 The value of $\int_0^{2\pi}\frac{x\sin^8 x}{\sin^8 x+\cos^8 x}dx$ is equal to –

[JEE MAIN 2020 (JAN)]

(A) 2π (B) 4π
(C) $2\pi^2$ (D) π^2

Q.78 The integral $\int\frac{dx}{(x+4)^{8/7}(x-3)^{6/7}}$ is equal to :

(where C is a constant of integration)

[JEE MAIN 2020 (JAN)]

(A) $\left(\frac{x-3}{x+4}\right)^{1/7}+C$ (B) $-\left(\frac{x-3}{x+4}\right)^{-1/7}+C$

(C) $\frac{1}{2}\left(\frac{x-3}{x+4}\right)^{3/7}+C$ (D) $-\frac{1}{13}\left(\frac{x-3}{x+4}\right)^{-13/7}+C$

Q.79 If for all real triplets (a, b, c), $f(x)=a+bx+cx^2$;

then $\int_0^1 f(x)\,dx$ is equal to : **[JEE MAIN 2020 (JAN)]**

(A) $\frac{1}{2}\left\{f(1)+3f\left(\frac{1}{2}\right)\right\}$ (B) $2\left\{3f(1)+2f\left(\frac{1}{2}\right)\right\}$

(C) $\frac{1}{6}\left\{f(0)+f(1)+4f\left(\frac{1}{2}\right)\right\}$ (D) $\frac{1}{3}\left\{f(0)+f\left(\frac{1}{2}\right)\right\}$

Q.80 If $\int\frac{d\theta}{\cos^2\theta(\tan 2\theta+\sec 2\theta)}=\lambda\tan\theta+2\log_e|f(\theta)|+C$

where C is a constant of integration, then the ordered pair (l, f (θ)) is equal to : **[JEE MAIN 2020 (JAN)]**

(A) (–1, 1 + tan θ) (B) (–1, 1 – tan θ)
(C) (1, 1 – tan θ) (D) (1, 1 + tan θ)

ANSWER KEY

EXERCISE - 1

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	C	B	C	B	C	A	B	A	C	A	B	A	B	D	A	C	C	B	C	C	A	B	A	A	A
Q	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
A	B	A	B	D	A	B	B	B	C	A	B	C	A	A	D	D	A	A	D	B	A	B	C	D	D
Q	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75
A	A	D	A	B	C	A	C	B	A	B	B	B	B	C	A	B	A	C	A	B	C	B	C	B	B
Q	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
A	A	D	B	A	A	C	A	A	C	C	A	B	A	D	A	A	C	B	C	A	B	A	A	C	B
Q	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120	121	122	123	124	125
A	A	C	D	A	A	A	C	A	C	C	D	D	B	C	B	D	A	B	A	B	A	B	D	A	A
Q	126	127	128	129	130	131	132	133																	
A	C	D	C	D	A	D	B	B																	

EXERCISE - 2

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	A	B	D	D	A	D	D	A	B	C	A	B	A	A	A	C	C	B	A	D	C	B	B	C	A
Q	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
A	C	A	B	C	A	C	A	A	B	A	B	D	B	A	A	A	A	B	D	C	A	A	D	B	D
Q	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75
A	D	D	B	B	B	D	B	A	A	B	C	D	C	C	C	B	D	A	B	D	A	A	C	A	A
Q	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97			
A	D	D	C	A	A	A	208	13	2250	7	2525	8	3	2	4	3	1	0	4	2	0	9			

EXERCISE - 3

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	C	B	A	A	C	A	B	A	C	D	D	A	C	D	B	D	B	A	C	B	A	D	D	B	D
Q	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
A	B	A	B	C	A	A	A	A	B	A	D	A	A	A	D	BC	C	D	D	B	C	B	A	A	D
Q	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75
A	D	B	D	A	D	D	C	A	D	C	D	A	D	C	D	A	A	A	A	D	C	D	A	A	B
Q	76	77	78	79	80																				
A	A	D	A	C	A																				

CHAPTER- 7: INTEGRATION
SOLUTIONS TO TRY IT YOURSELF
TRY IT YOURSELF-1

(1) $$\int \frac{(x^2+\sin^2 x)\sec^2 x}{1+x^2}dx = \int\left(\sec^2 x - \frac{1}{1+x^2}\right)dx$$
$$= \tan x - \tan^{-1} x + c$$

(2) Put $\tan^{-1} x^3 = t \Rightarrow \frac{3x^2}{1+x^6}dx = dt$

Integral becomes $\int \frac{1}{3}t\,dt = \frac{1}{6}t^2 + c = \frac{1}{6}(\tan^{-1} x^3)^2 + c$

(3) Put $xe^x = t \Rightarrow e^x(1+x)\,dx = dt$

Integral becomes $\int \frac{dt}{\cos t} = \int \sec t\,dt$

$= \ln(\sec t + \tan t) + c = \ln(\sec(xe^x) + \tan(xe^x)) + c$

(4) Put $e^x = t \Rightarrow e^x\,dx = dt$

Integral becomes
$$\int \frac{dt}{\sqrt{t^2-1}} = \ln(t+\sqrt{t^2-1}) + c = \ln(e^x + \sqrt{e^{2x}-1}) + c$$

(5) Let $I = \int \frac{5x+4}{\sqrt{x^2+2x+5}}dx$

Let $5x+4 = \lambda(2x+2) + \mu$

Comparing the coefficient's, we have

$2\lambda = 5$ and $2\lambda + \mu = 4$ gives $\lambda = 5/2$ and $\mu = -1$.

Hence, we have,
$$I = \frac{5}{2}\int \frac{2x+2}{\sqrt{x^2+2x+5}}dx - \int \frac{dx}{\sqrt{x^2+2x+5}}$$
$$= 5\sqrt{x^2+2x+5} - \int \frac{dx}{\sqrt{(x-1)^2+2^2}}$$
$$= 5\sqrt{x^2+2x+5} - \ln|x+1+\sqrt{x^2+2x+5}| + C$$

(6) Put $\ln x = t$ to get
$$I = \int \underset{I}{e^t}\ \underset{II}{\sin t}\ dt = e^t(-\cos t) - \int e^t(-\cos t)\,dt$$
$$= -\cos t\cdot e^t + \int e^t \cos t\,dt$$
$$= -\cos t\cdot e^t + [e^t \sin t - \int e^t \sin t\,dt]$$
$$\Rightarrow I = -\cos t\,.\,e^t + e^t \sin t - I \text{ or } I = \frac{1}{2}e^t(\sin t - \cos t) + c$$
$$= \frac{1}{2}x(\sin(\ln x) - \cos(\ln x)) + c$$

(7) Put $\ln x = t$ to get
$$\int e^t(\sin t + \cos t)\,dt = e^t\cdot \sin t + c = x\sin(\ln x) + c$$

(8) Put $\ln x = t$ to get
$$\int e^t\left(\ln t + \frac{1}{t^2}\right)dt = \int e^t\left(\ln t - \frac{1}{t} + \frac{1}{t} + \frac{1}{t^2}\right)dt$$
$$= e^t\left(\ln t - \frac{1}{t}\right) + c = e^{\ln x}\left(\ln(\ln x) - \frac{1}{\ln x}\right) + c$$
$$= x\left[\ln(\ln x) - \frac{1}{\ln x}\right] + c$$

(9) $$\int \frac{x^7 dx}{x^{10}\left(\frac{1}{x^2}-1\right)^5} = \int \frac{dx}{x^3\left(\frac{1}{x^2}-1\right)^5}$$

Put $\frac{1}{x^2} - 1 = t$ to get
$$\int \frac{-\frac{1}{2}dt}{t^5}dt = \frac{-1}{2}\cdot\frac{-1}{4}t^4 + c = \frac{1}{8}\frac{1}{(x^{-2}-1)^4} + c$$

(10) $$\int \frac{dx}{4-5\sin^2 x} = \int \frac{\operatorname{cosec}^2 x}{4\operatorname{cosec}^2 x - 5}dx = \int \frac{\operatorname{cosec}^2 x}{4\cot^2 x - 1}dx$$

Put cot x = 1 to get
$$\int \frac{-dt}{4t^2-1} = \frac{-1}{4}\int \frac{dt}{t^2 - \left(\frac{1}{2}\right)^2} = \frac{-1}{4}\ln\left|\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right| + c$$
$$= -\frac{1}{4}\ln\left(\frac{2\cot x - 1}{2\cot x + 1}\right) + c$$

(11) $$I = \int \frac{dx}{(3\sin x - 4\cos x)^2} = \int \frac{\sec^2 x\,dx}{(3\tan x - 4)^2}$$

Put $3\tan x - 4 = t \Rightarrow 3\sec^2 x\,dx = dt$
$$\Rightarrow\ I = \int \frac{\frac{1}{3}dt}{t^2} = -\frac{1}{3}\frac{1}{(3\tan x - 4)} + c$$

(12) $I = \int \frac{dx}{(x+2)\sqrt{x+1}}$ put $x + 1 = t^2$
$$\Rightarrow I = \int \frac{2t\,dt}{(t^2+1)\,t} = 2\tan^{-1} t + c = 2\tan^{-1}(\sqrt{x+1}) + c$$

(13) (i) $$\int \frac{\sqrt{x}}{x+1}dx = \int \frac{t}{t^2+1}\cdot 2t\,dt$$

[Putting $\sqrt{x} = t$ and $dx = 2t\,dt$]
$$= \int \frac{2t^2}{t^2+1}dt = 2\int \frac{t^2+1-1}{t^2+1}dt = 2\int dt - 2\int \frac{dt}{t^2+1}$$

$= 2t - 2\tan^{-1}t + C = 2\sqrt{x} - 2\tan^{-1}\sqrt{x} + C$

(ii) $\int \frac{dx}{e^x + e^{-x}} = \int \frac{1}{t + 1/t} \cdot \frac{dt}{t}$ [Putting $e^x = t$ and $dx = \frac{dt}{t}$]

$= \int \frac{dt}{t^2+1} = \tan^{-1} t + C = \tan^{-1}(e^x) + C$

(14) $I = \int \frac{\sin 2x}{a^2 + b^2 \sin^2 x} dx$ put $\sin^2 x = t \Rightarrow \sin 2x \, dx = dt$

$\Rightarrow I = \int \frac{dt}{a^2 + b^2 t^2} = \frac{1}{b^2}\int \frac{dt}{t^2 + (a/b)^2}$

$= \frac{1}{b^2}\left(\frac{1}{a/b}\right)\tan^{-1}\left(\frac{t}{a/b}\right) + c = \frac{1}{ab}\tan^{-1}\left(\frac{b\sin^2 x}{a}\right) + c$

TRY IT YOURSELF-2

(1) $\text{LHS} = \int_{-a^{1/3}}^{a^{1/3}} \lim_{n\to\infty}\left(1 - \frac{t^3}{n}\right)^n t^2 dt = \int_{-a^{1/3}}^{a^{1/3}} e^{-t^3} t^2 dt$

$= \left[-\frac{1}{3}e^{-t^3}\right]_{-a^{1/3}}^{a^{1/3}} = \frac{1}{3}[e^a - e^{-a}]$

$\Rightarrow e^a - e^{-a} = 2\sqrt{2}$ or $a = \ln(\sqrt{2} + \sqrt{3})$

(2) $\int \frac{\sin\sqrt{x+1}}{\sqrt{x+1}} dx = -2\cos\sqrt{x+1}$

$\Rightarrow \int_3^8 \frac{\sin\sqrt{x+1}}{\sqrt{x+1}} dx = [-2\cos\sqrt{x+1}]_3^8 = 2(\cos 2 - \cos 3)$

(3) $\int \frac{\sqrt{x^2-4}}{x^4} dx = \int \frac{\sqrt{1-4/x^2}}{x^3} dx$

Put $1 - \frac{4}{x^2} = t \Rightarrow 8x^{-3}dx = dt$ integral becomes

$\int \frac{1}{8} t^{1/2} dt = \frac{1}{12} t^{3/2} = \frac{1}{12}\left(1 - \frac{4}{x^2}\right)^{3/2}$

$\Rightarrow \int_2^4 \frac{\sqrt{x^2-4}}{x^4} dx = \left[\frac{1}{2}\left(1 - \frac{4}{x^2}\right)^{3/2}\right]_2^4 = \frac{\sqrt{3}}{32}$

(4) $\int_0^{\pi} |\cos x| dx = \int_0^{\pi/2} \cos x \, dx + \int_{\pi/2}^{\pi} -\cos x \, dx$

$= [\sin x]_0^{\pi/2} + [-\sin x]_{\pi/2}^{\pi} = 2$

(5) $\int_0^3 |5x - 9| dx = \int_0^{9/5} (9 - 5x) dx + \int_{9/5}^3 (5x - 9) dx$

$= \left[9x - \frac{5}{2}x^2\right]_0^{9/5} + \left[\frac{5x^2}{2} - 9x\right]_{9/5}^3 = \frac{15}{2}$

(6) Let $I = \int_1^4 \ln[x] dx = \int_1^2 \ln[x] dx + \int_2^3 \ln[x] dx + \int_3^4 \ln[x] dx$

$= 0 + \ln 2\int_2^3 dx + \ln 3\int_2^3 dx = \ln 2 + \ln 3 = \ln 6$

(7) **(A).** $I = \int_{-1}^3 \left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right) dx$

$= \int_{-1}^1 \left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right) dx$

$+ \int_1^3 \left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right) dx$

$= 0 + \int_1^3 \tan^{-1}\left(\frac{x}{x^2+1}\right) + \cot^{-1}\left(\frac{x}{x^2+1}\right) dx$

$= 0 + \int_1^3 \frac{\pi}{2} dx = 2 \times \frac{\pi}{2} = \pi$

(8) $I = \int_{50}^{100} \frac{\ln x}{\ln x + \ln(150 - x)} dx$ Using P-4 $[x \to 100 + 50 - x]$

$I = \int_{50}^{100} \frac{\ln(150 - x)}{\ln(150 - x) + \ln x} dx$

$I + I = \int_{50}^{100} 1 \, dx = 50 \Rightarrow I = 25$

(9) $I = \int_0^{\pi} \frac{dx}{1 + 2^{\tan x}}$ Using P-4 $[x \to \pi - x]$

$I = \int_0^{\pi} \frac{dx}{1 + 2^{-\tan x}} = \int_0^{\pi} \frac{2^{\tan x}}{1 + 2^{\tan x}} dx$

$I + I = \int_0^{\pi} 1 \, dx = \pi$ or $I = \frac{\pi}{2}$

(10) $\int_0^{\pi/2} \frac{dx}{1+\sin x} = \int_0^{\pi/2} \frac{dx}{1+\sin(\pi/2 - x)} = \int_0^{\pi/2} \frac{dx}{1+\cos x}$

$= \frac{1}{2}\int_0^{\pi/2} \sec^2 \frac{x}{2} dx = \frac{1}{2}\left[\frac{\tan x/2}{1/2}\right]_0^{\pi/2} = 1$

(11) Let $f(x) = \sin^4 x$ then $f(2\pi - x) = \sin^4(2\pi - x) = f(x)$

Using P-6, $I = 2\int_0^{\pi} \sin^4 x\, dx$,

Again using P-6, $I = 4\int_0^{\pi/2} \sin^4 x\, dx$

Using P-4, $I = 4\int_0^{\pi/2} \cos^4 x\, dx$; $k = 4$

(12) Using P-4, $I = \int_0^{2\pi} (2\pi - x)\cos^5 x\, dx$

$\Rightarrow \quad I + I = \int_0^{2\pi} (x + 2\pi - x)\cos^5 x\, dx$

or $\quad I = \int_0^{2\pi} \pi \cos^5 x\, dx = 2\pi\int_0^{\pi} \cos^5 x\, dx$ [Using P-4]

$= 2\pi \times 0$ [Using P-6 as $\cos^5(\pi - x) = -\cos^5 x$]

(13) We have, $\sqrt{1-\cos \pi x} = \sqrt{2}\left|\sin\frac{\pi x}{2}\right|$

which is a periodic function, having period

$T = \frac{1}{2} \times \frac{2\pi}{\pi/2} = 2$

Hence, we have

$\int_0^{10} \sqrt{1-\cos \pi x}\, dx = \int_0^{10} \sqrt{2}\left|\sin\frac{\pi x}{2}\right| dx = 5\sqrt{2}\int_0^{2} \left|\sin\frac{\pi x}{2}\right| dx$

$= 5\sqrt{2}\int_0^{2} \sin\left(\frac{\pi x}{2}\right) dx = 5\sqrt{2}\left[\frac{-\cos(\pi x/2)}{\pi/2}\right]_0^2 = \frac{20\sqrt{2}}{\pi}$

(14) $\lim_{x\to 0} \frac{\int_0^x (1-\cos 2x)\, dx}{x\int_0^x \tan x\, dx} = \lim_{x\to 0} \frac{\int_0^x (1-\cos 2x)\, dx}{x^3} \cdot \frac{x^2}{\int_0^x \tan x\, dx}$

$\lim_{x\to 0} \frac{\int_0^x (1-\cos 2x)\, dx}{x^3} \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{1-\cos 2x}{3x^2} = \frac{2}{3}$

and $\lim_{x\to 0} \frac{x^2}{\int_0^x \tan x\, dx} \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{2x}{\tan x} = 2$

Hence, we have L = 4/3.

(15) $S = \lim_{n\to\infty} \sum_{r=0}^{3n} \frac{1}{n+r} = \lim_{n\to\infty} \frac{1}{n}\sum_{r=0}^{3n} \frac{1}{1+\left(\frac{r}{n}\right)}$

$= \int_0^3 \frac{1}{1+x} dx = [\ln(1+x)]_0^3 = \ln 4$

CHAPTER- 7 : INTEGRATION
EXERCISE-1

(1) **(C).** $\int \frac{\cos x - x\sin x}{\sqrt{x\cos x}}dx$; put x cos x = t

$\Rightarrow (\cos x - x\sin x)\,dx = 2t\,dt$

$= \int \frac{2t\,dt}{t} = 2t + C = 2\sqrt{x\cos x} + C$

(2) **(B).** $\int\left[\left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x\right]dx = \frac{(2/5)^x}{\log(2/5)} + \frac{(3/5)^x}{\log(3/5)} + C$

(3) **(C).** $\tan 5x = \tan(3x+2x) = \frac{\tan 2x + \tan 3x}{1 - \tan 2x.\tan 3x}$

$\Rightarrow \tan 5x - \tan 2x\,.\,\tan 3x\,.\,\tan 5x = \tan 2x + \tan 3x$

$\Rightarrow \int \tan 2x.\tan 3x.\tan 5x\,dx$

$= \int(\tan 5x - \tan 2x - \tan 3x)\,dx$

$= \frac{1}{5}\log|\sec 5x| - \frac{1}{2}\log|\sec 2x| - \frac{1}{3}\log|\sec 3x| + C$

(4) **(B).** $\int \frac{\frac{\sin^4 x}{\cos^4 x}}{\frac{\cos^8 x}{\cos^4 x}}dx = \int \tan^4 x.\sec^4 x\,dx$

$= \int \tan^4 x\,(1+\tan^2 x)\sec^2 x\,dx$

$= \int \tan^4 x\,(1+\tan^2 x)\,d(\tan x)$

$= \int t^4(1+t^2)\,dt = \frac{t^5}{5} + \frac{t^7}{7} + C$

(5) **(C).** $\int e^x\left\{\frac{1}{(\cos^2 x)} + \frac{\sin x.\cos x}{\cos^2 x}\right\}dx$

$= \int e^x\{\sec^2 x + \tan^2 x\}\,dx = e^x \tan x + C$

(6) **(A).** $I = \int \frac{1}{1+9\tan^2 x}dx$ (N^r & d^r divided by $\cos^2 x$)

$I = \int \frac{\sec^2 x}{(1+\tan^2 x)(1+9\tan^2 x)}dx$

Let tan x = t

$\sec^2 x\,dx = dt$

$I = \int \frac{dt}{(1+t^2)(1+9t^2)} = \int\left(\frac{(-1/8)}{1+t^2} + \frac{(9/8)}{1+9t^2}\right)dt$

$= -\frac{1}{8}\tan^{-1}(t) + \frac{9}{8}\frac{\tan^{-1}(3t)}{3} + c$

$= \frac{1}{8}(3\tan^{-1}(3\tan x) - x) + c$

(7) **(B).** Put $\sqrt{x} = \cos\theta \;\therefore\; x = \cos^2\theta$

$\therefore\; dx = -2\cos\theta\sin\theta\,d\theta$

$\therefore \int\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx = -\int \tan\frac{\theta}{2}\,.\cos\theta\,.\sin\theta\,d\theta$

$= -4\int \sin^2\frac{\theta}{2}\cos\theta\,d\theta = -2\int(1-\cos\theta)\cos\theta\,d\theta$

$= -2\sin\theta + \int(1+\cos 2\theta)\,d\theta$

$= -2\sin\theta + \theta + \frac{\sin 2\theta}{2} + C$

$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x} + C$

(8) **(A).** $I = \int \frac{x + \sqrt[3]{x^2} + \sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)}dx$

Let $x = t^6$, $dx = 6t^5\,dt$

$I = \int \frac{\left(t^6 + t^4 + e\right)6t^5 dt}{t^6(1+t^2)} = 6\int \frac{t^5 + t^3 + 1}{(1+t^2)}dt$

$= 6\int \frac{t^3(t^2+1)+1}{(1+t^2)}dt$

$\Rightarrow I = 6\int t^3\,dt + 6\int \frac{dt}{1+t^2} = \frac{6t^4}{4} + 6\tan^{-1}t$

$\Rightarrow \frac{3}{2}x^{2/3} + 6\tan^{-1}\sqrt[6]{x} + c$

(9) **(C).** Put sin x = t

cos x dx = dt

$\int \frac{(1-t^2)+(1-t^2)^2}{t^2+t^4}dt = \int \frac{2-3t^2+t^4}{t^2(t^2+1)}dt$

$= \int\left(1 + \frac{2}{t^2} - \frac{6}{1+t^2}\right)dt = \sin x - \frac{2}{\sin x} - 6\tan^{-1}(\sin x) + c$

(10) **(A).** $I = \int\sqrt{\frac{x}{4-x^3}}\,dx = \int \frac{\sqrt{x}\,dx}{\sqrt{4-x^3}}$

Here integral of $\sqrt{x} = \frac{2}{3}x^{3/2}$ and $4 - x^3 = 4 - (x^{3/2})^2$

Put $x^{3/2} = t \Rightarrow \sqrt{x}\,dx = \frac{2}{3}dt$

So $I = \frac{2}{3}\int \frac{dt}{\sqrt{4-t^2}} = \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{2}\right) + c$

(11) **(B).** $I = \int \frac{(ax^2 - b)}{x} \frac{dx}{x\sqrt{c^2 - \left(\frac{ax^2+b}{x}\right)^2}}$

Let $ax + \frac{b}{x} = t \Rightarrow \left(a - \frac{b}{x^2}\right) dx = dt$

$I = \sin^{-1}\frac{t}{c} + C = \sin^{-1}\left(\frac{ax^2+b}{cx}\right) + C$

(12) **(A).** $\int x \tan^{-1} x\, dx$

$= \tan^{-1}x. \frac{x^2}{2} - \int\left(\frac{1}{1+x^2}\frac{x^2}{2}\right) dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2} dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left[1 - \frac{1}{1+x^2}\right] dx = \frac{1}{2}(x^2+1)\tan^{-1}x - \frac{1}{2}x + c$

(13) **(B).** Diff. both side

$f(x)\sin x \cos x = \frac{1}{2(b^2-a^2)}\frac{1}{f(x)} f'(x)$

$\sin 2x = \frac{1}{b^2-a^2}\frac{f'(x)}{f(x)}$ $\sin 2x = \frac{1}{b^2-a^2}\frac{f'(x)}{f^2(x)}$

Integrating, $f(x) = \frac{2}{(b^2-a^2)\cos 2x}$

(14) **(D).** $\int \frac{\sqrt{x}}{x(x+1)} dx = k \tan^{-1} m$

$= \int \frac{1/\sqrt{x}}{x+1} dx = 2\int \frac{d(\sqrt{x})}{(\sqrt{x})^2 + 1} = 2\tan^{-1}(\sqrt{x})$

$\therefore\ k = 2, m = \sqrt{x}$

(15) **(A).** $\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx = \int \cos^{-1}\left[\frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$

Put $x = \tan\theta$

$= \int \cos^{-1}(\cos 2\theta)\, dx = \int 2\theta\, dx$

$= 2\int \tan^{-1} x\, dx = 2\left[\tan^{-1} x.x - \int \frac{1.x}{1+x^2} dx\right]$

$= 2\int\left[x\tan^{-1}x - \frac{1}{2}\log(1+x^2)\right] + C$

$= 2x\tan^{-1}x - \log(1+x^2) + C$

(16) **(C).** $I_n = \int (\log x)^n dx = \int (\log x)^n .1\, dx$

$= (\log x)^n .x - \int n(\log x)^{n-1}.\frac{x}{x} dx = x(\log x)^n - n I_{n-1}$

$\therefore I_n + nI_{n-1} = x(\log x)^n$

(17) **(C).** Given integral can be expressed as

$\int \frac{\cot^{n-1} x}{\sin^2 x} dx = \frac{-\cot^n x}{n}$

(18) **(B).** $\int \frac{(x+1-2)e^x}{(x+1)^3} dx = \int e^x\left[\frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}\right] dx$

$= \frac{e^x}{(x+1)^2}$

(19) **(C).** $\int [(3x-1)\cos x + (1-2x)\sin x]\, dx = f(x)\cos x + g(x)\sin x + C$

$= (3x-1)\sin x - \int \sin x \cdot 3\, dx + (1-2n)(-\cos x)$

$+ \int \cos x\,(-2)\, dx$

$= (3x-1)\sin x + 3\cos x - \cos x + 2x\cos x - 2\sin x + C$

$= (3x-1-2)\sin x + (2+2x)\cos x + C$

$= 3(x-1)\sin x + 2(x+1)\cos x + C$

$\therefore\ f(x) = 2(x+1), g(x) = 3(x-1)$

(20) **(C).** $\int \frac{\sin 2x}{\sin^2 x + 2\cos^2 x} dx = \int \frac{\sin 2x}{1+\cos^2 x} dx$

$\because\ \sin^2 x = 1 - \cos^2 x$, put $\cos^2 x = t$, $1 + \cos^2 x = t$

$-2\cos x \sin x\, dx = dt$ or $\sin 2x\, dx = -dt$

$= -\int \frac{dt}{t} = -\log t + c = -\log(1 + \cos^2 x) + C$

(21) **(A).** $I = \int \frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}} = \int\left(1+\frac{1}{x^4}\right)^{-3/4} \cdot \frac{1}{x^5} dx$

$= \frac{1}{-4}\int\left(1+\frac{1}{x^4}\right)^{-3/4} d\left(1+\frac{1}{x^4}\right)$

$= -\frac{1}{4}\frac{\left(1+\frac{1}{x^4}\right)^{1/4}}{1/4} = -\frac{(x^4+1)^{1/4}}{x} + C$

(22) **(B).** $I = \int (1-\cos x)\, dx = x - \sin x + C$

(23) **(A).** $I = \int e^x\left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right) dx = e^x \tan\frac{x}{2} + C$

(24) **(A).** Putting $t = \frac{1}{u}$ in I_1, we get

$$I_1 \Rightarrow \int_{1/x}^{1} \frac{1}{1+\frac{1}{u^2}}\left(-\frac{1}{u^2}\right) du = -\int_{1/x}^{1} \frac{du}{u^2+1} = \int_{1}^{1/x} \frac{1}{1+u^2} du = I_2$$

(25) **(A).** we have $g(x) = \int_0^x \cos^4 t \ dt.$

$$\therefore g(x+\pi) = \int_0^{x+\pi} \cos^4 t \, dt = \int_0^{\pi} \cos^4 t \, dt + \int_{\pi}^{x+\pi} \cos^4 t \, dt$$

$= g(\pi) + \int_0^x \cos^4(u+\pi) \ du$, where $t = u + \pi$

$= g(\pi) + g(x)$

(26) **(B).** $I = \int_0^{\pi^2/4} \left(2\sin\sqrt{x} + \sqrt{x}\cos x\right) dx$

Let $f(x) = 2\sin\sqrt{x}, f'(x) = \frac{\cos\sqrt{x}}{\sqrt{x}}$

$x f'(x) = \sqrt{x}\cos\sqrt{x}$

$\therefore \quad I = 2x\sin\sqrt{x}\Big]_0^{\pi^2/4} = 2.\frac{\pi^2}{4} = \frac{\pi^2}{2}$

(27) **(A).** $I = \int_0^{2\pi} \ln(1+\sin x) \, dx$

$I = \int_0^{2\pi} \ln(1-\sin x) \, dx$

$2I = \int_0^{2\pi} \ln(\cos^2 x) \, dx$

$2I = 2\int_0^{\pi} \ln(\cos^2 x) \, dx$

$I = \int_0^{\pi} \ln(\cos^2 x) \, dx = 2\int_0^{\pi/2} \ln(\cos^2 x) \, dx = 4\int_0^{\pi/2} \ln(\cos x) \, dx$

$= 4\left(-\frac{\pi}{2}\ln 2\right) = -2\pi\ln 2$

(28) **(B).** $I = \int_{-\pi/4}^{\pi/4} \ln\sqrt{1+\sin 2x} \ dx$

$\{\cos x > \sin x \text{ in } (-\pi/4, \pi/4)\}$

$= \int_{-\pi/4}^{\pi/4} \ln(\cos x + \sin x) \, dx$

$I = \int_{-\pi/4}^{\pi/4} \ln(\cos x - \sin x) \, dx$;

hence $2I = \int_{-\pi/4}^{\pi/4} \ln(\cos 2x) \, dx$

$= \int_0^{\pi/2} \cos t \, dt = -\frac{\pi}{2}\ln 2 \Rightarrow I = -\frac{\pi}{4}\ln 2$

(29) **(D).** Put $x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}dt$

$\therefore$ After substitution

$I = \int_{2010}^{1/2010} t\sin\left(\frac{1}{t} - t\right)\left(-\frac{1}{t^2}\right) dt$

$I = \int_{2010}^{1/2010} -\frac{1}{t}\sin\left(\frac{1}{t} - t\right) = -\int_{1/2010}^{2010} \frac{1}{t}\sin\left(t - \frac{1}{t}\right) dt$

$I = -1$

$\Rightarrow 2I = 0 \Rightarrow I = 0$

(30) **(A).** $g(x+\pi) = \int_0^{x+\pi} \cos^4 t \ dt = \int_0^{\pi} \cos^4 t \ dt + \int_{\pi}^{x+\pi} \cos^4 t \ dt$

$= \int_0^{\pi} \cos^4 t \ dt + \int_0^{x} \cos^4 t \ dt \ = g(\pi) + g(x)$

(31) **(B).** $A - B = \int_0^{\pi/2} (\sin^2 x - \cos^2 x) \, dx$

$= \frac{1}{2}.\frac{\pi}{2} - \frac{1}{2}.\frac{\pi}{2}$ [Walli's formula]

(32) **(B).** Let $f(x) = x^3 \sin^4 x$. Then,

$f(-x) = (-x)^3 \sin^4(-x) = -x^3\{\sin(-x)\}^4$

$\Rightarrow f(-x) = -x^3(-\sin x)^4 = -x^3\sin^4 x = -f(x).$

So, $f(x)$ is an odd function

Hence, $\int_{-\pi/4}^{\pi/4} f(x)dx = 0 \Rightarrow \int_{-\pi/4}^{\pi/4} x^3\sin^4 x \, dx = 0$

(33) **(B).** $I = \int_{b-1}^{b} \frac{e^{-t}}{t-b-1}dt$; Putting $t = b - y$; $dt = -dy$

$= \int_1^0 \frac{e^{-(b-y)}}{-y-1} \times -dy = \int_0^1 \frac{e^{-b+y}}{-(y+1)}dy$

$= -e^{-b}\int_0^1 \frac{e^y}{y+1}dy = -e^{-b}a$

(34) (C). $\left[x^3 - 4014x^2 + (2007)^2 x + \frac{x}{2008}\right]_0^{2008}$

$= (2008)^3 - 4014(2008)^2 + \left((2007)^2 x + \frac{1}{2008}\right).2008$

$= (2008)\,[(2008)^2 - (4014)(2008) + (2007)^2] + 1$

$= (2008)\,[(2008)^2 - 2(2007)(2008) + (2007)^2] + 1$

$= 2008\,[(2008-2007)^2] + 1 = 2009$

(35) (A). $I = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{(1+e^x)(1+x^2)}$

Here, $f(x) = \frac{1}{(1+e^x)(1+x^2)}$

$\Rightarrow f(-x) = \frac{1}{(1+e^{-x})(1+(-x)^2)} = \frac{e^x}{(1+e^x)(1+x^2)}$

so $I = \int_0^{\sqrt{3}} \frac{dx}{1+x^2} = \tan^{-1} x \Big|_0^{\sqrt{3}} = \frac{\pi}{3}$

(36) (B). $I = \int_{-2\pi}^{5\pi} \cot^{-1}(\tan x)\, dx$

$= 7\int_0^{\pi} \cot^{-1}(\tan x)\, dx = 7\int_0^{\pi} \cot^{-1}\left(\cot\left(\frac{\pi}{2} - x\right)\right) dx$

$= 7\left\{\int_0^{\pi/2}\left(\frac{\pi}{2} - x\right) dx + \int_{\pi/2}^{\pi}\left(\pi + \frac{\pi}{2} - x\right) dx\right\}$

$= 7\left\{\left\{\frac{\pi}{2}x - \frac{x^2}{2}\right\}_0^{\pi/2} + \left\{\frac{3\pi}{2}x - \frac{x^2}{2}\right\}_{\pi/2}^{\pi}\right\} = \frac{7\pi^2}{2}$

(37) (C). $I = \int_0^{\pi/2} \frac{2^{\sin x}}{2^{\sin x} + 2^{\cos x}} dx$

$I = \int_0^{\pi/2} \frac{2^{\sin(\pi/2 - x)}}{2^{\sin(\pi/2-x)} + 2^{\cos(\pi/2-x)}} dx = \int \frac{2^{\cos x}}{2^{\cos x} + 2^{\sin x}} dx$

$\Rightarrow 2I = \int_0^{\pi/2} dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

(38) (A). Putting $\tan x = t^2$, then

$\sec^2 x\, dx = 2t\, dt \Rightarrow dx = \frac{2t\, dt}{1+t^4}$ .... (1)

$\therefore\ I = \int_0^1 \left(t + \frac{1}{t}\right)\frac{2t\, dt}{1+t^4}$

$= 2\int_0^1 \frac{t^2+1}{t^4+1} = dt = 2\int_0^1 \frac{1+1/t^2}{t^2+1/t^2} dt = 2\int_0^1 \frac{d(t-1/t)}{(t-1/t)^2+2}$

$= \sqrt{2}\left[\tan^{-1}\frac{1}{\sqrt{2}}\left(t - \frac{1}{t}\right)\right]_0^1 = \sqrt{2}\ [\tan^{-1}0 - \tan^{-1}(-\infty)]$

$= \sqrt{2}\ (\pi/2) = \pi/\sqrt{2}$

(39) (A). $I = \int_{\pi/4}^{3\pi/4} \frac{\phi}{1+\sin\phi} dx$ ...... (1)

$\Rightarrow I = \int_{\pi/4}^{3\pi/4} \frac{\pi - \phi}{1+\sin(\pi-\phi)} d\phi = \int_{\pi/4}^{3\pi/4} \frac{\pi-\phi}{1+\sin\phi} d\phi$ ...... (2)

$\therefore\ 2I = \int_{\pi/4}^{3\pi/4} \frac{\pi}{1+\sin\phi} d\phi = \pi\int_{\pi/4}^{3\pi/4} \frac{1-\sin\phi}{\cos^2\phi} d\phi$

$= \pi[\tan\phi - \sec\phi]_{\pi/4}^{3\pi/4} = 2\pi(\sqrt{2}-1)$

$\therefore\ I = \pi\ (\sqrt{2}-1)$

(40) (D). $I = \int_0^1 \log\left(\frac{1-x}{x}\right) dx$ ..(1)

$\Rightarrow\ I = \int_0^1 \log\left[\frac{1-(1-x)}{1-x}\right] dx$

$= \int_0^1 \log\left(\frac{x}{1-x}\right) dx = -\int_0^1 \log\left(\frac{1-x}{x}\right) dx = I$

$\Rightarrow 2I = 0 \Rightarrow I = 0$

(41) (D). $\int_0^1 (1+e^{-x})\, dx < \int_0^1 (1+e^{-x^2})\, dx < \int_0^1 2\, dx \quad \forall\, x \in (0,1)$

$2 - \frac{1}{e} < I < 2$

(42) (A). $I = \int_0^{\infty} \frac{e^{-x}}{e^{-x}+1} dx = -[\log(e^{-x}+1)]_0^{\infty}$

$= -[\log 1 - \log 2] = \log 2$

(43) (A). $\int_{-1}^{1} (x - [x])dx = \int_{-1}^{0} (x-[x])dx + \int_0^1 (x-[x])dx$

$= \int_{-1}^{0} (x+1)dx + \int_0^1 (x-0)dx$

$= \left[\frac{(x+1)^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2} + \frac{1}{2} = 1$

(44) $\because \frac{|x|}{x} = \begin{cases} -1 \text{ when } -1 < x < 0 \\ 1 \text{ when } 0 < x < 2 \end{cases}$

$\therefore I = \int_{-1}^{0} \frac{|x|}{x} dx + \int_{0}^{2} \frac{|x|}{x} dx = \int_{-1}^{0} (-1)\,dx + \int_{0}^{2} 1\,dx$

$= -[x]_{-1}^{0} + [x]_{0}^{2} = -1 + 2 = 1$

(45) **(B).** $\int_{0}^{\pi/2n} \frac{1}{1+\cos^n nx} dx$ [Let $nx = t$, $n\,dx = dt$]

$= \int_{0}^{\pi/2} \frac{1}{1+\cot^n t} \cdot \frac{dt}{n} = \frac{\pi}{4n}$

(46) **(A).** $\int_{0}^{\pi/4} \frac{\sin x + \cos x}{3 + \sin 2x} dx$

$\sin x + \cos x = \frac{d}{dx}[\sin x - \cos x]$
$(\sin x - \cos x)^2 = 1 - \sin 2x$
$4 - (\sin x - \cos x)^2 = 3 + 2\sin 2x$

$= \int_{0}^{\pi/4} \frac{d[\sin x - \cos x]}{4 - (\sin x - \cos x)^2} dx$

$= \frac{1}{4} \log\left[\frac{2 + (\sin x - \cos x)}{2 - (\sin x - \cos x)}\right]_{0}^{\pi/4} = \frac{1}{4} \log 3$

(47) **(B).**

$\int_{0}^{1} x(1-x)^{3/2} dx = \int_{0}^{1} (1-x)\, x^{3/2} dx = \int_{0}^{1} (x^{3/2} - x^{5/2})\, dx$

$= \left[\frac{2}{5} x^{5/2} - \frac{2}{7} x^{7/2}\right] = \frac{2}{5} - \frac{2}{7} = \frac{4}{35}$

(48) **(C).** $\int_{0}^{4} |x-1|\, dx = \int_{0}^{1} -(x-1)\, dx + \int_{1}^{4} (x-1)\, dx$

$= +\int_{0}^{1} (1-x)\, dx + \int_{1}^{4} (x-1)\, dx = x - \frac{x^2}{2}\Big|_{0}^{1} + \frac{x^2}{2} - x\Big|_{1}^{4}$

$= 1 - \frac{1}{2} + 8 - 4 - \frac{1}{2} + 1 = 5$

(49) **(D).** $I_n = \int_{0}^{4} \tan^n x\, dx = \int_{0}^{\pi/4} \tan^{n-2} x . \tan^2 x\, dx$

$= \int_{0}^{\pi/4} \tan^{n-2}(x)\, [\sec^2 x - 1]\, dx$

$I_n = \int_{0}^{\pi/4} \tan^{n-2}(x)\, dx - I_{n-2}$

$I_n + I_{n-2} = \frac{\tan^{(n-1)}(x)}{n-1}\Big|_{0}^{\pi/4}$

$I_n + I_{n-2} = \frac{1}{n-1} \quad \therefore\ I_{10} + I_8 = \frac{1}{9}$

(50) **(D).** $I = \int_{\pi/6}^{\pi/3} \frac{\sin^3 x}{\sin^3 x + \cos^3 x} dx$ .... (1)

$= \int_{\pi/6}^{\pi/3} \frac{\sin^3\left[\frac{\pi}{3} + \frac{\pi}{6} - x\right]}{\sin^3\left[\frac{\pi}{3} + \frac{\pi}{6} - x\right] + \cos^3\left[\frac{\pi}{3} + \frac{\pi}{6} - x\right]} dx$

$= \int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\cos^3 x + \sin^3 x} dx$ .... (2)

Eq. (1) + Eq. (2)

$2I = \int_{\pi/6}^{\pi/3} \frac{\sin^3 x + \cos^3 x}{\cos^3 x + \sin^3 x} dx = \int_{\pi/6}^{\pi/3} 1\, dx$

$\Rightarrow 2I = \pi/6 \Rightarrow I = \pi/12$

(51) **(A).** $\int_{0}^{11} [x]\, dx = \left(\int_{0}^{1} [x]\, dx + \int_{1}^{2} [x]\, dx + ... \int_{10}^{11} [x]\, dx\right)$

$= 0 + 1.x|_1^2 + 2.x|_2^3 + 3.x|_3^4 + ... + 10.x|_{10}^{11}$

$= (2-1) + 2(3-2) + 3(4-3) + \ldots 10(11-10)$

$= 1 + 2 + 3 + \ldots.. + 10 = \frac{10\,(11)}{2} = 5\,(11) = 55$

(52) **(D).** $I_1 = \int_{0}^{\pi/2} x \cdot \sin x\, dx = 1$; $I_2 = \int_{0}^{\pi/2} x \cdot \cos x\, dx = \frac{\pi}{2} - 1$

$I_1 + I_2 = \frac{\pi}{2}$

(53) **(A).** $I = \int_{-\alpha}^{(\pi-\alpha)} \sin|t|\, dt$ where $2x - \alpha = t \Rightarrow dx = \frac{dt}{2}$

$= \frac{1}{2} \int_{-\alpha}^{0} -\sin t\, dt + \frac{1}{2} \int_{0}^{\pi-\alpha} \sin t\, dt$

$= \frac{1}{2} \cos t\Big]_{-\alpha}^{0} - \frac{1}{2} \cos t\Big]_{0}^{\pi-\alpha} = \frac{1}{2}[1 - \cos\alpha] - \frac{1}{2}[-\cos\alpha - 1]$

$= \frac{1}{2}(1 - \cos\alpha) + \frac{1}{2}(1 + \cos\alpha) = 1$

(54) (B).

$$\int_0^{\pi}\frac{\cos^4 x}{\cos^4 x+\sin^4 x}dx = 2\int_0^{\pi/2}\frac{\cos^4 x}{\cos^4 x+\sin^4 x}dx = 2\times\frac{\pi}{4}=\frac{\pi}{2}$$

(55) (C). $I=\int_e^{\pi} x\ f(x)\ dx = \int_e^{\pi}(e+\pi-x)\ f\ (e+\pi-x)\ dx$

$$=\int_e^{\pi}(e+\pi-x)\ f\ (x)\ dx = \int_e^{\pi}(e+\pi)\ f\ (x) - I$$

$$2I=(e+\pi)\frac{2}{e+\pi}\Rightarrow\ I=1$$

(56) (A). The value of $\int_{-\pi/4}^{\pi/4}\log(\sec\theta-\tan\theta)\ d\theta=0$

$\therefore$ log (sec θ – tan θ) is an odd function

$\because$ If f (θ) = log (sec θ – tan θ)

then, f (–θ) = log [sec θ + tan θ]

= – log (sec θ – tan θ) = – f (θ)

(57) (C). $I=\int_0^{\pi/4}\log(\tan x+1)\ dx=\frac{\pi}{8}\log 2$

(58) (B). $I=2\int_0^{\pi/4}\frac{1}{1+\cos 2x}dx=2\int_0^{\pi/4}\frac{1}{2\cos^2 x}dx$

$$=\int_0^{\pi/4}\sec^2 x\ dx=\tan x\Big|_0^{\pi/4}=1$$

(59) (A). $\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left[\frac{1}{1+\sqrt{n}}+\frac{1}{2+\sqrt{2n}}+...+\frac{1}{n+\sqrt{n^2}}\right]$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r+\sqrt{rn}}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\cdot\frac{1}{\frac{r}{n}+\sqrt{\frac{r}{n}}}$$

$$=\int_0^1\frac{dx}{x+\sqrt{x}}=\int_0^1\frac{2t}{t+t^2}dt=2\int_0^1\frac{dt}{1+t}$$

[Putting $x=t^2$ so that $dx = 2t\ dt$]

$$=2\left[\log(1+t)\right]_0^1=2\log 2.=\log 4$$

(60) (B). $\lim_{x\to\infty}\frac{2^k(1+2^k+3^k+.......+n^k)}{n^{k+1}}$

$$=\lim_{x\to\infty}2^k\sum_{r=1}^{n}\frac{r^k}{n^{k+1}}$$

$$\text{Sum}=2^k\int_0^1 x^k dx=2^k\left(\frac{x^{k+1}}{k+1}\right)_0^1=\frac{2^k}{k+1}$$

(61) (B). Given question reduces to

$$\int_0^1\sqrt{\left(\frac{1+x}{1-x}\right)}=\int\frac{1+x}{\sqrt{(1-x^2)}}dx$$

$$=\left[\sin^{-1}x-\sqrt{(1-x^2)}\right]_0^1=\frac{\pi}{2}-(-1)=\frac{\pi}{2}+1$$

(62) (B). $\lim_{n\to\infty}\frac{1}{n}(e^{1/n}+e^{2/n}+e^{3/n}+....+e^{n/n})$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}e^{r/n}=\int_0^1 e^x dx=\left[e^x\right]_0^1=e-1$$

(63) $p=\lim_{n\to\infty}\left[\frac{\prod_{r=1}^{n}(n^3+r^3)}{n^{3n}}\right]^{1/n}$

$$\ln p=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\left(1+\left(\frac{r}{n}\right)^3\right)$$

$$=\int_0^1\ln(1+x^3)\ dx=\ln 2-3+3\lambda$$

(64) (C).

$f(x)\ \ =2x\ +\ 4x^3+6x^5+...........\infty$

$x^2 f(x)=\ \ \ \ \ \ \ 2x^3+4x^5+...........\infty$

– – – –

$(1-x^2)f(x)=2x+2x^3+2x^5+...........\infty$

$$(1-x^2)f(x)=\left(\frac{2x}{1-x^2}\right)\ \therefore f(x)=\frac{2x}{1-x^2}$$

then $\int f(x)\ dx=\int\frac{2x}{(1-x^2)^2}dx=\frac{1}{1-x^2}+C$

(65) (A). $I=\int_0^{\pi/2}e^x\left\{\cos(\sin x)\left(\frac{1+\cos x}{2}\right)+\sin(\sin x)\left(\frac{1-\cos x}{2}\right)\right\}dx$

$$=\frac{1}{2}\int_0^{\pi/2}e^x\ [\underbrace{\{\cos(\sin x)+\sin(\sin x)\}}_{f(x)}+\underbrace{\cos x\{\cos(\sin x)-\sin(\sin x)\}}_{f'(x)}]dx$$

$$=\frac{1}{2}\int_0^{\pi/2}e^x\ [f(x)+f'(x)]dx=\frac{1}{2}e^x f(x)\Big]_0^{\pi/2}$$

$$=\frac{1}{2}\left[e^x\{\cos(\sin x)+\sin(\sin x)\}\right]_0^{\pi/2}$$

$$=\frac{1}{2}[e^{\pi/2}(\cos 1+\sin 1)-1]$$

(66) (B). $I=\int \frac{1}{(1+\cos^2 x)\sin^2 x}dx$

$=\int \frac{(1+\cot^2 x)\,\text{cosec}^2 x\,dx}{(1+2\cot^2 x)}$

Let P = cot x ⇒ dp = – cosec²xdx $=-\int \frac{(1+p^2)\,dp}{1+2p^2}$

$=-\frac{1}{2}\int \frac{(2+2p^2)\,dp}{1+2p^2}=-\frac{1}{2}\int dp-\frac{1}{2}\int \frac{pdp}{1+2p^2}$

$=-\frac{1}{2}p-\frac{1}{4}\int \frac{dp}{1/2+p^2}=-\frac{1}{2}p-\frac{\sqrt{2}}{4}\tan^{-1}\left(p\sqrt{2}\right)$

$=-\frac{1}{2}\cot x-\frac{1}{2\sqrt{2}}\tan^{-1}\left(\cot x\sqrt{2}\right)+c$

(67) (A). $I=\int_0^{3\pi/4}(\sin x+\cos x)\,dx+\int_0^{3\pi/4}\underset{I}{x}\,\underbrace{(\sin x-\cos x)}_{II}\,dx$

$=\int_0^{3\pi/4}(\sin x+\cos x)\,dx+\underbrace{x(-\cos x-\sin x)\big|_0^{3\pi/4}}_{\text{zero}}+\int_0^{3\pi/4}(\sin x+\cos x)\,dx$

$=2\int_0^{3\pi/4}(\sin x+\cos x)\,dx=2(\sqrt{2}+1)$

(68) (C). $\int \frac{e^x(1+nx^{n-1}-x^{2n})}{(1-x^n)\sqrt{1-x^{2n}}}dx$

$=\int e^x\left(\sqrt{\frac{1+x^n}{1-x^n}}+\frac{nx^{n-1}}{(1-x^n)\sqrt{1-x^{2n}}}\right)dx=\frac{e^x\sqrt{1-x^{2n}}}{1-x^n}+C$

(69) (A). $\int 4\cos\left(x+\frac{\pi}{6}\right)\cos 2x.\cos\left(\frac{5\pi}{6}+x\right)dx$

$=2\int\left(\cos(2x+\pi)\cos\frac{2\pi}{3}\right)\cos 2x\,dx$

$=2\int\left(-\cos 2x-\frac{1}{2}\right)\cos 2x\,dx$

$=\int(-2\cos^2 2x-\cos 2x)\,dx=-\int(1+\cos 4x+\cos 2x)\,dx$

$=-x-\frac{\sin 4x}{x}-\frac{\sin 2x}{2}+c=-\left(x+\frac{\sin 4x}{4}+\frac{\sin 2x}{2}\right)+c$

(70) (B). Put $\sqrt{x}=t$ ∴ $\frac{1}{2\sqrt{x}}dx=dt$

$\therefore I=2\int t^2e^t dt=2[t^2e^t-(2t)e^t+2e^t]$

$=(2x-4\sqrt{x}+4)\,e^{\sqrt{x}}$

(71) (C). $I=\int(\cos x-\sin x)(3+4\sin 2x)dx$

Here integration of cos x – sin x = sin x + cos x
and 3 + 4 sin 2x = 3 + 4 ((sin x + cos x)² – 1)
Put sin x + cos x = 1 = (cos x – sin x)dx = dt

So, $I=\int(3+4(t^2-1)dt=\frac{t}{3}[4t^2-3]+c$

$=\left(\frac{\sin x+\cos x}{3}\right)[4(\sin x+\cos x)^2-3]$

$=\left(\frac{\sin x+\cos x}{3}\right)(1+4\sin 2x)+c$

(72) (B). $I=\int \frac{x^2\left(1-\frac{1}{x^2}\right)dx}{x^2\left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}\right)^{1/2}}$

Let $x+\frac{1}{x}=p\Rightarrow\left(1-\frac{1}{x^2}\right)dx=dp$;

$I=\int \frac{dp}{p\sqrt{p^2-2}}=\frac{1}{\sqrt{2}}\sec^{-1}\frac{p}{\sqrt{2}}=\frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{x^2+1}{\sqrt{2}x}\right)+c$

(73) (C). $I=\frac{1}{2}\int_0^{\pi/2}x\,|\cos 2x|\,dx$; $2x=t\Rightarrow dx=\frac{dt}{2}$

$I=\frac{1}{8}\int_0^{\pi}t\,|\cos t|\,dt$; $I=\frac{1}{8}\int_0^{\pi}(\pi-t)\,|\cos t|\,dt$

$2I=\frac{\pi}{8}\int_0^{\pi}|\cos t|\,dt=\frac{2\pi}{8}\Rightarrow I=\frac{\pi}{8}$

(74) (B). Using property : $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$

$I=\int_{-1}^{1}(1-x)^{1/2}(1+x)^{3/2}dx$

$2I=\int_{-1}^{1}(1+x)^{1/2}(1-x)^{1/2}[(1-x)+(1+x)]\,dx$

$2I=2\int_{-1}^{1}\sqrt{1-x^2}\,dx$; $I=2\int_0^1\sqrt{1-x^2}\,dx$ $(x=\sin\theta)$

$\Rightarrow dx=\cos\theta\,d\theta$

$I=2\int_0^{\pi/2}\cos^2\theta\,d\theta=\frac{\pi}{2}$

(75) $\int \frac{x+\sin x}{1+\cos x}dx = \int \frac{x+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}dx$

$= \int\left(\frac{1}{2}x\sec^2\frac{x}{2}+\tan\frac{x}{2}\right)dx = x\tan\frac{x}{2}+c$

(76) **(A).** $I = \int \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

dividing numerator and denominator by $\cos^2 x$

$\Rightarrow I = \int \frac{\sec^2 x\,dx}{a^2+b^2\tan^2 x}$

Let $b\tan x = t$, $\sec^2 x\,dx = \frac{dt}{b}$

$\Rightarrow I = \frac{1}{b}\int\frac{dt}{a^2+t^2} = \frac{1}{ab}+\tan^{-1}\frac{t}{a}$

$\Rightarrow \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan x\right)+c$

(77) **(D).** Let $I = \int\frac{x^3+x}{x^4-9}dx$

$= \int\frac{x^3}{x^4-9}dx + \int\frac{x}{x^4-9}dx = I_1+I_2+C$ (say)

where $I_1 = \int\frac{x^3}{x^4-9}dx$ and $I_2 = \int\frac{x}{x^4-9}dx$

Put $x^4-9=t$, so that $4x^3\,dx = dt$

$\therefore \quad I_1 = \frac{1}{4}\int\frac{1}{dt}dt = \frac{1}{4}\log_e|t| = \frac{1}{4}\log_e|x^4-9|$

$I_2 = \int\frac{x}{x^4-9}dx = \int\frac{x}{(x^2)^2-3^2}dx$

Put $x^2=t$ so that $2x\,dx = dt$

$I_2 = \frac{1}{2}\int\frac{dt}{t^2-3^2} = \frac{1}{2}\cdot\frac{1}{2\times3}\log_e\left|\frac{t-3}{t+3}\right|$

Hence, $I = \frac{1}{4}\log_e|x^4-9|+\frac{1}{12}\log_e\left|\frac{x^2-3}{x^2+3}\right|+C$

(78) **(B).** $\frac{2x}{(x-1)(x-4)} = \frac{C}{x-1}+\frac{D}{x-4}$

$2x = C(x-4)+D(x-1)$

$\therefore C=-2/3,\ D=8/3$

$\therefore \int\frac{e^{x-1}}{(x-1)(x-4)}2x\,dx = \int e^{x-1}\left(\frac{-2/3}{x-1}+\frac{8/3}{x-4}\right)dx$

$= -\frac{2}{3}F(x-1)+\frac{8}{3}e^3F(x-4)+c$

$\therefore A=-2/3,\ B=8/3\ e^3$

(79) **(A).** $I = \int\frac{a^x}{\sqrt{1-a^{2x}}}dx = \int\frac{a^x}{\sqrt{1^2-(a^x)^2}}dx$

Let $a^x = t$. Then, $d(a^x) = dt$

$\Rightarrow a^x\log_e a\,dx = dt \Rightarrow dx = \frac{dt}{a^x\log_e a}$

$\therefore I = \int\frac{a^x}{\sqrt{1^2-t^2}}\frac{dt}{a^x\log a} = \frac{1}{\log a}\int\frac{dt}{\sqrt{1^2-t^2}}$

$= \frac{1}{\log a}\sin^{-1}t + C = \frac{1}{\log a}\sin^{-1}(a^x)+C$

(80) **(A).** $I = \int_{\pi/2}^{3\pi/2}1.f(x)\,dx = xf(x)]_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2}f'(x).x\,dx$

$= \frac{3\pi}{2}f\left(\frac{3\pi}{2}\right)-\frac{\pi}{2}f\left(\frac{\pi}{2}\right)-\underbrace{\int_{\pi/2}^{3\pi/2}\frac{\cos x}{x}.x\,dx}_{-2}$

$= b.\frac{3\pi}{2}-a.\frac{\pi}{2}+2 = 2-\frac{\pi}{2}(a-3b)$

(81) **(C).** Put $x=\cos\theta$

$\therefore \frac{1-\cos\theta}{1+\cos\theta}=\tan^2\frac{\theta}{2}$ and $dx = -\sin\theta\,d\theta$

$I = \int\tan^{-1}[\tan(\theta/2)](-\sin\theta\,d\theta)]$

$= -\frac{1}{2}\int\theta\sin\theta d\theta = -\frac{1}{2}\int -\theta\cos\theta+\sin\theta]$

$= \frac{1}{2}[x\cos^{-1}x-\sqrt{(1-x^2)}]$

(82) **(A).** $\int x^2\sin x\,dx = x^2\int\sin x\,dx - \int(2x\int\sin x\,dx)dx$

$= -x^2\cos x+2[x\int\cos x\,dx-\int(1\int\cos x\,dx)dx$

$= -x^2\cos x + 2x\sin x - 2\cos x + c$

(83) **(A).** $I = \int\left(x+\frac{1}{x}\right)^{n+5}\left(\frac{x^2-1}{x^2}\right)dx$,

put $x+\frac{1}{x}=p$ then, $\left(1-\frac{1}{x^2}\right)dx = dp$

$\Rightarrow \int p^{n+5}\,dp = \frac{p^{n+6}}{n+6}+c = \frac{\left(x+\frac{1}{x}\right)^{n+6}}{n+6}+c$

(84) **(C).** Given integral $I=\int\left(1+\frac{1}{x^2-1}\right)dx$

$=\int dx+\int\frac{dx}{(x-1)(x+1)}=x+\frac{1}{2}\int\left(\frac{1}{x-1}-\frac{x}{x+1}\right)dx$

$=x+\frac{1}{2}\log\left(\frac{x-1}{x+1}\right)+c$

(85) **(C).** Let $I=\int\frac{1}{(x+1)\sqrt{x^2-1}}dx$

Put $x+1=\frac{1}{t}$ so that $dx=-\frac{1}{t^2}dt$

$\therefore\ I=\int\frac{1}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2-1}}\left(-\frac{1}{t^2}\right)dt$

$=-\int\frac{dt}{\sqrt{1-2t}}=-\int(1-2t)^{-1/2}dt$

$=-\frac{(1-2t)^{1/2}}{(-2)(1/2)}+C=\sqrt{1-2t}+C$

$=\sqrt{1-\frac{2}{x+1}}+C=\sqrt{\frac{x-1}{x+1}}+C$

(86) $\int\frac{dx}{x(x^n+1)}=\int\frac{dx}{x^{n+1}\left(1+\frac{1}{x^n}\right)}$

Put $1+\frac{1}{x^n}=t\ \Rightarrow\ \frac{1}{x^{n+1}}dx=dt$

$I=-\frac{1}{n}\int\frac{dt}{t}=\frac{1}{n}\ell n\left|\frac{x^n}{1+x^n}\right|+C$

(87) **(B).** $\int\frac{a^{\sqrt{x}}dx}{\sqrt{x}}$

Put $\sqrt{x}=t\Rightarrow\frac{dx}{2\sqrt{x}}=dt$; $I=2\int a^t dt=\frac{2a^{\sqrt{x}}}{\ell n\ a}+C$

(88) **(A).** $\int\frac{1}{9x^2-4}dx=\frac{1}{9}\int\frac{1}{x^2-(2/3)^2}dx$

$=\frac{1}{9}\cdot\frac{1}{2\times\frac{2}{3}}\log\left|\frac{x-\frac{2}{3}}{x+\frac{2}{3}}\right|+C\quad=\frac{1}{12}\log\left|\frac{3x-2}{3x+2}\right|+C$

(89) **(D).** $I=2\int_0^2 x^4.\sqrt{4-x^2}\,dx$

Put $x=2\sin\theta\ \Rightarrow\ 128\int_0^{\pi/2}\sin^4\theta.\cos^2\theta\,d\theta$.

Use Walli's formula to get 4π.

(90) **(A).** $\det.A=e^x.e^{e^x}.e^{e^{e^x}}$

$\therefore\ I=\int_0^1 e^x.e^{e^x}.e^{e^{e^x}}dx=e^{e^{e^x}}\Big|_0^1=e^{e^e}-e^e$

(91) **(A).** $I=\int\frac{x^2+1}{x^4+1}dx=\int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$

[Dividing the num. and denom. by x^2]

$=\int\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}dx$

Let $x-\frac{1}{x}=t\Rightarrow d\left(x-\frac{1}{x}\right)=dt\ \Rightarrow\left(1+\frac{1}{x^2}\right)dx=dt$

$\therefore I=\int\frac{dt}{t^2+(\sqrt{2})^2}=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)+C$

$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right)+C=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\ x}\right)+C$

(92) **(C).** $I=\int\frac{dx}{x\,(x^{2007}+1)}=\int\frac{x^{2007}+1-x^{2007}}{x\,(x^{2007}+1)}dx$

$=\int\left(\frac{1}{x}-\frac{x^{2006}}{1+x^{2007}}\right)dx=\ln x-\frac{1}{2007}\ln(1+x^{2007})$

$=\frac{\ln x^{2007}-\ln(1+x^{2007})}{2007}=\frac{1}{2007}\ln\left(\frac{x^{2007}}{1+x^{2007}}\right)+C$

$p+q+r=6021$

(93) **(B).** $\lim_{x\to0}\frac{\int_0^{x^2}\cos t^2dt}{x\sin x}\qquad\left(\frac{0}{0}\text{ form}\right)$

$=\lim_{x\to0}\frac{2x\cos x^4}{x\cos x+\sin x}$ [Using L'Hospital rule]

$= \lim_{x\to 0} \dfrac{2\cos x^4 - 8x^4 \sin x^4}{2\cos x - x\sin x}$ [Using L'Hospital rule again]

$= \dfrac{2-0}{2-0} = 1$

(94) **(C).** $I = \int_0^{\pi/2} \dfrac{d}{dx}((\sin x)^n)\,dx = (\sin x)^x \Big|_0^{\pi/2}$

$= 1 - \lim_{x\to 0} (\sin x)^x = 1 - 1 = 0$

(95) **(A).** Let $I = \int \dfrac{3 + 2\cos x}{(2+3\cos x)^2} dx$

Multiplying Nr. & Dr. by $\text{cosec}^2 x$

$\Rightarrow\ I = \int \dfrac{(3\,\text{cosec}^2 x + 2\cot x\,\text{cosec}\,x)}{(2\,\text{cosec}\,x + 3\cot x)^2} dx$

$= -\int \dfrac{-3\,\text{cosec}^2 x - 2\cot x\,\text{cosec}\,x}{(2\,\text{cosec}\,x + 3\cot x)^2} dx$

$= \dfrac{1}{2\,\text{cosec}\,x + 3\cot x} = \left(\dfrac{\sin x}{2 + 3\cos x}\right) + c$

(96) **(B).** Here $R(\sin x, \cos x) = \dfrac{1}{\sin x(2\cos^2 x - 1)}$

$R(\sin x, \cos x) = \dfrac{1}{-\sin x(2\cos^2 x - 1)}$

$= R - (\sin x, \cos x)$

So we put $\cos = t\ \Rightarrow\ -\sin x\,dx = dt$

$I = \int \dfrac{\sin dx}{(1-\cos^2 x)(2\cos^2 x - 1)} = \int \dfrac{dt}{(t^2-1)(2t^2-1)}$

$= \int \dfrac{dt}{t^2 - 1} - 2\int \dfrac{dt}{2t^2 - 1}$

$= \dfrac{1}{2}\ell n\left|\dfrac{\cos x - 1}{\cos x + 1}\right| - \dfrac{1}{\sqrt{2}}\ell n\left|\dfrac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1}\right| + C$

(97) **(A).** $I = \int_0^2 \dfrac{2x^3 - 6x^2 + 9x - 5}{x^2 - 2x + 5} dx$

(put $x - 1 = t$ as $x \to 0, t = -1$ and $x \to 2, t \to 1$)

$= \int_{-1}^{1} \dfrac{2(1+t)^3 - 6(1+t)^2 + 9(1+t) - 5}{t^2 + 4} dt$

$= \int_{-1}^{1} \dfrac{2t^3 + 3t}{t^2 + 4} dt \Rightarrow I = 0$

(98) **(A).** Here, $p = -\dfrac{7}{5}, q = -\dfrac{3}{5}$; $\dfrac{p + q - 2}{2} = -2$

$I = \int \sin^{-7/5} \cos^{-3/5} x\,dx = \int \dfrac{\cos^{-3/5} x}{\sin^{-3/5} x \sin^2 x} dx$

$= \int (\cot x)^{-3/5} \text{cosec}^2 x\,dx$

Put $\cot x = t \Rightarrow \text{cosec}^2 x = -dt$

So, $I = -\int t^{-3/5} dt = -\dfrac{5}{2}(\cot x)^{2/5} + C$

(99) **(C).** We have $\int_0^1 \dfrac{1}{\sqrt{1+x} + \sqrt{x}} dx$

$= \int_0^1 \dfrac{\sqrt{1+x} - \sqrt{x}}{(\sqrt{1+x} + \sqrt{x})(\sqrt{1+x} - \sqrt{x})} dx$

$= \int_0^1 (\sqrt{1+x} - \sqrt{x})\,dx = \left[\dfrac{2}{3}(1+x)^{3/2} - \dfrac{2}{3}x^{3/2}\right]_0^1$

$= \left[\dfrac{2}{3}(1+1)^{3/2} - \dfrac{2}{3}(1)^{3/2}\right] - \left[\dfrac{2}{3}(1+0)^{3/2} - \dfrac{2}{3}(0)^{3/2}\right]$

$= \dfrac{2}{3}[2^{3/2} - 1] - \dfrac{2}{3}[1 - 0] = \dfrac{2}{3}[2\sqrt{2} - 2] = \dfrac{4}{3}[\sqrt{2} - 1]$

(100) **(B).** We have $\int_0^{\pi/4} \sqrt{1 - \sin 2x}\,dx$

$\int_0^{\pi/4} \sqrt{\sin^2 x + \cos^2 x - 2\sin x\cos x}\,dx = \int_0^{\pi/4} \sqrt{(\cos x - \sin x)^2}$

$= \int_0^{\pi/4} |(\cos x - \sin x)|\,dx = \int_0^{\pi/4} (\cos x - \sin x)dx$

$= \int_0^{\pi/4} (\cos x - \sin x)dx = [\sin x + \cos x]_0^{\pi/4}$

$= \left(\sin\dfrac{\pi}{4} + \cos\dfrac{\pi}{4}\right) - (\sin 0 + \cos 0) = \left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}\right) - (0+1) = \dfrac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$

(101) **(A).** We have, $\int_0^4 \dfrac{1}{\sqrt{x^2 + 2x + 3}} dx$

$= \int_0^4 \dfrac{1}{\sqrt{(x^2+1)^2 + (\sqrt{2})^2}} dx$

$= \left[\log\left|x + 1 + \sqrt{(x+1)^2 + (\sqrt{2})^2}\right|\right]_0^4$

$= \left[\log\left|x + 1 + \sqrt{x^2 + 2x + 3}\right|\right]_0^4$

$= \log(5 + \sqrt{16 + 8 + 3}) - \log(1 + \sqrt{3})$

$= \log(5+3\sqrt{3}) - \log(1+\sqrt{3}) = \log\left(\frac{5+3\sqrt{3}}{1+\sqrt{3}}\right)$

(102) (C). $I = \int_1^2 1^x\,dx + \int_2^3 2^x\,dx = 1 + 4(\log 2)^{-1}$

(103) (D). $\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx = 2\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}dx$

$= -2\int_0^{\pi/2} \frac{(-\sin x)}{1+\cos^2 x}dx = -2\,(\tan^{-1}(\cos x))_0^{\pi/2}$

$= -2\,(\tan^{-1} 0 - \tan^{-1} 1) = \frac{\pi}{2}$

$\therefore \frac{\cos\alpha}{1+\sin^2\alpha} = \frac{1}{2}$ i.e, $2\cos\alpha = 2-\cos^2\alpha$

$\cos^2\alpha + 2\cos\alpha - 2 = 0$

$\cos\alpha = \frac{-2\pm\sqrt{4+8}}{2} = -1\pm\sqrt{3} = \sqrt{3}-1$

$\therefore$ α takes exactly one value in $(0, \pi/2)$

(104) (A). Let $f(x) = x^3 e^{x^4}$,

then $f(-x) = (-x)^3.\ e^{(-x)^4} = -x^3 e^{x^4} = -f(x)$

Hence f(x) is an odd function.

$\therefore \int_{-1}^{1} f(x)\,dx = 0$; or $\int_{-1}^{1} x^3 e^{x^4} dx = 0$

(105) (A). $I = \int_{\pi/4}^{3\pi/4} \frac{\cos(\pi-x)}{1-\cos(\pi-x)}dx = \int_{\pi/4}^{3\pi/4} \frac{-\cos x}{1+\cos x}dx$

$2I = \int_{\pi/4}^{3\pi/4} \cos x\left(\frac{1}{1-\cos x} - \frac{1}{1+\cos x}\right)dx$

$= 2\int_{\pi/4}^{3\pi/4} \cot^2 x\,dx = 2\left(2-\frac{\pi}{2}\right)$ $\quad\therefore I = 2-\frac{\pi}{2}$

(106) (A). $\lim_{n\to\infty}\left[\frac{1}{na} + \frac{1}{na+1} + ... + \frac{1}{na+n(b-a)}\right]$

$= \lim_{n\to\infty} \sum_{r=1}^{n(b-a)} \frac{1}{na+r} = \lim_{n\to\infty} \sum_{r=1}^{n(b-a)} \frac{1}{a+(r/n)}\frac{1}{n}$

$= \int_0^{b-a} \frac{1}{a+x}dx = \ln\left(\frac{b}{a}\right)$

(107) (C). Let $I = \int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}dx$ ......(i)

Then, $I = \int_1^2 \frac{\sqrt{3-x}}{\sqrt{3-(3-x)}+\sqrt{3-x}}dx$

$\Rightarrow I = \int_1^2 \frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}dx$ .... (ii)

Adding (i) and (ii), we get

$2I = \int_1^2 \frac{\sqrt{x}+\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}dx = \int_1^2 1.dx = [x]_1^2 = 2-1 = 1 \Rightarrow I = 1/2$

(108) (A). We have $\int_0^{\pi/4} \sin 3x \sin 2x\,dx$

$= \frac{1}{2}\int_0^{\pi/4} (2\sin 3x \sin 2x)\,dx = \frac{1}{2}\int_0^{\pi/4} (\cos x - \cos 5x)\,dx$

$= \frac{1}{2}\left[\sin x - \frac{\sin 5x}{5}\right]_0^{\pi/4}$

$= \frac{1}{2}\left[\left(\sin\frac{\pi}{4} - \frac{\sin\frac{5\pi}{4}}{5}\right) - \left(\sin 0 - \frac{\sin 0}{5}\right)\right]$

$= \frac{1}{2}\left[\frac{1}{\sqrt{2}} + \frac{1}{(\sqrt{2})5}\right] = \frac{1}{2(5\sqrt{2})} = \frac{3\sqrt{2}}{10}$

(109) (C). $g(x) = \int_0^x f(x)\,dx = \begin{cases} \frac{ax^2}{2} - x & , 0 \le x < 1 \\ \frac{a}{2} - 1 + \frac{x^3}{3} + bx - \frac{1}{3} - b, & x \ge 1 \end{cases}$

$\because g(1^-) = g(1) = g(1^+)$ $\quad\therefore g'(1^+) = g'(1^-)$
i.e., $a = 2+b$

(110) (C). We have $\int_0^{\pi} \sin^3 x\,dx$

$= \int_0^{\pi} \frac{3\sin x - \sin 3x}{4}dx$ $\quad[\because \sin 3x = 3\sin x - 4\sin^3 x]$

$= \frac{1}{4}\int_0^{\pi} (3\sin x - \sin 3x)\,dx = \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right]_0^{\pi}$

$= \frac{1}{4}\left[\left(-3\cos\pi + \frac{\cos 3\pi}{3}\right) - \left(-3\cos 0 + \frac{\cos 0}{3}\right)\right]$

$= \frac{1}{4}\left[\left(3-\frac{1}{3}\right) - \left(-3+\frac{1}{3}\right)\right] = \frac{4}{3}$

(111) $I = \int_{-\log 2}^{\log 2} \sin\left\{\frac{e^x - 1}{e^x + 1}\right\} dx$

If $f(x) = \sin\left\{\frac{e^x - 1}{e^x + 1}\right\}$

$f(-x) = \sin\left\{\frac{1 - e^x}{1 + e^x}\right\} = -\sin\left\{\frac{e^x - 1}{e^x + 1}\right\} = -f(x)$

Hence f (x) is an odd function of x $\therefore I = 0$

(112) **(D).** $I = \int_0^{2\pi} \frac{e^{|\sin x|} \cos x}{1 + e^{\tan x}} dx$ .... (1)

$I = \int_0^{2\pi} \frac{e^{|\sin x|} \cos x}{1 + e^{-\tan x}} dx$ .... (2)

Using $\int_0^a f(x) dx = \int_0^a f(a - x)\, dx$

Adding (1) and (2) we get

$2I = \int_0^{2\pi} e^{|\sin x|} \cos x\, dx = 2\int_0^{\pi} e^{\sin x} \cos x\, dx$

$\{\because f(2a - x) = f(x)\}$

$\therefore I = 0$

$f(a - x) = -f(x)$

(113) **(B).** $f(x) = \{2x\}$ is a periodic function with period 1/2

Let $I = \int_{-3/2}^{10} \{2x\} dx = \int_{-3(1/2)}^{20(1/2)} \{2x\} dx$

$= 23\int_0^{1/2} 2x\, dx$ (as $\{2x\} = 2x - [2x]$ and when

$x \in [0, 1/2), [2x] = 0) = 23x^2 \Big|_0^{1/2} = \frac{23}{4}$

(114) **(C).** Let $\sin^{-1} x = \theta$ or, $x = \sin\theta$.
Then, $dx = d(\sin\theta) = \cos\theta\, d\theta$.

Now, $x = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$ and $x = \frac{1}{\sqrt{2}}$

$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$\therefore \int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1 - x^2)^{3/2}} dx = \int_0^{\pi/4} \frac{\theta}{\cos^3\theta} \cos\theta\, d\theta = \int_0^{\pi/4} \underset{I}{\theta} \underset{II}{\sec^2\theta}\, d\theta$

$= [\theta \tan\theta]_0^{\pi/4} - \int_0^{\pi/4} 1.\tan\theta\, d\theta$

$= [\theta \tan\theta]_0^{\pi/4} + [\log\cos\theta]_0^{\pi/4}$

$= \left(\frac{\pi}{4} - 0\right) + \left(\log\left(\frac{1}{\sqrt{2}}\right) - \log 1\right) = \frac{\pi}{4} - \frac{1}{2}\log 2$

(115) **(B).** Let $I = \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} dx$

Let cos x = t and – sin x dx = dt.
Now, $x = 0 \Rightarrow t = \cos 0 = 1$ and

$x = \frac{\pi}{2} \Rightarrow t = \cos\frac{\pi}{2} = 0$

$\therefore I = \int_1^0 \frac{\sin x}{1 + t^2}\left(\frac{-dt}{\sin x}\right) = -\int_1^0 \frac{dt}{1 + t^2}$

$= -\left[\tan^{-1} t\right]_1^0 = -[\tan^{-1} 0 - \tan^{-1}] = -\left[0 - \frac{\pi}{4}\right] = \frac{\pi}{4}$

(116) **(D).** $\int_{-1}^{1} f(x) dx = \int_{-1}^{0} f(x) dx + \int_0^1 f(x) dx$

$= \int_{-1}^{0} (1 - 2x)\ dx + \int_0^1 (1 + 2x)\ dx$

$= [x - x^2]_{-1}^0 + [x + x^2]_0^1 = 4$

(117) **(A).** $\int \frac{dx}{\sqrt{2 - x^2}} = \sin^{-1}\frac{x}{\sqrt{2}} + c$

$\int_0^1 \frac{dx}{\sqrt{2 - x^2}} = \sin^{-1}\frac{x}{\sqrt{2}}\Big|_0^1 = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + c - \sin^{-1}(0)$

$-c = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

(118) **(B).** $\int_0^4 \{\sin\{x\}\} dx = 4\int_0^4 \{\sin x\} dx$

$(\therefore \int_0^T f(x) dx = n\int_0^T f(x) dx \quad \because f(x + T) = f(x))$

$= 4\int_0^4 \sin x\, dx = -4[\cos x]_0^1 = 4(1 - \cos 1)$

(119) **(A).** Let $I = \int_0^{\pi/2} \frac{\phi(x)}{\phi(x) + \phi\left(\frac{\pi}{2} - x\right)} dx$

then $I = \int_0^{\pi/2} \frac{\phi\left(\frac{\pi}{2} - x\right)}{\phi\left(\frac{\pi}{2} - x\right) + \phi(x)} dx$

Adding, $2I = \int_0^{\pi/2} \frac{\phi(x)+\phi\left(\frac{\pi}{2}-x\right)}{\phi\left(\frac{\pi}{2}-x\right)+\phi(x)} dx = \int_0^{\pi/2} 1.dx = [x]_0^{\pi/2}$

$= \pi/2 \Rightarrow I = \pi/4$

(120) **(B).** $I = \int_0^{\pi} x \sin x \cos^4 x dx$

$\int_0^{\pi} (\pi - x)\sin(\pi - x)\cos^4(\pi - x) dx$

$= \int_0^{\pi} (\pi - x)\sin x \cos^4 x dx = \pi\int_0^{\pi} \sin x \cos^4 x\ dx - I$

$\Rightarrow 2I = \pi\left[\frac{-\cos^5 x}{5}\right]_0^{\pi} = \frac{\pi}{5}(1+1) = \frac{2\pi}{5} \Rightarrow I = \pi/5$

(121) **(A).** Let $f(x) = |\sin x|$.
Then $f(-x) = |\sin(-x)| = |-\sin x| = |\sin|x = f(x)$
So, $f(x)$ is an even function.

$\therefore \int_{-\pi/2}^{\pi/2} |\sin x| dx = 2\int_0^{\pi/2} |\sin x| dx = 2\int_0^{\pi/2} \sin x dx$

$\left[\because \sin x \ge 0 \text{ for } 0 \le x \le \frac{\pi}{2}\right]$

$\Rightarrow \int_{-\pi/2}^{\pi/2} |\sin x| dx = 2[-\cos x]_0^{\pi/2}$

$= 2\left(-\cos\frac{\pi}{2} + \cos 0\right) = 2$

(122) **(B).** Let $I = \int_0^{\pi/2} \log \tan x dx$ .... (i)

Then, $I = \int_0^{\pi/2} \log\tan\left(\frac{\pi}{2} - x\right) dx$

$\Rightarrow I = \int_0^{\pi/2} \log\cot x dx$ .... (ii)

Adding (i) and (ii), we get

$2I = \int_0^{\pi/2} (\log\tan x + \log\cot x)dx = \int_0^{\pi/2} \log(\tan x.\cot x)dx$

$= \int_0^{\pi/2} \log 1.dx = \int_0^{\pi/2} 0.dx = 0 \Rightarrow I = 0$

(123) **(D).** We have $\int_0^3 [x]dx$

$= \int_0^1 [x]dx + \int_1^2 [x]dx + \int_2^3 [x]dx = \int_0^1 0\,dx + \int_1^2 1.dx + \int_2^3 2.dx$

$= 0 + [x]_1^2 + [2x]_2^3 = (2-1) + (6-4) = 3$

(124) **(A).** We know that |sinx| is a periodic function of π

Hence $\int_0^{4\pi} |\sin x| dx = 4\int_0^{\pi} |\sin x| dx = 4\int_0^{\pi} \sin x dx$

$= 4[-\cos x]_0^{\pi} = 8$

(125) **(A).**

$f(x) = e^x + \int_0^1 e^x f(x)\,dt = e^x + ke^x$ where $k = \int_0^1 f(t)\,dt$

$\therefore k = \int_0^1 (e^t + ke^t)\,dt = e + ke - 1 - k \quad \therefore k = \frac{e-1}{2-e}$

Thus $f(x) = e^x\left(1 + \frac{e-1}{2-e}\right) = \frac{e^x}{2-e}$

(126) **(C).** Since, $0 < x < 1$ then $x > x^2 \Rightarrow -x < -x^2$

$e^{-x} < e^{-x^2}$

$e^{-x}\cos^2 x < e^{-x^2}\cos^2 x$

$\therefore \int_0^1 e^{-x}\cos^2 x\,dx < \int_0^1 e^{-x^2}\cos^2 x$

$I_1 < I_2$

(127) **(D).** $\cos^2(\pi + x) = \cos^2 x$

$I_1 = \int_0^{3\pi} f(\cos^2 x)dx = 3\int_0^{\pi} f(\cos^2 x)dx = 3I_2$

$\therefore I_1 = 3I_2$

(128) **(C).** $I_{10} = \int_1^e 1.(\ln x)^{10} dx = \left[(\ln x)^{10} x\right]_1^e - \int_1^e 10(\ln x)^9.\frac{1}{x}.x\,dx$

$= e - 0 - 10 = \int_1^e (\ln x)^9 dx = e - 10I_9$

$I_{10} + 10I_9 = e$

(129) **(D).** $I = \int_0^{\pi/2} 2\cos\theta\cos 2\theta\,d\theta$

$= 2\int_0^{\pi/2} (1 - 2\sin^2\theta)\cos\theta\,d\theta = 2\left(\sin\theta - \frac{2}{3}\sin^3\theta\right)_0^{\pi/2} = \frac{2}{3}$

(130) $f(x) = \int_1^x \sqrt{2-t^2}\,dt$

$f'(x) = \sqrt{2-x^2}\,.\,1 - \sqrt{2-1}\,.\,0 = \sqrt{2-x^2}$

$\therefore \quad x^2 = f'(x) = \sqrt{2-x^2}$

or $x^4 + x^2 - 2 = 0$ or $(x^2+2)(x^2-1) = 0 \therefore x = \pm 1$ (only real)

(131) (D). $I = \int_0^{\pi/2} e^{-\alpha} \sin x\, dx \;;\; x \in \left[0, \frac{\pi}{2}\right]$

$\Rightarrow 0 < \sin x < 1 \Rightarrow -\alpha < -\alpha \sin x < 0$

$\Rightarrow e^{-\alpha} < e^{-\alpha \sin x} < e^0 = 1$

$\therefore \; I < \int_0^{\pi/2} 1\, dx = \frac{\pi}{2}$ and $I > \int_0^{\pi/2} e^{-\alpha} dx = \frac{\pi}{2} e^{-\alpha} > 0$

(132) (B). $I = \int_1^4 (\{x\})^{[x]} dx = \int_1^4 (x-[x])^{[x]} dx$

$= \int_1^2 (x-[x])^{[x]} dx + \int_2^3 (x-[x])^{[x]} dx + \int_3^4 (x-[x])^{[x]} dx$

$= \int_1^2 (x-1)^1 dx + \int_2^3 (x-2)^2 dx + \int_3^4 (x-3)^3 dx$

$= \left[\frac{(x-1)^2}{2}\right]_1^2 + \left[\frac{(x-2)^3}{3}\right]_2^3 + \left[\frac{(x-3)^4}{4}\right]_3^4$

$= \left[\frac{1}{2} - 0\right] + \left[\frac{1}{3} - 0\right] + \left[\frac{1}{4} - 0\right] = \frac{13}{12}$

(133) (B). $I = \int_{-1}^{1} \frac{dx}{(1+e^x)(1+x^2)}$ .... (1)

$= \int_{-1}^{1} \frac{dx}{1+e^{-x}} \cdot \frac{1}{1+x^2}$

$I = \int_{-1}^{1} \frac{e^x\, dx}{(1+e^x)(1+x^2)}$ .... (2)

Adding eq. (1) and eq. (2)

$2I = \int_{-1}^{1} \frac{(1+e^x)\, dx}{(1+e^x)(1+x^2)} = \int_{-1}^{1} \frac{dx}{(1+x^2)} = 2\int_0^1 \frac{dx}{(1+x^2)}$

$I = \int_0^1 \frac{dx}{(1+x^2)} = \tan^{-1}(1) = \frac{\pi}{4}$

[convert it into value of definite integral 'I' is same as]

EXERCISE-2

(1) $I = \lim_{x \to 0} \frac{f(x)}{g(x)}$, where $f(0) = 0$, $g(0) = 0$

$\therefore \quad I = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$ where

$f'(x) = \sin\sqrt{x^2}\, \frac{d}{dx}(x^2) - 0 = 2x \sin x$

$\therefore \quad I = \lim_{x \to 0} \frac{2x \sin x}{3x^2} = \frac{2}{3} \lim_{x \to 0} \frac{\sin x}{x} = \frac{2}{3}$

(2) (B). $\int \frac{dx}{\sqrt{2ax - x^2}} = \int \frac{dx}{\sqrt{a^2 - (x^2 - 2ax + a^2)}}$

$= \int \frac{dx}{\sqrt{a^2 - (x-a)^2}} = \sin^{-1}\left(\frac{x-a}{a}\right) + C$

$\therefore f(x) = \sin^{-1} x$ and $g(x) = \frac{x-a}{a}$

(3) (D). Given : $f(x) = \int_x^2 \frac{dy}{\sqrt{1+y^3}}$; $f(2) = 0$

Also, $f'(x) = -\frac{1}{\sqrt{1+x^3}}$

Now, let $I = \int_0^2 \underset{II}{x}\, \underset{I}{\underbrace{f(x)}}\, dx = f(x).\frac{x^2}{2}\Big|_0^2 - \frac{1}{2}\int_0^2 f'(x)\, x^2 dx$

$= 0 + \frac{1}{2}\int_0^2 \frac{x^2}{\sqrt{1+x^3}} dx$

Put $1 + x^3 = t^2$

$\Rightarrow 3x^2\, dx = 2t\, dt = \frac{1}{2} \cdot \frac{2}{3} \int_1^3 \frac{t\, dt}{t} = \frac{1}{3}(3-1) = \frac{2}{3}$

(4) (D). $\ln x = \ln\left(\frac{x}{2}\right) + \ln 2$

$\Rightarrow \quad \tan(\ln x) = \frac{\tan(\ln x/2) + \tan(\ln 2)}{1 - \tan(\ln x/2)\tan(\ln 2)}$

$\Rightarrow \tan(\ln x) \tan\left(\ln \frac{x}{2}\right) \tan(\ln 2)$

$= \tan(\ln x) - \tan\left(\ln \frac{x}{2}\right) - \tan(\ln 2)$

$\therefore \quad I = \int \frac{\tan(\ln x)}{x} dx - \int \frac{\tan(\ln x/2)}{x} dx - \int \frac{\tan(\ln 2)}{x} dx$

$$= \ln \sec(\ln x) - \ln \sec\left(\ln \frac{x}{2}\right) - \tan(\ln 2)\ln x$$

$$= \ln\left\{\frac{\sec(\ln x)}{\sec(\ln x/2)x^{\tan \ln 2}}\right\} + C$$

(5) **(A).** $I = \int_0^{\pi/3} \ln\left(\frac{\cos x + \sqrt{3}\sin x}{\cos x}\right) dx$

$$= \int_0^{\pi/3} \ln 2\cos\left(x - \frac{\pi}{3}\right) dx - \int_0^{\pi/3} \ln \cos x \, dx$$

$$= \int_0^{\pi/3} \ln(2\cos x)\, dx - \int_0^{\pi/3} \ln(\cos x)\, dx$$

$$= \int_0^{\pi/3} \ln 2\, dx - \int_0^{\pi/3} \ln(\cos x)\, dx - \int_0^{\pi/3} \ln(\cos x)\, dx = \frac{\pi}{3}\ln 2$$

Alternatively,

$$I = \int_0^{\pi/3} \ln\left(1 + \sqrt{3}\tan\left(\frac{\pi}{3} - x\right)\right) dx$$

$$= \int_0^{\pi/3} \ln\left(1 + \sqrt{3}\left(\frac{\sqrt{3} - \tan x}{1 + \sqrt{3}\tan x}\right)\right) dx$$

$$= \int_0^{\pi/3} \ln\left(\frac{1 + \sqrt{3}\tan x + \sqrt{3} - \sqrt{3}\tan x}{1 + \sqrt{3}\tan x}\right) dx$$

$$= \int_0^{\pi/3} \ln\left(\frac{4}{1 + \sqrt{3}\tan x}\right) dx$$

$$2I = \left(\frac{\pi}{3}\right).2\ln 2 \Rightarrow I = \frac{\pi}{3}\ln 2$$

(6) **(D).** Let $I = \int_0^{2(2+\sqrt{3})} \frac{16}{(4+x^2)^2} dx$

Put $x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\, d\theta$

$$\therefore \quad I = \int_0^{5\pi/12} \frac{16 \, . \, 2\sec^2\theta}{16\sec^2\theta.\sec^2\theta} d\theta$$

$$= \int_0^{5\pi/12} 2\cos^2\theta\, d\theta = \int_0^{5\pi/12} (1 + \cos 2\theta)\, d\theta$$

$$= \theta + \frac{1}{2}\sin 2\theta \Big|_0^{5\pi/12} = \frac{5\pi}{12} + \frac{1}{4} = \frac{3 + 5\pi}{12}$$

(7) $I = \int_{-\pi/2}^{\pi/2} \frac{\ln(\cos x)}{1 + e^x . e^{\sin x}} dx = \int_{-\pi/2}^{\pi/2} \frac{\ln(\cos x)}{1 + e^{-(x + \sin x)}} dx$

$$2I = \int_{-\pi/2}^{\pi/2} \frac{e^{(x+\sin x)}\ln(\cos x)}{1 + e^{x + \sin x}} dx$$

$$= \int_{-\pi/2}^{\pi/2} \frac{\ln(\cos x)}{1 + e^{x+\sin x}}(1 + e^{(x + \sin x)})\, dx = \int_{-\pi/2}^{\pi/2} \ln(\cos x)\, dx$$

$$2I = 2\int_0^{\pi/2} \ln(\cos x)\, dx \Rightarrow I = -\frac{\pi}{2}\ln 2$$

(8) **(A).** Here $f'(1) = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$,

$f'(2) = \tan\frac{\pi}{3} = \sqrt{3}$, $f'(3) = \tan\frac{\pi}{4} = 1$

Now, $\int_2^3 f'(x) f''(x)\, dx = \left(\frac{(f'(x))^2}{2}\right)_2^3$

$$= \frac{1}{2}[(f'(3))^2 - (f'(2))^2] = \frac{1}{2}[1 - 3] = -1$$

And, $\int_1^3 f''(x)\, dx = (f'(x))_1^3 = f'(3) - f'(1) = \left(1 - \frac{1}{\sqrt{3}}\right)$

$\therefore$ Value $= -1 + 1 - \frac{1}{\sqrt{3}} = \frac{-1}{\sqrt{3}}$

(9) **(B).** Let $I = \int_0^{\pi/2} \frac{\cos x + 4}{3\sin x + 4\cos x + 25} dx$ ..... (1)

and $J = \int_0^{\pi/2} \frac{\sin x + 3}{3\sin x + 4\cos x + 25} dx$ ...... (2)

Now we multiply the eq. (1) by 4 and (2) by 3 and add to get $4I + 3J = \int_0^{\pi/2} \frac{4\cos x + 3\sin x + 25}{3\sin x + 4\cos x + 25} dx = \frac{\pi}{2}$ .... (3)

and we multiply the eq. (1) by 3 and (2) by 4 and subtract to get

$3I - 4J = \int_0^{\pi/2} \frac{3\cos x - 4\sin x}{3\sin x + 4\cos x + 25} dx = \ln\left(\frac{25}{29}\right)$ .... (4)

Again multiply eq. (3) by 4 and eq. (4) by 3 and add to get

$$25I = 2\pi + 3\ln\frac{28}{29}$$

$\Rightarrow a + b + c + d = (2 + 3 + 28 + 29) = 62$

(10) **(C).** $a_n - a_{n-1} = \int_0^{\pi/2}\left(\frac{\sin^2(nx)}{\sin x} - \frac{\sin^2(n-1)x}{\sin x}\right)dx$

$= \frac{1}{2}\int_0^{\pi/2}\left(\frac{\cos 2(n-1)x - \cos 2nx}{\sin x}\right)dx$

$= \int_0^{\pi/2}\sin(2n-1)x\,dx = \left[-\frac{\cos(2n-1)x}{(2n-1)}\right]_0^{\pi/2} = \frac{1}{2n-1}$

$\Rightarrow a_2 - a_1 = \frac{1}{3}, a_3 - a_2 = \frac{1}{5}, a_4 - a_3 = \frac{1}{7}$ .... are in H.P.

(11) **(A).** Let g (x) = t then $I = \int_{g(1)}^{g(2)}\frac{1}{f(t)}, f'(t)\,dt$

$[\log f(t)]_{g(1)}^{g(2)} = \log f(g(2)) - \log f(g(1)) = 0$

$[\because g(1) = g(2)]$

(12) **(B).** In order to remove x, apply Prop.IV

$\therefore I = \int_0^{\pi}\frac{(\pi - x)\,dx}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}$ ..(1)

or $I = \int_0^{\pi}\frac{(\pi - x)\,dx}{a^2\cos^2 x + b^2\sin^2 x}$ ..(2)

Adding (1) and (2), we get

$2I = \int_0^{\pi}\frac{(x + \pi - x)\,dx}{a^2\cos^2 x + b^2\sin^2 x} = \pi\int_0^{\pi}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

$\therefore I = \frac{\pi}{2}\int_0^{\pi}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

$= 2.\frac{\pi}{2}\int_0^{\pi/2}\frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

Divide above and below by $\cos^2 x$.

$\therefore I = \pi\int_0^{\pi/2}\frac{\sec^2 x\,dx}{a^2 + b^2\tan^2 x}$

Put b tanx = t $\therefore b\sec^2 x\,dx = dt$

$\therefore I = \frac{\pi}{b}\int_0^{\infty}\frac{dt}{a^2 + t^2} = \frac{\pi}{b}.\frac{1}{a}\left[\tan^{-1}\frac{t}{a}\right]_0^{\infty}$

$= \frac{\pi}{ab}\left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{2ab}$

(13) **(A).** $\int(x^{3m} + x^{2m} + x^m)\left(\frac{(2x^{3m} + 3x^{2m} + 6x^m)^{1/m}}{x}\right).dx$

$\int(x^{3m-1} + x^{2m-1} + x^{m-1})\,(2x^{3m} + 3x^{2m} + 6x^m)^{1/m}.dx$

$= \frac{1}{6}\int(2x^{3m} + 3x^{2m} + 6x^m)^{1/m}(6x^{3m-1} + 6x^{2m-1} + 6x^{m-1})\,dx$

$= \frac{1}{6m}\int(2x^{3m} + 3x^{2m} + 6x^m)^{1/m}d(2x^{3m} + 3x^{2m} + 6x^m)$

$= \frac{1}{6(m+1)}(2x^{3m} + 3x^{2m} + 6x^m)^{\frac{1}{m}+1}$

(14) **(A).** $I = \int \underset{II}{1}.\underset{I}{\tan^{-1}\sqrt{x}}\,dx$

$= \tan^{-1}\sqrt{x}.x - \int\frac{1}{1+x}\times\frac{1}{2\sqrt{x}}.x\,dx$

$= x\tan^{-1}\sqrt{x} - \frac{1}{2}\int\frac{\sqrt{x}}{1+x}dx$

Let $x = t^2$, $dx = 2t\,dt$

$= x\tan^{-1}\sqrt{x} - \frac{1}{2}\int\frac{t.2t\,dt}{(1+t^2)} = x\tan^{-1}\sqrt{x} - \int\frac{(1+t^2)-1}{(1+t^2)}dt$

$= x\tan^{-1}\sqrt{x} - [t - \tan^{-1}(t)] + c$

$= x\tan^{-1}\sqrt{x} - [\sqrt{x} - \tan^{-1}\sqrt{x}] + c$

(15) **(A).** $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$I = \int_{2-\log 3}^{3+\log 3}\frac{\log(4+5-x)}{\log(4+5-x) + \log(9-5+x)}dx$

$I = \int_{2-\log 3}^{3+\log 3}\frac{\log(9-x)}{\log(9-x) + \log(4+x)}dx$

$\Rightarrow 2I = \int_{2-\log 3}^{3+\log 3}\frac{\log(4+x) + \log(9-x)}{\log(9-x) + \log(4+x)}dx = \int_{2-\log 3}^{3+\log 3}1.dx$

$2I = (3 + \log 3 - 2 + \log 3) = 1 + 2\log 3$

$I = \left(\frac{1}{2} + \log 3\right)$

(16) **(C).** $f(x) = \int_1^x |t|\,dt \Rightarrow f'(x) = |x|.1 = \begin{cases} -x; -\frac{1}{2} \le x \le 0 \\ x; 0 \le x \le \frac{1}{2} \end{cases}$

At x = 0, neither max. nor min.

$f\left(\frac{1}{2}\right) = \int_1^{1/2}|t|\,dt = \left(\frac{t^2}{2}\right)_1^{1/2} = \left(\frac{1}{8} - \frac{1}{2}\right) = -\frac{3}{8}$

$f\left(-\frac{1}{2}\right) = \int_1^{-1/2}|t|\,dt = -\int_{-1/2}^{1}|t|\,dt$

$$= -\left\{ \int_{-1/2}^{0} |t|\, dt + \int_{0}^{1} |t|\, dt \right\}$$

$$= -\left\{ -(t^2/2)_{-1/2}^{0} + (t^2/2)_0^1 \right\}$$

$$= -\left\{ -\left(0 - \frac{1}{8}\right) + \left(\frac{1}{2}\right) \right\} = -\left\{ \left(\frac{1}{8} + \frac{1}{2}\right) \right\} = -\frac{5}{8}$$

Hence greatest value of f (x) = $-\frac{3}{8}$

(17) **(C).** We have , $\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right) = \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

$$\therefore \int_{-1}^{1} \frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right) dx = \int_{-1}^{1} -\frac{1}{1+x^2} dx = -2\int_0^1 \frac{1}{1+x^2} dx$$

$$= -2\left[\tan^{-1} x\right]_0^1 = -2(\pi/4) = -\pi/2$$

Note that $\int_{-1}^{1} \frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right) dx = \left[\tan^{-1}\frac{1}{x}\right]_{-1}^{1}$

$= \tan^{-1} 1 - \tan^{-1}(-1) = \pi/2$

is incorrect, because $\tan^{-1}\frac{1}{x}$ is not an antiderivative primitive of $\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)$ on the interval [–1, 1].

(18) **(B).** Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2} dx = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{(1+\tan^2\theta)} \sec^2\theta\, d\theta$

[Putting $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$]

$$= \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta = \int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - \theta\right)\right] d\theta$$

$$= \int_0^{\pi/4} \left(1 + \frac{1-\tan\theta}{1+\tan\theta}\right) d\theta = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$= \log 2 \int_0^{\pi/4} 1\, d\theta - I \Rightarrow 2I = \frac{\pi}{4}\log 2 \quad \therefore I = \frac{\pi}{8}\log 2$$

(19) **(A).** $\int_{-2}^{2} [x^2]\, dx = 2\int_0^2 [x^2]\, dx$ [$\because$ integrand is even]

$$= 2\left[\int_0^1 [x^2]dx + \int_1^{\sqrt{2}} [x^2]dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2]dx + \int_{\sqrt{3}}^{2} [x^2]dx\right]$$

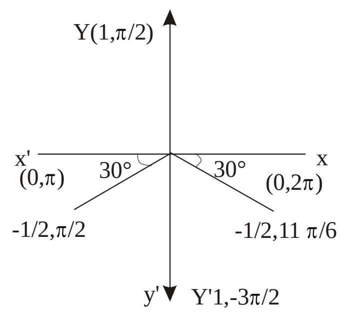
$$\left[\begin{array}{l}\because [x^2] = 0 \text{ if } 0 \le x < 1; 1 \text{ if } 1 \le x < \sqrt{2}; \\ \quad 2 \text{ if } \sqrt{2} \le x < \sqrt{3}; 3 \text{ if } \sqrt{3} \le x < 2\end{array}\right]$$

$$= 2\left[\int_0^1 0\,dx + \int_1^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\,dx + \int_{\sqrt{3}}^{2} 3\,dx\right]$$

$$= 2[x]_1^{\sqrt{2}} + 4[x]_{\sqrt{2}}^{\sqrt{3}} + 6[x]_{\sqrt{3}}^{2}$$

$$= (10 - 2\sqrt{3} - 2\sqrt{2})$$

(20) **(D).** It is a question of greatest integer function. We subdivide the interval π to 2π as under keeping in view that we have to evaluate [2sinx]



We know that $\sin\frac{\pi}{6} = \frac{1}{2}$

$$\sin\left(\pi + \frac{\pi}{6}\right) = \sin\frac{7\pi}{6} - \frac{1}{2}$$

$$\sin\frac{11\pi}{6} = \sin\left(2\pi - \frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}$$

$$\sin\frac{9\pi}{6} = \sin\frac{3\pi}{2} = -1$$

Hence we divide the interval π to 2π as

$$\left(\pi, \frac{7\pi}{6}\right), \left(\frac{7\pi}{6}, \frac{11\pi}{6}\right), \left(\frac{11\pi}{6}, 2\pi\right)$$

$$\sin x = \left(0, -\frac{1}{2}\right), \left(-1, -\frac{1}{2}\right), \left(0, -\frac{1}{2}\right)$$

2 sinx = (0 – 1), (–2, – 1), (0 – 1)

[2sinx] = – 1 – 2 – 1

$\therefore I = I_1 + I_2 + I_3$

$= \int -1dx + \int -2dx + \int -1dx$

between proper limits

$$= -\frac{\pi}{6} - 2\left(\frac{4\pi}{6}\right) - \frac{\pi}{6} = -\frac{10\pi}{6} = -\frac{5\pi}{3}$$

(21) (C).

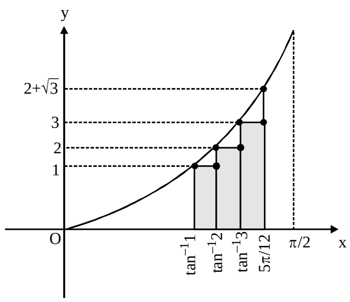
Let $I = \int_{-1}^{0} [x[1+\sin \pi x]+1]\, dx + \int_{0}^{1} [x[1+\sin \pi x]+1]\, dx$

Now, $[1 + \sin \pi x] = 0$ if $-1 < x < 0$

and $[1 + \sin \pi x] = 1$ if $0 < x < 1$

$\therefore\ I = \int_{-1}^{0} 1\, dx + \int_{0}^{1} [x+1]\, dx = 2$

(22) (B). Method 1 :



$\int_{0}^{5\pi/12} [\tan x]\, dx$ = Area of shaded region

$= 0 + 1\,(\tan^{-1} 2 - \tan^{-1} 1) + 2\,(\tan^{-1} 3 - \tan^{-1} 2)$

$+ 3\left(\frac{5\pi}{12} - \tan^{-1} 3\right)$

$= \pi - \tan^{-1}(-1) = 5\pi/4$

Method 2 : Put $\tan x = t$; $\sec^2 x\, dx = dt$

$dx = \frac{dt}{(1+t^2)}\quad \therefore\ I = \int_{0}^{2+\sqrt{3}} \frac{[t]\, dt}{(1+t^2)}$

(23) (B). Let $\frac{3x-4}{3x+4} = t \Rightarrow \frac{6x}{8} = \left(\frac{t+1}{1-t}\right) \Rightarrow x = \frac{4}{3}\left(\frac{1+t}{1-t}\right)$

$\therefore\ f(t) = \frac{4}{3}\left(\frac{1+t}{1-t}\right) + 2 = \frac{10-2t}{3(1-t)} = \frac{8+2(1-t)}{3(1-t)}$

$f(t) = \frac{8}{3(1-t)} + \frac{2}{3}$

$\therefore\ \int f(x)\, dx = \frac{8}{3}\int \frac{1}{(1-x)}\, dx + \frac{2}{3}\int 1\, dx$

$= -\frac{8}{3}\log|1-x| + \frac{2}{3}x + C$

(24) (C). $I = \int_{1}^{1.5} x\,[2x]\,\mathrm{sgn}\,(x-2)dx + \int_{1.5}^{2} x\,[2x]\,\mathrm{sgn}\,(x-2)\, dx$

$+ \int_{2}^{2.5} x\,[2x]\,\mathrm{sgn}\,(x-2)\, dx$

$= \int_{1}^{1.5} x\,(2)\,(-1)\, dx + \int_{1.5}^{2} x\,(3)\,(-1) + \int_{2}^{2.5} x\,(4)\,(1)\, dx$

$= -(x^2)_1^{1.5} - 3\left(\frac{x^2}{2}\right)^2_{1.5} + 2(x^2)\,(x^2)_2^{2.5} = \frac{5}{8}$

(25) (A). $I_1 = \int_{0}^{\pi/2} \cos\theta . f\,(\sin\theta + 1 - \sin^2\theta)\, d\theta$ .... (1)

Put $\sin\theta = t \Rightarrow \cos\theta\, d\theta = dt$

$I_2 = \int_{0}^{1} 2t.f\,(t+1-t^2)\, dt$ .... (2)

$= \int_{0}^{1} (1-t).f\,(1-t+1-(1-t)^2)\, dt$

$I_2 = 3\int_{0}^{1} (1-t).f\,(1+t-t^2)\, dt$

$I_2 = 2\left[\int_{0}^{1} f\,(1+t-t^2)\, dt - \int_{0}^{1} t.f\,(1+t-t^2)\, dt\right]$ .... (3)

Eq. (2) + eq. (3),

$2I_2 = 2\int_{0}^{1} f\,(1+t-t^2)\, dt \Rightarrow I_1 = I_2 \Rightarrow \frac{I_1}{I_2} = 1$

(26) (C). $f'(x) = \frac{1}{\sqrt{1+g^2(x)}}\, g'(x)$

$f'\left(\frac{\pi}{2}\right) = \frac{g'(\pi/2)}{\sqrt{1+g^2(\pi/2)}}$; $g\left(\frac{\pi}{2}\right) = 0 = g'\left(\frac{\pi}{2}\right)$

Now $g'(x) = [1 + \sin(\cos^2 x)]\,(-\sin x)$

$g'\left(\frac{\pi}{2}\right) = 1\,(-1) = -1$

Hence, $f'\left(\frac{\pi}{2}\right) = -1$ as $h'(0^+) = -1$

(27) (A).

$L = \lim_{n\to\infty}\left[\frac{1}{\sqrt{4n^2-1}} + \frac{1}{\sqrt{4n^2-4}} + \frac{1}{\sqrt{4n^2-9}} + \ldots\ldots \frac{1}{\sqrt{3n^2}}\right] = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{4n^2-r^2}}$

$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{(1-0)}{n}\,\frac{1}{\sqrt{4-\left(0+r\left(\frac{1-0}{n}\right)\right)^2}}$

Which is of the form

$$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{b-a}{n}f\left(a+r\left(\frac{b-a}{n}\right)\right)$$

Here b = 1, a = 0 and $f(x)=\dfrac{1}{\sqrt{4-x^2}}$

$$\text{So } L=\int_0^1\frac{dx}{\sqrt{4-x^2}}=\sin^{-1}\frac{x}{2}\Big|_0^1=\frac{\pi}{6}$$

(28) **(B).** Putting x = – t in m,

$$m=\int_2^0\frac{|\sin(-t)|}{\left[\frac{-t}{\pi}\right]+\frac{1}{2}}(-dt)=\int_0^2\frac{|\sin t|}{\left[\frac{-t}{\pi}\right]+\frac{1}{2}}dt$$

$$=\int_0^2\frac{|\sin t|}{-\left[\frac{t}{\pi}\right]-1+\frac{1}{2}}dt$$

If $x\notin I$, then $[-x]=-[x]-1=-\displaystyle\int_0^2\frac{|\sin t|}{\left[\frac{t}{\pi}\right]+\frac{1}{2}}dt=-n$

(29) **(C).** In first integral putting $t=\sin^2 y$ and in second putting $t=\cos^2 z$, we have

$$I=\int_0^x y\sin 2y\,dy-\int_{\pi/2}^{x} z\sin 2z\,dz$$

$$=\int_0^x y\sin 2y\,dy+\int_x^{\pi/2} z\sin 2z\,dz=\int_0^{\pi/2}\theta\sin 2\theta\,d\theta$$

$$=\left[\theta\left(-\frac{\cos 2\theta}{2}\right)+\frac{\sin 2\theta}{4}\right]_0^{\pi/2}=\pi/4$$

(30) **(A).** Let $y=\sqrt{x+y}$

$$y^2-y-x=0$$

$$y=\frac{1\pm\sqrt{1+4x}}{2\times 1}\quad y>1\ ;\quad y=\frac{1+\sqrt{1+4x}}{2}$$

$$\int_0^2\frac{1+\sqrt{1+4x}}{2}dx=\left[\frac{x}{2}+\frac{(1+4x)^{3/2}}{\frac{3}{2}.2.4}\right]_0^2$$

$$=\left[\left(1+\frac{27}{12}\right)-\left(0+\frac{1}{12}\right)\right]=1+\frac{26}{12}=\frac{19}{6}$$

(31) **(C).** Here

$$\int_0^\infty\frac{x\log x}{(1+x^2)^2}dx=\int_0^1\frac{x\log x}{(1+x^2)^2}dx+\int_1^\infty\frac{x\log x}{(1+x^2)^2}dx$$

$$I=I_1+I_2$$

Putting x = 1/t in second integrand ; $dx=-\dfrac{1}{t^2}dt$

$$\therefore\ I_2=\int_1^0\frac{\frac{1}{t}\log\left(\frac{1}{t}\right)}{\left(1+\frac{1}{t^2}\right)^2}\left(-\frac{1}{t^2}\right)dt=-\int_0^1\frac{t\log t}{(1+t^2)^2}dt=-I_1$$

$$I=I_2+I_1=-I_1+I_1=0$$

(32) **(A).** Let f(x)

$$=\frac{x-1}{(2x+1)(x-2)(x-3)}=\frac{A}{2x+1}+\frac{B}{x-2}+\frac{C}{x-3}$$

$$-A=\frac{x-1}{(x-2)(x-3)}\Bigg]_{x=-\frac{1}{2}}=-\frac{6}{35}$$

$$B=\frac{x-1}{(2x+1)(x-3)}\Bigg]_{x=2}=-\frac{1}{5}\ ;$$

$$C=\frac{x-1}{(2x+1)(x-2)}\Bigg]_{x=3}=\frac{2}{7}$$

$$\int f(x)\,dx=\frac{-6}{35}\int\frac{dx}{2x+1}-\frac{1}{5}\int\frac{dx}{x-2}+\frac{2}{7}\int\frac{dx}{x-3}$$

$$=\frac{3}{35}\ln|2x+1|-\frac{1}{5}\ln|x-2|+\frac{2}{7}\ln|x-3|+C$$

(33) **(A).** $I=\displaystyle\int_1^e\Bigg[\underbrace{\underbrace{(e^x+e^{-x})}_{f(x)}+\underbrace{(e^x-e^{-x})}_{f(x)}}_{II}\Bigg].\underbrace{\ln x}_{I}\,dx$

$$\left(\int f(x)+xf'(x))\,dx=x\,f(x)\right)$$

$$I=\ln x.x\,f(x)]_1^e-\int_1^e\frac{x\,f(x)}{x}dx=e\,f(e)-\int_1^e(e^x+e^{-x})\,dx$$

$$=e\,(e^e+e^{-e})-[e^x-e^{-x}]_1^e$$

$$=(e^{e+1}+e^{1-e})-[(e^e-e^{-e})-(e-e^{-1})]$$

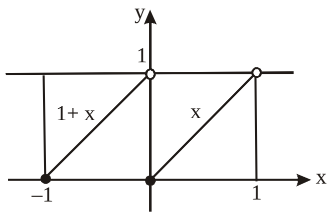
$$=e^{e+1}+e^{1-e}-e^e+e^{-e}+e-e^{-1}$$

(34) **(B).** $I_1=\displaystyle\int_{-1}^{1}[\{x\}\{x\}^2+\{x^2\}\{x^3\}]\,dx$

[Using $\{x+n\}=\{x\},n\in I$]

$$I_1=\int_{-1}^{1}\{x^2\}+(\{x\}+\{x^3\})\,dx$$

Now, $\{x\} = \begin{cases} x & \text{if } 0 \le x < 1 \\ 1+x & \text{if } -1 \le x < 0 \end{cases}$



Also, for $-1 < x < 1$, $\{x^2\} = x^2$

and $\{x^3\} = \begin{cases} x^3 & \text{if } 0 \le x < 1 \\ 1+x^3 & \text{if } -1 \le x < 0 \end{cases}$

$$I_1 = \int_{-1}^{0} x^2[(1+x)+(1+x^3)]\,dx + \int_0^1 x^2\,(x+x^3)\,dx$$

$$I_1 = \int_{-1}^{0} (2x^2+x^3+x^5)\,dx + \int_0^1 (x^5+x^3)\,dx$$

$$= 2\int_{-1}^{0} x^2 dx = 2.\frac{x^3}{10}\Big]_{-1}^{0} = \frac{2}{3}$$

Note that $\int_{-1}^{0} (x^3+x^5)\,dx + \int_0^1 (x^3+x^5)\,dx$

$$= \int_{-1}^{1} (x^3+x^5)\,dx = 0$$

(35) **(A).** Let $5\sin x + 6 = A(\sin x + 2\cos x + 3) + B(\cos x - 2\sin x) + C$

Equating the coefficients of sin x, cos x and constant term, we get

$$\left.\begin{aligned} A - 2B &= 5 \\ 2A + B &= 0 \\ 3A + C &= 6 \end{aligned}\right\} \Rightarrow A = 1, B = -2, C = 3$$

$$I = \int dx - 2\int \frac{(\cos x - 2\sin x)dx}{\sin x + 2\cos x + 3} + 3\int \frac{dx}{\sin x + 2\cos x + 3}$$

$$= x - 2\ell n\,|\sin x + 2\cos x + 3| + 3\ell_1$$

Put $\tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}dx = 2dt$

So $\ell_1 = \int \frac{2dt}{t^2+2t+5} = \int \frac{2dt}{(t+1)^2+4} = \tan^{-1}\left(\frac{t+1}{2}\right) + C$

$$= \tan^{-1}\left(\frac{1+\tan(x/2)}{2}\right) + C$$

$$I = x - 2\ell n\,|\sin x + 2\cos x + 3| + 3\tan^{-1}\left(\frac{1+\tan(x/2)}{2}\right) + C$$

(36) **(B).** $I = \int \frac{dx}{x^{29}\left(1 - \frac{6}{x^7}\right)}$

Let $\left(1 - \frac{6}{x^7}\right) = p \Rightarrow \frac{42}{x^8}dx = dp$ and $x^7 = \left(\frac{6}{1-p}\right)$

$$I = \frac{1}{42}\int \frac{(1-p)^3}{(6)^3 p}dp = \frac{1}{(42)(216)}\int \frac{1 - p^3 - 3p + 3p^2}{p}\,dp$$

$$= \frac{1}{54432}\,[\ln p^6 + 9p^2 - 2p^3 - 18p] + c$$

(37) **(D).** Let $I = \int \frac{dx}{\sqrt[3]{x+1} + \sqrt{x+1}}$

$$\Rightarrow I = \int \frac{dx}{(x+1)^{1/3} + (x+1)^{1/2}}$$

The least common multiple of 2 and 3 is 6.
So substitute $x + 1 = t^6 \Rightarrow dx = 6t^5\,dt$

$$\Rightarrow I = \int \frac{6t^5dt}{t^2+t^3} = 6\int \frac{t^3dt}{1+t} \Rightarrow I = 6\int t^2 - t + 1 - \frac{1}{1+t}dt$$

$$\Rightarrow I = 6\left(\frac{t^3}{3} - \frac{t^2}{2} + t - \log(t+1)\right) + C$$

On substituting $t = (1 + x)^{1/6}$, we get

$$I = 6\left(\frac{(1+x)^{1/2}}{3} - \frac{(1+x)^{1/3}}{2} + (1+x)^{1/6} - \log\left((x+1)^{1/6}+1\right)\right) + C$$

(38) **(B).** $I = \int_{-\infty}^{\infty} \frac{1}{(1+e^x)\sqrt[4]{e^{x^2}\sqrt{e^{2x^2}\sqrt{e^{3x^2}\,......\infty}}}}\,dx =$

$$2I = \int_{-\infty}^{\infty}\left(\frac{1}{1+e^x} + \frac{1}{1+e^{-x}}\right).\frac{1}{e^{x^2/4}.e^{2x^2/8}.e^{3x^2/16}.....\infty}\,dx$$

$$\int_{-\infty}^{\infty} \frac{1}{e^{x^2}}dx \Rightarrow I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{e^{x^2}}dx = \frac{1}{2}\sqrt{\pi}$$

(39) **(A).** $\int_0^{2\pi} \ln(\sec^2 x)\,dx = 2\int_0^{\pi} \ln(\sec^2 x)\,dx$

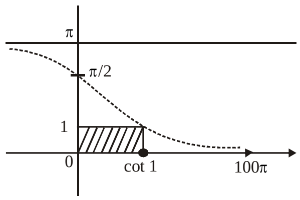
$$= 4\int_0^{\pi/2} \ln(\sec^2 x)\,dx = 8\int_0^{\pi/2} \ln(\sec x)\,dx$$

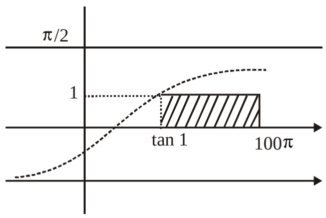
$= -8\int_0^{\pi/2} \ln(\cos x)\,dx = -8\left(-\frac{\pi}{2}\ln 2\right)$

$= 4\pi \ln 2 = 2k^2 \sin^{-1}\left(\frac{k}{2}\right) \ln k \qquad \therefore k = 2$

$\therefore [k] = 2$

(40) (A).





$\int_0^{100\pi} ([\cot^{-1} x] + [\tan^{-1} x])\,dx = \cot 1 + (100\pi - \tan 1)$

$= 100\pi + \frac{1 - \tan^2 1}{\tan 1} = 100\pi + 2\cot 2$

(41) (A). $I_{NR} = \int_0^{\pi/2} \underset{I}{x}\, \underset{II}{\cos x . e^{\sin x}}\,dx + \int_0^{\pi/2} e^{\sin x}\,dx$

$= x e^{\sin x}\Big|_0^{\pi/2} - \int_0^{\pi/2} e^{\sin x}dx + \int_0^{\pi/2} e^{\sin x}dx = \frac{e\pi}{2}$ ...... (1)

Again, $I_{DR} = \int_0^{\pi/2} \underset{I}{x}\, \underset{II}{\sin x . e^{\cos x}}\,dx - \int_0^{\pi/2} e^{\cos x}dx$

$= -x e^{\cos x}\Big|_0^{\pi/2} + \int_0^{\pi/2} e^{\cos x}dx - \int_0^{\pi/2} e^{\cos x}dx = -\frac{\pi}{2}$ ...... (2)

$\therefore \left|\frac{I_{Nr}}{I_{Dr}}\right| = \frac{e\pi . 2}{2\pi} = e$

(42) (A). $I = \int_0^{\pi} \frac{x|\sin x|}{1 + |\cos x|}dx$ .... (1)

or $I = \int_0^{\pi} \frac{(\pi - x)|\sin x|}{1 + |\cos x|}dx$ .... (2)

Add eq. (1) and (2), $2I = \pi\int_0^{\pi} \frac{|\sin x|}{1 + |\cos x|}dx$

$\left(\text{Using } \int_0^{na} f(x)\,dx = n\int_0^{a} f(x)\,dx\right)$

$I = \frac{\pi}{2}.2\int_0^{\pi/2} \frac{\sin x\,dx}{1 + \cos x} = \pi\int_0^{\pi/2} \frac{\sin x\,dx}{1 + \cos x} = \pi\int_0^{\pi/2} \frac{\cos x\,dx}{1 + \sin x}$

$= \pi . \ln(1 + \sin x)\Big]_0^{\pi/2} = \pi \ln 2$

(43) (B).

Let $I = \int_{-1}^{1} \cot^{-1}\left(\frac{1}{\sqrt{1 - x^2}}\right).\left(\cot^{-1}\frac{x}{\sqrt{1 - (x^2)^{|x|}}}\right)dx$ ...... (1)

$I = \int_{-1}^{1} \left(\cot^{-1}\frac{1}{\sqrt{1 - x^2}}\right).\left(\cot^{-1}\frac{-x}{\sqrt{1 - (x^2)^{|x|}}}\right)dx$ ...... (2)

On adding,

$2I = \int_{-1}^{1} \cot^{-1}\frac{1}{\sqrt{1 - x^2}}\left\{\cot^{-1}\frac{x}{\sqrt{1 - (x^2)^{|x|}}} + \pi - \cot^{-1}\frac{x}{\sqrt{1 - (x^2)^{|x|}}}\right\}dx$

$2I = \int_{-1}^{1} \pi \cot^{-1}\frac{1}{\sqrt{1 - x^2}}\,dx = \int_{-1}^{1} \pi \tan^{-1}\sqrt{1 - x^2}\,dx$

$= 2\pi\int_0^1 \tan^{-1}\sqrt{1 - x^2}\,dx$ ...... (3)

(As $\tan^{-1}\sqrt{1 - x^2}$ is even function)

$\therefore I = \pi\int_0^1 \underset{II}{1}\, \underset{I}{\tan^{-1}(\sqrt{1 - x^2})}\,dx$ ...... (4)

Intetrating by parts

$I = \underbrace{\pi \tan^{-1}(\sqrt{1 - x^2}).x\Big]_0^1}_{\text{Zero}} - \int_0^1 \frac{x}{(1 + 1 - x^2)}\frac{(-x)}{\sqrt{1 - x^2}}dx$

$= \pi\int_0^1 \frac{x^2}{(2 - x^2)\sqrt{1 - x^2}}dx$

Let $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$.

$I = \pi\int_0^{\pi/2} \frac{\sin^2\theta}{(2 - \sin^2\theta)}d\theta = -\pi\int_0^{\pi/2} \frac{2 - \sin^2\theta - 2}{2 - \sin^2\theta}d\theta$

$$\therefore \quad I = 2\pi\int_0^{\pi/2}\frac{d\theta}{2-\sin^2\theta}-\frac{\pi^2}{2} = 2\pi\int_0^{\pi/2}\frac{\sec^2\theta d\theta}{2+\tan^2\theta}-\frac{\pi^2}{2}$$

Put $\tan\theta = t$

$$I = 2\pi\int_0^{\infty}\frac{dt}{2+t^2}-\frac{\pi^2}{2} = 2\pi\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}\bigg]_0^{\infty}-\frac{\pi^2}{2}$$

$$= \frac{\pi^2}{\sqrt{2}}-\frac{\pi^2}{2} = \frac{\pi^2(\sqrt{2}-1)}{2} = \frac{\pi^2(\sqrt{a}-\sqrt{b})}{\sqrt{c}}$$

$\Rightarrow a = 2, b = 1$ and $c = 4$

$\Rightarrow a + b + c = 2 + 1 + 4 = 7$

(44) **(D).** $A\int\frac{x^3}{y}dx + B\int\frac{x^2}{y}dx + c\int\frac{x}{y}dx = x^2y$

Differentiating both sides

$$\frac{Ax^3+Bx^2+cx}{y} = 2xy + x^2\frac{dy}{dx}$$

$$Ax^3+Bx^2+Cx = 2xy^2 + x^2y\frac{dy}{dx}$$

$$Ax^3+Bx^2+Cx = 2x(x^2-x+1)+x^2\frac{2x-1}{2}$$

$$= 2x^3-2x^2+2x+x^3-\frac{x^2}{2}$$

$A = 3, B = -5/2, C = 2$

(45) **(C).** We have, $f\left(\frac{1}{x}\right)+x^2f(x) = 0 \Rightarrow f(x) = -\frac{1}{x^2}f\left(\frac{1}{x}\right)$

$$\therefore I = \int_{\sin\theta}^{\text{cosec}\theta}f(x)dx = \int_{\sin\theta}^{\text{cosec}\theta}-\frac{1}{x^2}f\left(\frac{1}{x}\right)dx = \int_{\text{cosec}\theta}^{\sin\theta}f(t)dt,$$

where $t = \frac{1}{x}$

$$= -\int_{\sin\theta}^{\text{cosec}\theta}f(t)dt = I \qquad \therefore 2I = 0 \Rightarrow I = 0$$

(46) **(A).** $I(x) = \int_{1/e}^{\tan x}\frac{tdt}{1+t^2}+\int_{1/e}^{\cot x}\frac{dt}{t(1+t^2)}$

Diff. w.r.t x.

$$\frac{dI(x)}{dx} = \frac{\tan x}{(1+\tan^2x)}\sec^2x + \frac{1}{\cot x(1+\cot^2x)}(-\text{cosec}^2x) = 0$$

$I(x) =$ constant. Let $x = \pi/4$

$$\text{Thus } I\left(\frac{\pi}{4}\right) = \int_{1/e}^{1}\frac{tdt}{1+t^2}+\int_{1/e}^{1}\frac{dt}{t(1+t^2)}$$

$$= \int_{1/e}^{1}\frac{t^2+1}{t(t^2+1)}dt = \int_{1/e}^{1}\frac{dt}{t} = [\ln t]_{1/e}^{1} = 1$$

(47) (A), (48) (D), (49) (B).

Differentiate both sides

$$\frac{x^3-6x^2+11x-6}{\sqrt{x^2+4x+3}} = (Ax^2+Bx+c)\frac{(x+2)}{\sqrt{x^2+4x+3}} + (2Ax+B)\sqrt{x^2+4x+3}+\frac{\lambda}{\sqrt{x^2+4x+3}}$$

$$x^3-6x^2+11x-6 = (Ax^2+Bx+c)(x+2) + (2Ax+B)(x^2+4x+3)+\lambda$$

Comparing coefficients of like powers of x

x^3: $1 = A + 2A \Rightarrow A = 1/3$

x^2 : $-6 = 2A + B + 8A + B$

$$2B = -6-10.\frac{1}{3} \Rightarrow B = -\frac{14}{3}$$

$x : 11 = 2B + C + 6A + 4B$

$$C = 11-6\left(-\frac{14}{3}\right)-6.\frac{1}{3} = 11+28-2 = 37$$

Constant terms : $-6 = 2C + 3B + \lambda$

$$\lambda = -6-2.37-3\left(-\frac{14}{3}\right) = -6-74+14 = -66$$

(50) **(D).** Here, $\sum_{k=0}^{n}\frac{{}^nC_k}{n^k(k+3)}$ $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k+3}\,{}^nC_k\frac{1}{nk}$

$$= \lim_{n\to\infty}\sum_{k=0}^{n}{}^nC_k\frac{1}{nk}\int_0^1 x^{k+2}dx \quad \left(\because \frac{1}{k+3} = \int_0^1 x^{k+2}dx\right)$$

$$= \int_0^1\left(x^2\lim_{n\to\infty}\sum_{k=0}^{n}{}^nC_k\left(\frac{x}{n}\right)^k\right)dx$$

$$= \int_0^1 x^2\left\{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\right\}dx$$

$$= \int_0^1 x^2e^x\,dx = \int_0^1 x^2.e^x\,dx = (e-2)\text{ (By parts)}$$

(51) **(D).** $f(x) + f(x + 4) = f(x + 2) + f(x + 6)$ ....(1)

Put $x = (x + 2)$

$f(x + 2) + f(x + 6) = f(x + 4) + f(x + 8)$ ....(2)

From equation (1) & (2)

$\Rightarrow f(x) + f(x + 4) = f(x + 4) + f(x + 8)$

$\Rightarrow f(x) = f(x + 8)$

$\therefore$ Period is $= 8$

$g(x) = 8(f(1) - f(0))$

$g'(x) = 0$

(52) **(D).** $f(x+\pi)=\int_0^{x+\pi}(\sin^4 t+\cos^4 t)\,dt$

$$=\int_0^{x+\pi}(\sin^4 t+\cos^4 t)\,dt$$

$$=\int_0^{x}(\sin^4 t+\cos^4 t)\,dt+\int_0^{x+\pi}(\sin^4 t+\cos^4 t)\,dt$$

$$=f(x)+\int_0^{\pi}(\sin^4 t+\cos^4 t)\,dt=f(x)+f(\pi)$$

$$\text{Or}=f(x)+2\int_0^{\pi/2}(\sin^4 t+\cos^4 t)\,dt=f(x)+2f(\pi/2)$$

(53) **(B).** $I=\int 2\sin 2x.\cos 2x\, e^{\tan^2 x}dx$

$$=4\int \sin x.\cos x\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)e^{\tan^2 x}dx$$

$$=5\int \tan x.\sec^2 x\cos^6 x(1-\tan^2 x)\,e^{\tan^2 x}dx$$

Let $\tan^2 x=t$; $2\tan x\sec^2 x\,dx=dt$

$$=2\int\frac{(1-t)\,e^t}{(1+t)^3}dt$$

$$=2\int\frac{\{(t+1)-1\}.e^t}{(t+1)^3}=-2\int e^t\left\{\frac{1}{(t+1)^2}-\frac{2}{(t+1)^3}\right\}dt$$

$$=-\frac{2e^t}{(t+1)^2}+C=-2\cos^4 x.e^{\tan x}+C$$

(54) **(B).**

$$F(x)=\frac{1}{2}\int\frac{(x^2+1)-(x-1)^2}{(x^2+1)(x-1)}dx=\frac{1}{2}\ln|x-1|-\frac{1}{2}\int\frac{x-1}{x^2+1}dx$$

$$=\frac{1}{2}\ln|x-1|+\frac{1}{4}\ln(x^2+1)+\frac{1}{2}\tan^{-1}x+C$$

$\therefore$ discontinuous at x = 1

Note that f (x) = $\int\frac{dx}{x^{1/3}}=\frac{3}{2}x^{2/3}+C$ is continuous

although $\frac{1}{x^{1/3}}$ is discontinuous at x = 0.

(55) **(B).** For $x>0$

$$\because \sin x<x\Rightarrow\frac{\sin x}{x}<1\Rightarrow\int_0^{\pi/2}\frac{\sin x}{x}dx<\int_0^{\pi/2}1\,dx$$

$$\Rightarrow\int_0^{\pi/2}\frac{\sin x}{x}dx<\frac{\pi}{2}\ \text{and}\ \lim_{x\to 0}\frac{\sin x}{x}=1$$

Both S-1 and S-2 are individually true but S-2 is not the correct explanation of S-1.

(56) **(D).** $\cot^{-1}x=\frac{\pi}{2}-\tan^{-1}x\ \therefore\ d(\cot^{-1}x)=-d(\tan^{-1}x)$

Thus, $\int 2^{\tan^{-1}x}d(\cot^{-1}x)$

$$=-\int 2^{\tan^{-1}x}d(\tan^{-1}x)=-\frac{2^{\tan^{-1}x}}{\ln 2}+C$$

$\therefore$ S-1 is false, S-2 is true

(57) **(B).** $\because \sin^6 x+\cos^6 x=(\sin^2 x+\cos^2 x)^3-3\sin^2 x\cos^2 x$

$$=1-\frac{3}{4}\sin^2 2x\qquad(\therefore\ \text{period}=\frac{\pi}{2})$$

$\therefore$ Least and greatest value of $(\sin^6 x+\cos^6 x)$ are 1/4 & 1.

$$\left(\frac{\pi}{2}-0\right)\times\frac{1}{4}<\int_0^{\pi/2}(\sin^6 x+\cos^6 x)\,dx<\left(\frac{\pi}{2}-0\right)\times 1$$

$$\Rightarrow\frac{\pi}{8}<\int_0^{\pi/2}(\sin^6 x+\cos^6 x)\,dx<\frac{\pi}{2}$$

(58) **(A).** Since $y=f^{-1}(x)$, therefore $x=f(y)$.

$\therefore\ dx=f'(y)\,dy$. Thus $\int f^{-1}(x)\,dx=\int yf'(y)\,dy+c$

$\therefore$ Statement 2 is true.

Statement 1 is true for $f^{-1}(x)=\sin^{-1}x$ in statement-2.

(59) **(A).** Statement 2 : $\int e^{g(x)}(g'(x)f(x)+f'(x))\,dx$

$$=\int f(x).e^{g(x)}.g'(x)dx+\int e^{g(x)}.f'(x)dx$$

$$=f(x).e^{g(x)}-\int f'(x).e^{g(x)}dx+\int e^{g(x)}.f'(x)dx=e^{g(x)}f(x)+c$$

Statement 1 : $g(x)=\sin^{-1}x,\ f(x)=\sqrt{1-x^2}$

$$\int e^{\sin^{-1}x}\left(1-\frac{x}{\sqrt{1-x^2}}\right)dx$$

$$=e^{\sin^{-1}x}\sqrt{1-x^2}+c\quad\text{(By statement-2)}$$

(60) **(B).** Statement 1 : $\int_0^n\{x\}\,dx=n\int_0^1 x\,dx=n\left[\frac{x^2}{2}\right]_0^1=\frac{n}{2}$

Statement 2 : $\int_0^n\{x\}\,dx=\int_0^n(x-\{x\}\,dx=\int_0^n x\,dx-\int_0^n\{x\}\,dx$

$$=\left[\frac{x^2}{2}\right]_0^n-n\int_0^1 x.\,dx=\frac{n^2}{2}-n\left[\frac{x^2}{2}\right]_0^1=\frac{n}{2}(n-1)$$

(61) **(C).** $\int_a^b x\, f(x)\, dx = \int_a^b (a+b-x)\, f\,(a+b-x)\, dx$

$$= (a+b)\int_a^b f\,(a+b-x)\, dx - \int_a^b xf\,(a+b-x)\, dx$$

∴ Statement-2 is true only when $f(a+b-x) = f(x)$ which holds in statement-1.

∴ Statement-2 is false and statement-1 is true.

(62) (D), (63) (C), (64) (C).

(i) $\int_0^1 \frac{x^2 e^x}{x+1}\, dx = \int_0^1 \frac{(x^2-1+1)\, e^x}{x+1}\, dx$

$$= \int_0^1 (x-1)\, e^x dx + \int_0^1 \frac{e^x}{x+1}\, dx$$

$$= ((x-1)\, e^x - e^x)_0^1 + A = ((-e)-(-1-1)) + A = 2 - e + A$$

(ii) $\int_0^1 \left(\frac{x}{x+1}\right)^2 e^x\, dx = \int_0^1 \left(\frac{(x+1)^2-(2x+1)}{(x+1)^2}\right) e^x\, dx$

$$I = \int_0^1 e^x\, dx - 2\int_0^1 \frac{e^x}{x+1} dx + \int_0^1 \frac{1}{(x+1)^2} e^x dx$$

$$I = (e-1) - 2A + \left(-\frac{e}{2}+1+A\right) = \frac{e}{2} - A$$

(iii) $\int_0^1 \frac{x^3}{(x+1)^3}\, e^x\, dx = \int_0^1 \frac{x^3+1-1}{(x+1)^3}\, e^x\, dx$

$$= \int_0^1 \frac{(x+1)^3 - 3x(x+1) - 1}{(x+1)^3}\, e^x\, dx$$

$$= 1 - 3\int_0^1 \frac{x+1-1}{(x+1)^2}\, e^x dx - 1\int_0^1 \frac{1}{(x+1)^3}\, e^x dx$$

$$= 1 - 3\int_0^1 \frac{e^x}{x+1}\, dx + 3\int_0^1 \frac{e^x}{(x+1)^2}\, dx$$

$$-\left[\left(\frac{-1}{2\,(x+1)^2} e^x\right)_0^1 - \int_0^1 \frac{-1}{2\,(x+1)^2} e^x dx\right]$$

$$= 1 - 3A + 3\left(-\frac{e}{2}+1+A\right) - \left[\left(\frac{-e}{8}+\frac{1}{2}\right) + \frac{1}{2}\left(\frac{-e}{2}+1+A\right)\right]$$

$$= 1 - 3A - \frac{3e}{2} + 3 + 3A + \frac{e}{8} - \frac{1}{2} + \frac{e}{4} - \frac{1}{2} - \frac{A}{2}$$

$$= 3 - \frac{A}{2} - \frac{9e}{8}$$

(65) (C), (66) (B), (67) (D).

$$I = \int_0^{10\pi} \frac{\cos 6x.\cos 7x.\cos 8x.\cos 9x}{1+e^{2\sin^3 4x}}\, dx \quad \text{........ (1)}$$

$$I = \int_0^{10\pi} \frac{\cos 6x.\cos 7x.\cos 8x.\cos 9x}{1+e^{-2\sin^3 4x}}\, dx$$

$$I = \int_0^{10\pi} \frac{e^{2\sin^3 4x}(\cos 6x.\cos 7x.\cos 8x.\cos 9x}{1+e^{2\sin^3 4x}}\, dx \quad \text{....... (2)}$$

Add eq. (1) + eq. (2)

$$2I = \int_0^{10\pi} \underbrace{\cos 6x.\cos 7x.\cos 8x.\cos 9x}_{f\,(x)}\, dx$$

f (x) repeats after 2π

$$2I = 5\int_0^{2\pi} f\,(x)\, dx, \text{ again } f\,(2a-x) = f\,(x)$$

$$2I = 10\int_0^{\pi} f\,(x)\, dx, \text{ again } f\,(\pi-x) = f\,(x)$$

$$2I = 20\int_0^{\pi/2} f\,(x)\, dx \Rightarrow I = 10\int_0^{\pi/2} f\,(x)\, dx$$

$$\therefore\ I = 10\int_0^{\pi/2} \cos 6x.\cos 7x.\cos 8x.\cos 9x\, dx \quad \text{....... (3)}$$

$$I = 10\int_0^{\pi/2} \cos\,(3\pi-6x)\cos\left(\frac{7\pi}{2}-7x\right)\cos\,(4\pi-8x)\cos\left(\frac{9\pi}{2}-9x\right) dx$$

$$I = 10\int_0^{\pi/2} \cos 6x\,.\cos 7x\,\cos 8x\,.\cos 9x\, dx \quad \text{........ (4)}$$

Eq. (3) + Eq. (4)

$$2I = 10\int_0^{\pi/2} \cos 6x\,.\cos 8x\,[\cos 7x\,.\cos 9x + \sin 7x.\sin 9x]\, dx$$

$$2I = 10\int_0^{\pi/2} \cos 6x\,.\cos 8x\,.\cos 2x\, dx$$

$$I = 5\int_0^{\pi/2} \cos 6x\,.\cos 8x\,.\cos 2x\, dx$$

$$I=5\int_0^{\pi/2}\frac{\cos 8x}{2}(\cos 8x+\cos 4x)\,dx$$

$$=\frac{5}{2}\int_0^{\pi/2}\cos 8x\,(\cos 4x+\cos 2x)\,dx$$

$$=\frac{5}{2}\int_0^{\pi/2}\cos^2 8x\,dx+\frac{5}{2}\int_0^{\pi/2}\cos 4x\cos 2x\,dx$$

$$=\frac{1}{8}\cdot\frac{5}{2}\int_0^{4\pi}\cos^2 t\,dt\quad(8x=t)=\frac{5}{4}\int_0^{\pi}\cos^2 t\,dt$$

$$I=\frac{5}{2}\int_0^{\pi/2}\cos^2 t\,dt=\frac{5}{2}\int_0^{\pi/2}\sin^2 t\,dt$$

$$2I=\frac{5}{2}\int_0^{\pi/2}dt\Rightarrow I=\frac{5}{4}\cdot\frac{\pi}{2}\Rightarrow I=\frac{5\pi}{8}$$

(68) (A). $a_n=\int_0^{\pi/2}(1-\sin t)^n\sin 2t\,dt$

Let $1-\sin t=u\Rightarrow -\cot t\,dt=du$

$$=2\int_0^1 u^n(1-u)\,du=2\left(\int_0^1 u^n du-\int_0^1 u^{n+1}du\right)=2\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$$

Hence, $\frac{a_n}{n}=2\left(\frac{1}{n(n+1)}-\frac{1}{n(n+2)}\right)$

$$\lim_{n\to\infty}\sum_{n=1}^{n}\frac{a_n}{n}=2\left(\sum\left(\frac{1}{n}-\frac{1}{n+1}\right)\right)-\frac{1}{2}\sum\left(\frac{1}{n}-\frac{1}{n+2}\right)$$

$$=2\left(\sum_{n=1}^{n}\left(\frac{1}{n}-\frac{1}{n+1}\right)\right)-\sum_{n=1}^{n}\left(\frac{1}{n}-\frac{1}{n+2}\right)$$

$$=2(1)-\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+.....\right]=2-\frac{3}{2}=\frac{1}{2}$$

(69) (B). $I=\int_0^{\pi}(a\sin x+b\sin 2x)^2dx$

$$I=\int_0^{\pi}(a\sin x-b\sin 2x)^2\,dx$$

Add $2I=2\int_0^{\pi}(a^2\sin^2 x+b^2\sin^2 2x)\,dx$

$$I=2\int_0^{\pi/2}(a^2\sin^2 x)\,dx+2\int_0^{\pi/2}(b^2\sin^2 2x)\,dx$$

$$=2a^2\frac{\pi}{4}+2b^2\underbrace{\int_0^{\pi/2}\sin^2 2x\,dx}_{J}$$

Let $J=\int_0^{\pi/2}\sin^2 2x\,dx$, put $2x=t$

$$\frac{1}{2}\int_0^{\pi}\sin^2 t\,dt=\int_0^{\pi/2}\sin^2 t\,dt=\frac{\pi}{4}$$

Hence, $I=\frac{\pi a^2}{2}+\frac{\pi b^2}{2}=\frac{\pi}{2}(a^2+b^2)$

$$I(a)=\frac{\pi}{2}[a^2+(1-a^2)=\frac{\pi}{2}[2a^2-2a+1]$$

$$=\pi\left[a^2-a+\frac{1}{2}\right]=\pi\left[\left(a-\frac{1}{2}\right)^2+\frac{1}{4}\right]$$

$\therefore$ I (a) is minimum when a = 1/2 and minimum value $\pi/4$

(70) (D). $I=\int\frac{dx}{\left(x+\sqrt{x^2-4}\right)^{5/3}}$

Put $x+\sqrt{x^2-4}=t\Rightarrow\left(1+\frac{x}{\sqrt{x^2-4}}\right)dx=dt$

$$\Rightarrow x+\sqrt{x^2-4}=t\ \Rightarrow\sqrt{x^2-4}=t-x$$

$$\Rightarrow\ x=\frac{t^2+4}{2t}\ \Rightarrow x^2-4=\left(\frac{t^2+4}{2t}\right)^2-4$$

$$=\frac{t^4+16+8t^2-16t^2}{4t^2}=\left(\frac{t^2-4}{2t}\right)^2$$

so, $I=\int\left(\frac{t^2-4}{2t^2}\right)\frac{1}{t^{5/3}}dt\ =\frac{1}{2}\int t^{-5/3}dt-2\int t^{-11/3}dt$

$$=\frac{1}{2}\left(\frac{t^{-2/3}}{-2/3}\right)-2\left(\frac{t^{-8/3}}{-8/3}\right)+C\ =\frac{3}{4}t^{-8/3}[1-t^2]+c$$

Where $t=\left(x+\sqrt{x^2-4}\right)$

(71) (A). Let $\frac{x-1}{(x+1)(x-2)}=\frac{A}{x+1}+\frac{B}{x-2}$ .... (i)

$\Rightarrow x-1=A(x-2)+B(x+1)$ .... (ii)

Putting $x-2=0$ or, $x=2$ in (ii), we get

$1=3B\Rightarrow B=1/3$

Putting $x+1=0$ or $x=-1$ in (ii) we get

$-2=-3A\Rightarrow A=2/3$

Substituting the values of A and B in (i), we get

$$\frac{x-1}{(x+1)(x-2)} = \frac{2}{3}.\frac{1}{x+1} + \frac{1}{3}.\frac{1}{x-2}$$

$$\therefore \int \frac{x-1}{(x+1)(x-2)}\,dx = \frac{2}{3}\int \frac{1}{x+1}dx + \frac{1}{3}\int \frac{1}{x-2}dx$$

$$= \frac{2}{3}\log|x+1| + \frac{1}{3}\log|x-2| + C$$

(72) **(A).** Let $I = \int_0^{\pi/4} \log(1+\tan x)dx$ .... (i)

Then $I = \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$

$$\Rightarrow I = \int_0^{\pi/4} \log\left(1+\frac{\tan\pi/4 - \tan x}{1+\tan\pi/4\tan x}\right)dx$$

$$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$$

$$= \int_0^{\pi/4} \log\left(\frac{1+\tan x+1-\tan x}{1+\tan x}\right)dx = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$$

$$= \int_0^{\pi/4} \{\log 2 - \log(1+\tan x)\}dx$$

$$= \int_0^{\pi/4} \log 2dx - \int_0^{\pi/4} \log(1+\tan x)dx$$

$$\Rightarrow I = (\log 2)\,[x]_0^{\pi/4} - I \Rightarrow I = \frac{\pi}{4}\log 2 \Rightarrow I = \frac{\pi}{8}\log 2$$

(73) **(C).** $I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}dx = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$ ....(i)

Then, $I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx$

$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$ .....(ii)

Adding (i) and (ii), we get

$$\Rightarrow 2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$$

$$\Rightarrow 2I = \int_{\pi/6}^{\pi/3} 1.dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6} \quad \Rightarrow I = \pi/12$$

(74) **(A).** $L = \lim_{n\to\infty}\left[\frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \dots\dots\dots + \frac{1}{64n}\right]$

$$= \lim_{n\to\infty}\sum_{r=1}^{3n}\frac{n^2}{(n+r)^3}$$

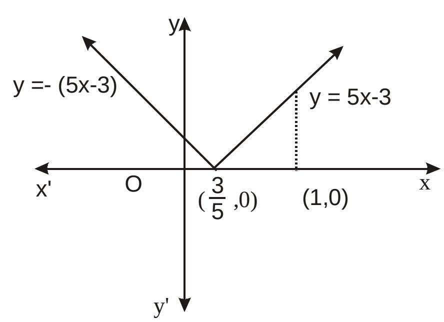
Put 3n = m, we get

$$L = \lim_{n\to\infty}\sum_{r=1}^{m}\frac{m^2/9}{\left(\frac{m}{3}+r\right)^3} = \lim_{n\to\infty}\sum_{r=1}^{m}\frac{3}{m}\left(\frac{1}{\left(1+\frac{3r}{m}\right)}\right)^3$$

$$= \int_0^3 \frac{dx}{(1+x)^3} = \left.\frac{-1}{2(1+x)^2}\right|_0^3 = \frac{15}{32}$$

(75) **(A).** $\int_0^1 |5x-3| = \begin{cases} -(5x-3) \text{ when } 5x-3 < 0 \text{ i.e., } x < 3/5 \\ 5x-3 \text{ when } 5x-3 \geq 0 \text{ i.e., } x \geq 3/5 \end{cases}$

$$\therefore \int_0^1 |5x-3|\,dx = \int_0^{3/5}|5x-3|\,dx + \int_{3/5}^1 |5x-3|\,dx$$



$$= \int_0^{3/5} -(5x-3)dx + \int_{3/5}^1 (5x-3)dx$$

$$= \left[3x - \frac{5x^2}{2}\right]_0^{3/5} + \left[\frac{5x^2}{2} - 3x\right]_{3/5}^1$$

$$= \left(\frac{9}{5} - \frac{9}{10}\right) + \left(-\frac{1}{2} + \frac{9}{10}\right) = \frac{13}{10}$$

(76) **(D).** $I = \int_{-\pi}^{\pi} (\cos^2 px + \sin^2 qx - 2\cos px\ \sin qx)dx$

$\because$ $\sin^2 qx$, $\cos^2 px$ are even functions of x and cospx .sinqx is an odd function.

$$\therefore \int_{-\pi}^{\pi} \cos^2 px\,dx = 2\int_0^{\pi} \cos^2 px dx$$

$$\int_{-\pi}^{\pi} \sin^2 qx\,dx = 2\int_0^{\pi} \sin^2 qx\,dx \text{ and } \int_{-\pi}^{\pi} \cos px \sin qx\ dx = 0$$

$$\therefore\ I = 2\int_0^{\pi} \cos^2 px\ dx + 2\int_0^{\pi} \sin^2 qx\,dx = 0$$

$$= 2\int_0^{\pi}\left(\frac{1+\cos 2px}{2}\right)dx + 2\int_0^{\pi}\left(\frac{1-\cos 2qx}{2}\right)dx$$

$$= \int_0^{\pi}(1+\cos 2px)\,dx + \int_0^{\pi}(1-\cos 2qx)\,dx$$

$$= \left[x + \frac{\sin 2px}{2p}\right]_0^{\pi} + \left[x - \frac{\sin 2qx}{2q}\right]_0^{\pi} = 2\pi$$

(77) **(D).** $\lim_{n\to\infty} \frac{1}{\sqrt{4n^2}} + \frac{1}{\sqrt{4n^2-1^2}} + \frac{1}{\sqrt{4n^2-2^2}} + \ldots + \frac{1}{\sqrt{4n^2-(n-1)^2}}$

$$= \lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{\sqrt{4n^2-r^2}} = \lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{n}\frac{1}{\sqrt{4-\left(\frac{r}{n}\right)^2}}$$

$$= \int_0^1 \frac{dx}{\sqrt{4-x^2}} = \left(\sin^{-1}\frac{x}{2}\right)_0^1 = \frac{\pi}{6}$$

(78) **(C).** $\int_{-2}^{3} |x^2-1|\,dx$

$$= \int_{-2}^{-1}|x^2-1|\,dx + \int_{-1}^{1}|x^2-1|\,dx + \int_{1}^{3}|x^2-1|\,dx$$

(Here modulus function will change at the points, when $x^2 - 1 = 0$ i.e. at $x = \pm 1$)

$$\text{So } I = \int_{-2}^{-1}(x^2-1)dx + \int_{-1}^{1}(1-x^2)dx + \int_{1}^{3}(x^2-1)dx$$

$$= \left.\frac{x^3}{3} - x\right|_{-2}^{-1} + \left. x + \frac{x^3}{3}\right|_{-1}^{1} + \left.\frac{x^3}{3} - x\right|_{1}^{3}$$

$$= \frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+6+\frac{2}{3} = \frac{28}{3}$$

(79) **(A).** Applying the formula

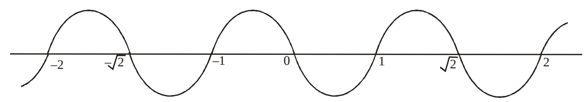
$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = f(b(x)).b'(x) - f(a(x)).a'(x)$$

We have, $f'(x) = (x^2-1)(x^2-2)(x^2-4)^3 . 2x$

$$= 2(x+2)^3(x+\sqrt{2})(x+1)\,x(x-1)(x-\sqrt{2})(x-2)^3$$

So that the possible external points are

$-2, -\sqrt{2}, -1, 0, 1, \sqrt{2}, 2$ one can see that it is time consuming to verify whether f ”(x) is –ve or +ve at these points in order to classify them as points of maxima or minima. Instead one can find out whether f '(x) changes its sign from +ve to –ve or from –ve to +ve as one moves from left to right of a point. In the first case it is a point of maximum and in the second case it is a point of minimum.



In the neighbourhood of $-\sqrt{2}$, it can be seen that f ' (x) changes its sign from +ve to –ve as one moves from left to right of $-\sqrt{2}$. It is a point of maximum, Similarly $0, \sqrt{2}$ are also points of maximum.

(80) **(A).** Let $f(x) = \frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)}$

$$\Rightarrow f(x) = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$\Rightarrow 1 = A(x^2 - x + 1) + (Bx + C)(x + 1)$

Comparing the coefficients of x^2, x, and constants

$0 = A + B,\ 0 = -A + B + C,\ 1 = A + C$

$\Rightarrow A = 1/3,\ B = -1/3\ \&\ C = 2/3$

$$\Rightarrow f(x) = \frac{\frac{1}{3}}{x+1} + \frac{-\frac{x}{3}+\frac{2}{3}}{x^2-x+1}$$

Let $I_1 = \frac{1}{3}\int\frac{dx}{x+1} = \frac{1}{3}\log|x+1| + C_1$

Let $I_2 = \int\frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}dx = \frac{1}{3}\int\frac{2-x}{x^2-x+1}dx$

Express the numerator in terms of derivative of denominator.

$$\Rightarrow I_2 = -\frac{1}{6}\int\frac{2x-4}{x^2-x+1}dx$$

$$\Rightarrow I_2 = -\frac{1}{6}\int\frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int\frac{dx}{x^2-x+1}$$

$$\Rightarrow I_2 = -\frac{1}{6}\log|x^2-x+1| + \frac{1}{2}\int\frac{dx}{x^2-x+1}$$

$$\Rightarrow I_2 = -\frac{1}{6}\log|x^2-x+1| + \frac{1}{2}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$\Rightarrow I_2 = -\frac{1}{6}\log|x^2-x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C_2$$

$$\Rightarrow I_2 = -\frac{1}{6}\log|x^2-x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C_2$$

$$\Rightarrow \int\frac{dx}{x^3+1} = \int f(x)\,dx = I_1 + I_2$$

$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log|x^2-x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

$$= \frac{1}{3}\log\left|\frac{x+1}{\sqrt{x^2-x+1}}\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

(81) **(A).** Consider $I = \int_0^1 (by + a(1-y))^x\,dy$

$$= \int_0^1 (a+(b-a)y)^x\,dy = \left[\frac{(a+(b-a)y)^{x+1}}{(x+1)}\cdot\frac{1}{b-a}\right]_0^1$$

$$I = \frac{1}{(x+1)(b-a)}(b^{x+1}-a^{x+1}) = \frac{1}{(x+1)}\left(\frac{b^{x+1}-a^{x+1}}{b-a}\right)$$

$$\text{Now, } L = \lim_{x\to 0}\left(\frac{b^{x+1}-a^{x+1}}{b-a}\right)^{1/x}\cdot\left(\frac{1}{(x+1)}\right)^{1/x}$$

$$L = \underbrace{\lim_{x\to 0}\left(\frac{1}{(x+1)}\right)^{1/x}}_{1^\infty}\cdot\underbrace{\lim_{x\to 0}\left(\frac{b^{x+1}-a^{x+1}}{b-a}\right)^{1/x}}_{1^\infty}$$

$$\left(\lim_{x\to 0}(x+1)^{1/x} = e^{\lim_{x\to 0}\frac{1}{x}(x+1-1)} = e \Rightarrow \frac{1}{(x+1)^{1/x}} = \frac{1}{e}\right)$$

$$\therefore L = \frac{1}{e}\cdot\underbrace{\lim_{x\to 0}\left(\frac{b^{x+1}-a^{x+1}}{b-a}\right)^{1/x}}_{\ell}$$

$$\text{Now, } \ell = e^{\lim_{x\to 0}\frac{1}{x}\left(\frac{b^{x+1}-a^{x+1}-b+a}{b-a}\right)}$$

$$= e^{\frac{1}{b-a}\lim_{x\to 0}\frac{b(b^x-1)-a(a^x-1)}{x}} = e^{\frac{1}{b-a}(b\ln b - a\ln a)}$$

$$= e^{\ln\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}} = \left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} \qquad \therefore L = \frac{1}{e}\cdot\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

(82) Let $I = \int_0^1 {}^{207}C_7\cdot\underbrace{x^{200}}_{II}\cdot\underbrace{(1-x)^7}_{I}\,dx$

$$I = {}^{207}C_7\left[\underbrace{\left|(1-x)^7\cdot\frac{x^{201}}{201}\right|}_{zero} + \frac{7}{201}\int_0^1 (1-x)^6.x^{201}dx\right]$$

$$= {}^{207}C_7\cdot\frac{7}{201}\int_0^1(1-x)^6.x^{201}dx$$

I.B.P. again 6 more times

$$= {}^{207}C_7\cdot\frac{7!}{201.202.203.204.205.206.207}\int_0^1 x^{207}dx$$

$$= \frac{(207)!}{7!(200)!}\frac{7!}{201.202.......207}\cdot\frac{1}{208}$$

$$= \frac{(207)!}{(207)!\,7!}\cdot\frac{7!}{208} = \frac{1}{208} = \frac{1}{k} \quad \Rightarrow k = 208$$

(83) **13.** The polynomial is an everywhere differentiable function. Therefore, the points of extremum can only be roots of the derivative. Furthermore, the derivative of a polynomial is a polynomial. The polynomial of the least degree with roots $x_1 = 1$ and $x_2 = 3$ has the from $a(x-1)(x-3)$.

Hence $P'(x) = a(x-1)(x-3) = a(x^2-4x+3)$ since at the point $x = 1$, there must be $P(1) = 6$, we have

$$P(x) = \int_1^x P'(x)\,dx + 6 = a\int_1^x (x^2-4x+3)\,dx + 6$$

$$= a\left(\frac{x^3}{3} - 2x^2 + 3x - \frac{4}{3}\right) + 6$$

The coefficient ‘a’ is determined from the condition $P(3) = 2$, whence $a = 3$.

Hence $P(x) = x^3 - 6x^2 + 9x + 2$

Now $P(2) = 8 - 24 + 18 + 2 = 28 - 24 = 4$

Also $P'(x) = 3(x^2-4x+3) \Rightarrow P'(0) = 9$

$\therefore P(2) + P'(0) = 4 + 9 = 13$

(84) **2250.** We have $F(x) + F\left(x+\frac{1}{2}\right) = 3$ .... (1)

Replace x by $x+\frac{1}{2}$ in (1),

we get $F\left(x+\frac{1}{2}\right) + F(x+1) = 3$ .... (2)

$\therefore$ From (1) and (2), we get

$F(x) = F(x+1)$ .... (3)

$\Rightarrow F(x)$ is periodic function

Now consider, $I = \int_0^{1500} F(x)\,dx = 1500\int_0^1 F(x)\,dx$

$$= 1500\left[\int_0^{1/2} F(x)\,dx + \int_{1/2}^{1} F(x)\,dx\right]$$

(Using property of periodic function)

Put $x = y + \frac{1}{2}$ in 2nd integral, we get

$$I = 1500\left[\int_0^{1/2} F(x)\,dx + \int_0^{1/2} F\left(y+\frac{1}{2}\right)dy\right]$$

$$= 1500\int_0^{1/2}\left(F(x)\,dx + F\left(x+\frac{1}{2}\right)\right)dx$$

$$= 1500\int_0^{1/2} 3\,dx \quad \text{[Using (i)]}$$

Hence, $I = 1500\,(3)\left(\frac{1}{2}\right) = 750\times 3 = 2250$

(85) **7.** Using $\sin 2x = \frac{2\tan x}{1+\tan^2 x}$

$$I = \int_0^{\pi/2}\frac{1-\frac{2\tan x}{1+\tan^2 x}}{1+\frac{2\tan x}{1+\tan^2 x}}dx = \int_0^{\pi/2}\frac{(1-\tan x)^2}{(1+\tan x)^4}\cdot(1+\tan^2 x)\,dx$$

$$= \int_0^{\pi/2}\frac{(1-\tan x)^2}{(1+\tan x)^4}\cdot\sec^2 x\,dx$$

put $y = \tan x \Rightarrow dy = \sec^2 x\,dx$

$$\therefore I = \int_0^{\infty}\frac{(1-y)^2}{(1+y)^4}dy$$

now put $1 + y = z \Rightarrow dy = dz$

$$\therefore I = \int_1^{\infty}\frac{(2-z)^2}{z^4}dz = -\left.\frac{3z^2-6z+4}{3z^3}\right|_1^{\infty} = \frac{1}{3}$$

$\Rightarrow a = 1,\ b = 3 \Rightarrow 1 + 3 + 3 = 7$ Ans.]

<u>Alternatively:</u> $I = \int_0^{\pi/2}\frac{(\cos x-\sin x)^2}{(\cos x+\sin x)^4}dx$

$$I = -\frac{1}{3}\int_0^{\pi/2}\underbrace{(\cos x-\sin x)}_{I}\cdot\underbrace{\left(\frac{d}{dx}\left(\frac{1}{(\cos x+\sin x)^3}\right)\right)}_{II}dx$$

integrating by parts

$$= -\frac{1}{3}\left[\left.\frac{(\cos x-\sin x)}{(\cos x+\sin x)^3}\right|_0^{\pi/2} + \int_0^{\pi/2}\frac{(\sin x+\cos x)}{(\cos x+\sin x)^3}dx\right]$$

$$= -\frac{1}{3}\left[\{(-1)-(1)\} + \int_0^{\pi/2}\frac{dx}{1+\sin 2x}\right]$$

using $\sin 2x = \frac{2\tan x}{1+\tan^2 x}$ $= \frac{2}{3} - \frac{1}{3}\int_0^{\pi/2}\frac{\sec^2 x}{(1+\tan x)^2}dx$

$$= \frac{2}{3} - \frac{1}{3}\int_1^{\infty}\frac{dt}{t^2} = \frac{2}{3} + \frac{1}{3}\left[t^t\right]_0^{\infty}$$

$$= \frac{2}{3} + \frac{1}{3}[(0)-(1)] = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

$\Rightarrow a = 1,\ b = 3 \Rightarrow 1 + 3 + 3 = 7$

(86) **2525.**

$$I = \int_0^{\infty}\frac{dx}{x^2+\frac{1}{x^2}+(a^2-2)} = \int_0^{\infty}\frac{x^2\,dx}{x^4+(a^2-2)x^2+1}$$

$(a^2 - 2 = k \geq 0)$

$$= \int_0^{\infty}\frac{x^2\,dx}{x^4+kx^2+1} = \frac{1}{2}\int_0^{\infty}\frac{(x^2+1)+(x^2-1)}{x^4+kx^2+1}dx$$

$$= \underbrace{\frac{1}{2}\int_0^{\infty}\frac{1+(1/x^2)}{x^2+(1/x^2)+k}dx}_{I_1} + \underbrace{\frac{1}{2}\int_0^{\infty}\frac{1-(1/x^2)}{x^2+(1/x^2)+k}dx}_{I_2}$$

now proceed, $I_1 = \frac{\pi}{2a}$ and $I_2 = 0$

$$\therefore \boxed{I = \frac{\pi}{2a}}\ ;\ \frac{\pi}{2a} = \frac{\pi}{5050} \Rightarrow a = 2525$$

(87) **8.** Let $\theta = \frac{\pi}{4} + x \Rightarrow d\theta = dx$

or $4\theta = \pi + 4x \Rightarrow \pi - 4\theta = -4x$

$$= \int_{-\pi/2}^{0}\frac{(-4x)\tan\left(\frac{\pi}{4}+x\right)}{1-\tan\left(\frac{\pi}{4}+x\right)}dx = -4\int_{-\pi/2}^{0}\frac{x\frac{(1+\tan x)}{1-\tan x}}{1-\frac{1+\tan x}{1-\tan x}}dx$$

$$= -4\int_{-\pi/2}^{0}\frac{x(1+\tan x)}{1-\tan x}\cdot\frac{(1-\tan x)}{(-2)\tan x}dx$$

$$= 2\int_{-\pi/2}^{0}\frac{x(1+\tan x)}{\tan x}dx = 2\int_{-\pi/2}^{0}\left(\frac{x}{\tan x}+x\right)dx$$

$$I = x^2\Big]_{-\pi/2}^{0} + \int_{-\pi/2}^{0}\frac{x}{\tan x}dx$$

$$I = -\frac{\pi^2}{4} + 2\int_0^{\pi/2} \frac{t}{\tan t}dt \qquad x = -t$$

now $I_1 = \int_0^{\pi/2} \underset{I}{t}\,\underset{II}{\cot t}\,dt = t\ln\sin t\Big]_0^{\pi/2} - \int_0^{\pi/2} \ln\sin t\,dt$

$$I_1 = 0 + \frac{\pi}{2}\ln 2$$

Hence, $2\cdot\frac{\pi}{2}\ln 2 - \frac{\pi^2}{4} = \pi\ln 2 - \frac{\pi^2}{4} \Rightarrow k = 2, w = 4$

$\Rightarrow kw = 8$

(88) **3.** $g(1) = 5$ and $\int_0^1 g(t)\,dt = 2$

$$2f(x) = \int_0^x (x^2 - 2xt + t^2)\,g(t)\,dt$$

$$= x^2\int_0^x g(t)\,dt - 2x\int_0^x t\,g(t)\,dt + \int_0^x t^2 g(t)\,dt$$

Differentiating

$$2f'(x) = x^2 g(x) + \int_0^x g(t)dt\cdot 2x - 2\left[x^2 g(x) + \left(\int_0^x t\,g(t)\,dt\right)\right] + x^2 g(x)$$

$$2f'(x) = 2x\int_0^x g(t)dt - 2\int_0^x t\,g(t)\,dt$$

$$f''(x) = x\,g(x) + \int_0^x g(t)dt - x\,g(x) = \int_0^x g(t)dt$$

hence $f''(1) = \int_0^1 g(t)\,dt = 2$

also $f'''(x) = g(x) \Rightarrow f'''(1) = g(1) = 5$

$\therefore f'''(1) - f''(1) = 5 - 2 = 3$

(89) **2.** $f(x) = \begin{cases} e^{\cos x}\sin x & \text{for } |x| \le 2 \\ 2 & \text{otherwise}\end{cases}$

$$f(x) = \begin{cases} e^{\cos x}\sin x & \text{for } -2 \le x \le 2 \\ 2 & \text{otherwise}\end{cases}$$

$$\int_{-2}^{3} f(x)\,dx = \int_{-2}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx$$

$$= \int_{-2}^{2} e^{\cos x}\sin x\,dx + \int_2^3 2\,dx = 0 + 2[x]_2^3$$

[$\because$ $e^{\cos x}\sin x$ is an odd function]

$= 2[3 - 2] = 2$

$$\therefore \int_{-2}^{3} f(x)\,dx = 2$$

(90) **4.** $I = \int_{-2}^{0} [x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1)\,dx$

$$\left[\frac{x^4}{4} + x^3 + \frac{3x^2}{2} + 3x + (x+1)\sin(x+1) + \cos(x+1)\right]_{-2}^{0}$$

$= (\sin 1 + \cos 1) - (4 - 8 + 6 - 6 + \sin 1 + \cos 1) = 4$

(91) **3.** Given that $\int_{\sin x}^{1} t^2(f(t))\,dt = 1 - \sin x$

$$\Rightarrow \frac{d}{dx}\int_{\sin x}^{1} t^2(f(t))\,dt = \frac{d}{dx}(1 - \sin x)$$

$\Rightarrow -\sin^2 x\, f(\sin x)\,.\cos x = -\cos x$

$\Rightarrow f(\sin x) = 1/\sin^2 x$

$$\Rightarrow f(1/\sqrt{3}) = \frac{1}{(1/\sqrt{3})^2} = 3$$

(92) **1.** $$\lim_{x\to a} \frac{\int_a^x f(x)\,dx - \left(\frac{x-a}{2}\right)(f(x) + f(a))}{(x-a)^3} = 0$$

$$\lim_{h\to 0} \frac{\int_a^{a+h} f(x)\,dx - \frac{h}{2}(f(a+h) + f(a))}{h^3} = 0$$

$$\lim_{h\to 0} \frac{f(a+h) - \frac{1}{2}[f(a) + f(a+h) - \frac{h}{2}(f(a+h))}{3h^2} = 0$$

[Using L'Hospital rule]

$$\Rightarrow \lim_{h\to 0} \frac{\frac{1}{2}f(a+h) - \frac{1}{2}f(a) - \frac{h}{2}f'(a+h)}{3h^2} = 0$$

$$\Rightarrow \lim_{h\to 0} \frac{\frac{1}{2}f'(a+h) - \frac{1}{2}f'(a+h) - \frac{h}{2}f''(a+h)}{6h} = 0$$

[Using L'Hospital rule]

$$\Rightarrow \lim_{h\to 0} \frac{-f''(a+h)}{12} = 0 \Rightarrow f''(x) = 0, \forall a \in R$$

$\Rightarrow$ f(x) must be of max. degree 1.

(93) **0.** From given integral equation, $f(0) = 0$.

Also differentiating the given integral equation w.r.t. x

$f'(x) = f(x)$

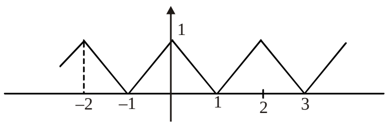
If $\ldots \neq 0 \Rightarrow \frac{f'(x)}{f(x)} = 1 \Rightarrow f(x) = e^c e^x$

$\because$ $f(0) = 0 \Rightarrow e^c = 0$, a contradiction

$\therefore$ $f(x) = 0 \ \forall\, x \in R$

$\Rightarrow$ $f(\ln 5) = 0$

(94) **4.** $f(x) = \begin{cases} x-1, & 1 \le x < 2 \\ 1-x, & 0 \le x < 1 \end{cases}$



f (x) is periodic with period 2

$\therefore \quad I = \int_{-10}^{10} f(x) \cos \pi x \, dx$

$= 2\int_0^{10} f(x) \cos \pi x \, dx = 2 \times 5 \int_0^2 f(x) \cos \pi x \, dx$

$= 10\left[\int_0^1 (1-x) \cos \pi x \, dx + \int_1^2 (x-1) \cos \pi x \, dx\right]$

$= 10\,(I_1 + I_2)$

$I_2 = \int_1^2 (x-1) \cos \pi x \, dx$ put $x - 1 = t$

$I_2 = -\int_1^0 t \cos \pi t \, dt$

$I_1 = \int_0^1 (1-x) \cos \pi x \, dx = \int_0^1 x \cos(\pi x)\, dx$

$\therefore \ I = 10\left[-2\int_0^1 x \cos \pi x \, dx\right] = -20\left[x \frac{\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2}\right]_0^1$

$= -20\left[-\frac{1}{\pi^2} - \frac{1}{\pi^2}\right] = \frac{40}{\pi^2} \qquad \therefore \ \frac{\pi^2}{10} I = 4$

(95) **2.** Using by parts

$4x^2 \cdot \frac{d\,(1-x^2)^5}{dx}\Bigg]_0^1 - \int_0^1 d\,(1-x^2)^5 .12x^2 dx$

$= -12x^2 \cdot (1-x^2)^5 \Big]_0^1 + \int_0^1 24x\,(1-x^2)^5 dx$

$1 - x^2 = t$; $-2x\,dx = dt$

$= -\int_1^0 12t^5 dt = 2$

(96) **0.** $I = \int_{-1}^{2} \frac{x\,[x^2]}{2 + [x+1]} dx = \int_{-1}^{2} \frac{x\,[x^2]}{3 + [x+1]} dx$

$= \int_{-1}^{0} \frac{0}{3-1} dx = \int_{0}^{1} \frac{0}{3+0} dx + \int_{1}^{\sqrt{2}} \frac{x.1}{3+1} dx$

$= \frac{1}{4}\left[\frac{x^2}{2}\right]_1^{\sqrt{2}} = \frac{2-1}{8} = \frac{1}{4} \quad \therefore \ 4I - 1 = 0$

(97) **9.** $\alpha = \int_0^1 (e^{9x + 3\tan^{-1} x})\left(\frac{12 + 9x^2}{1 + x^2}\right) dx$

$\Rightarrow \quad \alpha = (e^{9x + 3\tan^{-1} x})_0^1 = e^{9 + \frac{3\pi}{4}} - 1$

$\Rightarrow \quad \ln(1 + \alpha) = 9 + \frac{3\pi}{4}$

Alter : $\alpha = \int_0^1 (e^{9x + 3\tan^{-1} x})\left(\frac{12 + 9x^2}{1 + x^2}\right) dx$

Let $9x + 3\tan^{-1}x = t$

$\Rightarrow \quad \left(9 + \frac{3}{1+x^2}\right) dx = dt \Rightarrow \left(\frac{12 + 9x^2}{1+x^2}\right) dx = dt$

$\Rightarrow \quad \alpha = \int_0^{9+3\pi/4} e^t dt = (e^t)_0^{9+3\pi/4} = e^{9+3\pi/4} - 1$

Now, $\log_e |1 + \alpha| - \frac{3\pi}{4} = \log_e e^{(9+3\pi/4)} - \frac{3\pi}{4} = 9$

EXERCISE-3

(1) **(C).** $\int \frac{\cos 2x - 1}{\cos 2x + 1} dx = \int \frac{\cos 2x + 1 - 2}{\cos 2x + 1} dx = \int \left(1 - \frac{2}{1 + \cos x}\right) dx$

$= \int \left(1 - \frac{2}{2\cos^2 x}\right) dx = \int (1 - \sec^2 x)\, dx = x - \tan x + c$

(2) **(B).** $\int \frac{(\log x)}{x^2} dx$

Integrating by parts

$\int I\,dx = I \int II\,dx - \int \left[\frac{dI}{dx} \int II\,dx\right] dx$

$\therefore \ \int \frac{(\log x)}{x^2} dx = \log x \int \frac{1}{x^2} dx - \int \left[\frac{d}{dx}(\log x) \int \frac{1}{x^2} dx\right] dx$

$= \log x.\left(-\frac{1}{x}\right) - \int \frac{1}{x} \times \left(-\frac{1}{x}\right) dx = -\log x \times \frac{1}{x} + \int \frac{1}{x^2} dx$

$= -\log x \times \frac{1}{x} - \frac{1}{x} + c = -\left[\frac{\log x + 1}{x}\right] + c$

(3) **(A).** $I_n = \int_0^{\pi/4} \tan^n x\, dx$

$I_{n-1} + I_{n-1} = \int_0^{\pi/4} (\tan^{n-1} x + \tan^{n+1} x)\, dx$

$= \int_0^{\pi/4} \tan^{n-1} x\, (1 + \tan^2 x)\, dx = \int_0^{\pi/4} \tan^{n-1} x \sec^2 x\, dx$

Put $\tan x = t$

$\sec^2 x\, dx = dt$

$\Rightarrow$ At $x = 0$ then $t = 0$ and $x = \pi/4$ then $t = 1$

$= \int_0^1 t^{n-1}\, dt = \left(\frac{t^{n-1+1}}{n-1+1}\right)_0^1 = \frac{1}{n}$

Now, $n.(I_{n-1} + I_{n+1}) = n.\frac{1}{n} = 1$

(4) **(A).** $I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\, dx$

$I = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\, dx + \int_{-\pi}^{\pi} \frac{2x \sin x}{1+\cos^2 x}\, dx$

$\left\{ \because \int_{-a}^{a} f(x) = 0 \text{ if } f \text{ is odd function} \right.$

$\qquad = 2\int_{-a}^{a} f(x) : f(x) \text{ is even}$

$\because \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\, dx$ is an odd function

and $\int_{-\pi}^{\pi} \frac{2x \sin x}{1+\cos^2 x}\, dx$ is even function

$I = 4\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$ .... (1)

$\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

$I = 4\int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\, dx = 4\int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\, dx$ .... (2)

Adding eq. (1) and (2), $2I = 4\int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}\, dx$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt \Rightarrow \sin dx = -dt$

$\therefore 2I = 4\pi \int_1^{-1} \frac{-dt}{1+t^2} \Rightarrow I = 2\pi \int_{-1}^{1} \frac{dt}{1+t^2}$

$= 2\pi\, [\tan^{-1} t]_{-1}^{1} = 2\pi\, (\tan^{-1}(1) - \tan^{-1}(-1))$

$= 2\pi\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = 2\pi \times \frac{\pi}{2} = \pi^2$

(5) **(C).**

$\int_{\pi}^{10\pi} |\sin x|\, dx = \int_0^{\pi} |\sin x|\, dx + \int_{\pi}^{10\pi} |\sin x|\, dx - \int_0^{\pi} |\sin x|\, dx$

$= \int_0^{10\pi} |\sin x|\, dx - \int_0^{\pi} |\sin x|\, dx$

{$\because$ $|\sin x|$ is periodic with perod π and in $[0, \pi]$ $\sin x \ge 0$

and $\int_0^{na} f(x)\, dx = n\int_0^{a} f(x)\, dx$ if a is period of function}

$= 10\int_0^{\pi} |\sin x|\, dx - \int_0^{\pi} |\sin x|\, dx$

$= 9\int_0^{\pi} |\sin x|\, dx = 9\int_0^{\pi} \sin x\, dx$

$= 9(-\cos x)_0^{\pi} = 9(1+1) = 9 \times 2 = 18$

(6) **(A).** $\int_0^{\sqrt{2}} [x^2]\, dx = \int_0^{1} [x^2]\, dx + \int_1^{\sqrt{2}} [x^2]\, dx$

$= \int_0^1 0\, dx + \int_1^{\sqrt{2}} 1\, dx = 0 + [x]_1^{\sqrt{2}} = \sqrt{2} - 1$

(7) **(B).** $\lim_{n\to\infty} \frac{1^P + 2^P + 3^P + \ldots + n^P}{n^{P+1}}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{r^P}{n^{P+1}} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{r}{n}\right)^P$

Replace $\frac{r}{n} \to x$, $\frac{1}{n}$ by dx, $\sum \to \int$

If $\frac{r}{n} = x$ if $r = 1$ and $n \to \infty$ then $x = 0$

and r = n then $x = \frac{n}{n} \Rightarrow x = 1$

$$\therefore \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{r}{n}\right)^P = \int_0^1 x^P dx = \left[\frac{x^{P+1}}{P+1}\right]_0^1$$

$$= \frac{1}{P+1} - 0 = \frac{1}{P+1}$$

(8) **(A).** $\frac{d}{dx}F(x) = \frac{e^{\sin x}}{x}, x > 0$

Integrating, $F(x) = \int \frac{e^{\sin x}}{x} dx$ ..... (1)

Also, $\int_1^4 \frac{3e^{\sin x^3}}{x} dx = F(k) - F(1)$

Let $x^3 = t \Rightarrow 3x^2\, dx = dt \Rightarrow 3\, dx = \frac{dt}{x^2}$

$$\Rightarrow \int_1^{64} \frac{e^{\sin t}}{x} \times \frac{dt}{x^2} = F(k) - F(1)$$

$$\Rightarrow \int_1^{64} \frac{e^{\sin t}}{x} dt = F(k) - F(1)$$

$\Rightarrow [F(x)]_1^{64} = F(k) - F(1)$ [Using (1),

$\int \frac{e^{\sin x}}{x} dx = F(x)]$

$\Rightarrow F(64) - F(1) = F(k) - F(1) \Rightarrow k = 64$

(9) **(C).** $I = \int_a^b x\, f(x)\, dx$ .... (1)

$$= \int_a^b (a+b-x)\, f(a+b-x)\, dx$$

$I = \int_a^b (a+b-x)\, f(x)\, dx$ .... (2)

$\because$ given $f(a+b-x) = f(x)\}$

Adding eq. (1) and eq. (2)

$2I = \int_a^b (a+b)\, f(x)\, dx$; $I = \frac{a+b}{2}\int_a^b f(x)\, dx$

(10) **(D).** $I = \int_0^1 x(1-x)^n dx$

Let $1 - x = t \Rightarrow -dx = dt \Rightarrow dx = -dt$

$\therefore I = -\int_1^0 (1-t)\, t^n dt = \int_0^1 (1-t)t^n dt = \int_0^1 (t^n - t^{n+1})\, dt$

$$= \left[\frac{t^{n+1}}{n+1} - \frac{t^{n+2}}{n+2}\right]_0^1 = \frac{1}{n+1} - \frac{1}{n+2}$$

(11) **(D).** $\lim_{x\to 0} \frac{\int_0^{x^2} \sec^2 t\, dt}{x \sin x}$ $\left(\frac{0}{0}\text{ form}\right)$

Applying L' Hospital rule

$\left\{\because \frac{d}{dx}\int_{\phi(x)}^{\psi(x)} f(t)\, dt = f(\psi x) - \psi'(x) - f(\phi x)\, \phi'(x)\right.$

$$\lim_{x\to 0} \frac{\sec^2 x^2 .2x}{x\cos x + \sin x}$$

Again applying L' Hospital rule

$$= \lim_{x\to 0} \frac{2x.2\sec^2 x^2.\tan x^2.2x + 2\sec^2 x^2}{-x\sin x + \cos x + \cos x} = \frac{0 + 2\sec^2 0}{0 + 2\cos 0} = 1$$

(12) **(A).**

$$\lim_{n\to\infty} \frac{1 + 2^4 + 3^4 + ... + n^4}{n^5} - \lim_{n\to\infty} \frac{1 + 2^3 + 3^3 + ... + n^3}{n^5}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^4}{n^5} - \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^3}{n^5}$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^4 - \lim_{n\to\infty} \frac{n^2(n+1)^2}{4n^5}$$

$$= \int_0^1 x^4 dx - \frac{1}{4}\lim_{n\to\infty} \frac{(n+1)^2}{n^3} = \left[\frac{x^5}{5}\right]_0^1 - \frac{1}{4}\lim_{n\to\infty} \frac{n^2(1+1/n)^2}{n^3}$$

$$= \left(\frac{1}{5} - 0\right) - \frac{1}{4}\lim_{n\to\infty}\left(\frac{1}{n} + \frac{2}{n^2} + \frac{1}{n^3}\right) = \frac{1}{5}$$

(13) **(C).** $\because f(y) = e^y$

$\therefore f(t-y) = e^{t-y}$ .... (1) and $g(y) = y$ .... (2)

and $F(t) = \int_0^t f(t-y)\, g(y)\, dy$

$$= \int_0^t e^{t-y}.y\, dy = e^t \int_0^t e^{-y}.y\, dy$$

$$= e^t \left[y\int e^{-y} dy - \int\left(\frac{d}{dy}y\int e^{-y} dy\right) dy\right]_0^t$$

$$= e^t\left[-y.e^{-y} - \int -e^{-y}dy\right]_0^t$$

$$= e^t\left[-y.e^{-y} + (-e^{-y})\right]_0^t = e^t\left[-ye^{-y} - e^{-y}\right]_0^t$$

$$= e^t\,[-t\,e^{-t} - e^{-t} - (-0.e^{-0} - e^{-0})]$$

$$= e^t[-te^{-t} - e^{-t} + 1] = e^t(-te^{-t} - e^{-t}) + e^t$$

$$= e^t e^{-t}(-t-1) + e^t = e^t - (t+1)$$

(14) **(D).** Here $f'(x) = f(x) \Rightarrow \dfrac{f'(x)}{f(x)} = 1$

$$\Rightarrow \int \frac{f'(x)}{f(x)}\,dx = \int dx = \log f(x) = x + c$$

$\Rightarrow f(x) = e^{x+c}$ .... (1)

$\therefore\ f(0) = 1\ \therefore\ e^{0+c} = 1$

$\Rightarrow e^c = 1$ .... (2)

$\therefore\ f(x) = e^x e^c$

$f(x) = e^x$ .... (3)

Now, $f(x) + g(x) = x^2 \Rightarrow g(x) = x^2 - f(x) = x^2 - e^x$

$$\therefore \int_0^1 f(x)\,g(x)\,dx = \int_0^1 e^x.(x^2 - e^x)\,dx$$

$$= \int_0^1 (x^2e^x - e^{2x})\,dx = \int_0^1 x^2e^x dx - \int_0^1 e^{2x}dx$$

$$= \left[(x^2 - 2x + 2)\,e^x\right]_0^1 = \left[\frac{e^{2x}}{2}\right]_0^1$$

$$= (e^1 - 2) - \left(\frac{e^2}{2} - \frac{1}{2}\right) = e - \frac{e^2}{2} - \frac{3}{2}$$

(15) **(B).** $\displaystyle\int \frac{\sin x}{\sin(x-\alpha)}\,dx = Ax + B\log\sin(x-\alpha) + C$

$$\Rightarrow \int \frac{\sin(x-\alpha+\alpha)}{\sin(x-\alpha)}\,dx = Ax + B\log(x-\alpha) + C$$

$$\Rightarrow \int \frac{\sin(x-\alpha)\cos\alpha + \cos(x-\alpha)\sin\alpha}{\sin(x-\alpha)}\,dx$$

$= Ax + B\log\sin(x-\alpha) + C$

$$\Rightarrow \int \cos\alpha + \sin\alpha\cot(x-\alpha)\,dx = Ax + B\log\sin(x-\alpha) + C$$

$\Rightarrow x\cos\alpha + \sin\alpha\log\sin(x-\alpha) + C$

$= Ax + B\log\sin(x-\alpha) + C$

On comparing, $A = \cos\alpha,\ B = \sin\alpha$

(16) **(D).** $\displaystyle\int \frac{dx}{\cos x - \sin x} = \frac{1}{\sqrt{2}}\int \frac{dx}{\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x}$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\cos x\cos\frac{\pi}{4} - \sin x\sin\frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\cos\left(x + \frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}}\int \sec\left(x + \frac{\pi}{4}\right)dx$$

$$= \frac{1}{\sqrt{2}}\log\left|\tan\frac{\pi}{4} + \left(\frac{x + \frac{\pi}{4}}{2}\right)\right| + C = \frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2} + \frac{3\pi}{8}\right)\right| + C$$

Since $\displaystyle\int \sec dx = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + C\right|$

(17) **(B).** $\displaystyle\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}e^{r/n}$ Replace $\sum \to \int$

$\dfrac{1}{n} \to dx$; $\dfrac{r}{n} = x \Rightarrow$ if $r = 1$ and $n \to \infty$ then $x = 0$

and $r = n$ then $x = 1$

$$\therefore \int_0^1 e^x dx = [e^x]_0^1 = (e^1 - 1)$$

(18) **(A).** $\displaystyle\int_{-2}^{3} |1 - x^2|\,dx$

$$= \int_{-2}^{-1}|1-x^2|\,dx + \int_{-1}^{1}|1-x^2|\,dx + \int_{1}^{3}|1-x^2|\,dx$$

$$= \int_{-2}^{-1} -(1-x^2)\,dx + \int_{-1}^{1}(1-x^2)\,dx + \int_{1}^{3} -(1-x^2)\,dx$$

$$= \left[-x + \frac{x^3}{3}\right]_{-2}^{-1} + \left(x - \frac{x^3}{3}\right)_{-1}^{1} + \left(-x + \frac{x^3}{3}\right)_{1}^{3}$$

$$= -\frac{1}{3} + 1 + \frac{8}{3} - 2 + \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) + \left(\frac{27}{3} - 3 - \frac{1}{3} + 1\right)$$

$$= \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + 9 - 3 + \frac{2}{3} = \frac{28}{3}$$

(19) **(C).** $\displaystyle I = \int_0^{\pi/2} \frac{(\sin x + \cos x)^2}{\sqrt{1 + \sin 2x}}\,dx$

$$= \int_0^{\pi/2} \frac{(\sin x + \cos x)^2}{\sqrt{\sin^2 x + \cos^2 x + 2\sin x\cos x}}\,dx$$

$$= \int_0^{\pi/2} (\sin x + \cos x)\, dx$$

$$= [-\cos x + \sin x]_0^{\pi/2} = [-0 + 1 - (-1 + 0)] = 2$$

(20) **(B).** $\int_0^{\pi} xf\ (\sin x)\, dx = A \int_0^{\pi/2} f\ (\sin x)\, dx$

Let $I = \int_0^{\pi} xf\ (\sin x)\, dx$ .... (1) $\left\{\because \int_0^a f\ (x)\, dx = \int_0^a f\ (a - x)\, dx\right.$

$$= \int_0^{\pi} (\pi - x)\ f\ \sin\ (\pi - x)\, dx$$

$$I = \int_0^{\pi} (\pi - x)\ f\ \sin\ (x)\, dx \quad .... (2)$$

Adding eq. (1) and eq. (2)

$$2I = \int_0^{\pi} xf\ (\sin x) + (\pi - x)\ f\ (\sin x)\, dx = \int_0^{\pi} \pi f\ (\sin x)\, dx$$

$$\Rightarrow\ I = \frac{\pi}{2} \int_0^{\pi} f\ (\sin x)\, dx$$

$\left\{\because \int_0^{2a} f\ (x)\, dx = 2\int_0^{a} f\ (x)\, dx \right.$ if $f(2a - x) = f(x)$

here $\sin(\pi - x) = \sin x\}$

$$= \frac{\pi}{2}.2 \int_0^{\pi/2} f\ (\sin x)\, dx = \pi \int_0^{\pi/2} f\ (\sin x)\, dx \quad \therefore\ A = \pi$$

(21) **(A).** Given that $f(x) = \frac{e^x}{1 + e^x} \Rightarrow f(a) = \frac{e^a}{1 + e^a}$

and $f(a) + f(-a) = 1 \Rightarrow f(a) = 1 - f(-a)$

Let $f(-a) = t \Rightarrow f(a) = 1 - t$

Now, $I_1 = \int_{f(-a)}^{f(a)} xg\,[x(1 - x)]\, dx$

$\left\{\because \int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx\right. = \int_t^{1-t} x\ g\ (x(1 - x))\, dx$ .... (1)

$$I_1 = \int_t^{1-t} (1 - x)\ g\ ((1 - x)x)\, dx \quad .... (2)$$

Adding eq. (1) and eq. (2)

$$2I_1 = \int_t^{1-t} g\ (x(1 - x))\,[x + 1 - x]\, dx$$

$\left\{\because I_2 = \int_{f(-a)}^{f(a)} g\ (x(1 - x))\, dx = \int_t^{1-t} g\ (x(1 - x))\, dx\ \text{(given)}\right.$

$$2I_1 = \int_t^{1-t} g\ (x(1 - x))\, dx\ ;\ \ 2I_1 = I_2 \Rightarrow \frac{I_2}{I_1} = 2$$

(22) **(D).** $\int \left\{\frac{(\log x - 1)}{1 + (\log x)^2}\right\}^2 dx$

Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t\, dt$

$$\int \left(\frac{t - 1}{1 + t^2}\right)^2 e^t dt = \int \frac{t^2 - 2t + 1}{(1 + t^2)^2} e^t dt$$

$$= \int \frac{(t^2 + 1) - 2t}{(1 + t^2)^2} e^t dt = \int \left(\frac{1}{(1 + t^2)} + \frac{-2t}{(1 + t^2)^2}\right) e^t dt$$

$$\because\ \int (f(t) + f'(t))\, e^t dt = e^t f(t) + C$$

If $f(t) = \frac{1}{t^2 + 1}\quad \therefore f'(t) = \frac{-2t}{(t^2 + 1)^2}$

$$\therefore \int \left(\frac{1}{1 + t^2} + \frac{-2t}{(1 + t^2)}\right) e^t dt = e^t \times \frac{1}{1 + t^2} + C = \frac{x}{1 + (\log x)^2} + C$$

(23) **(D).** $\lim_{n \to \infty} \left[\frac{1}{n^2}\sec^2 \frac{1}{n^2} + \frac{2}{n^2}\sec^2 \frac{4}{n^2} + ... + \frac{1}{n}\sec^2 1\right]$

$\lim_{n \to \infty} \sum_{r=1}^{n} \frac{r}{n^2}\sec^2 \frac{r^2}{n^2}$ Replace $\sum \to \int$; $\frac{r}{n} \to x$

If $r = 1$ and $n \to \infty$, $x = 0$ and $r = n$ then $x = 1$ and $\frac{1}{n} \to dx$

$$\therefore\ \lim_{n \to \infty} \sum_{r=1}^{n} \left(\frac{r}{n}\right) \frac{1}{n} \sec^2 \left(\frac{r}{n}\right)^2 = \int_0^1 x \sec^2 x^2 dx$$

Put $x^2 = t$

$2x\, dx = dt \Rightarrow x\, dx = \frac{dt}{2}$

$$= \frac{1}{2}\int_0^1 \sec^2 t\, dt = \frac{1}{2}[\tan t]_0^1 = \frac{1}{2}[\tan 1 - \tan 0] = \frac{1}{2}\tan 1$$

(24) **(B).** $I_1 = \int_0^1 2^{x^2} dx,\ I_2 = \int_0^1 2^{x^3} dx$

$I_3 = \int_1^2 2^{x^2} dx,\ I_4 = \int_1^2 2^{x^3} dx$

For $0 < x < 1$, we have $x^2 > x^3$
For $1 < x < 2$, we have $x^3 > x^2$

$\therefore\ 2^{x^2} > 2^{x^3}$ for $0 < x < 1$

and $2^{x^2} < 2^{x^3}$ for $1 < x < 2$

$\therefore\ \int_0^1 2^{x^2} dx > \int_0^1 2^{x^3} dx$ and $\int_1^2 2^{x^2} dx < \int_1^2 2^{x^3} dx$

$\therefore I_1 > I_2$ and $I_3 < I_4$

(25) **(D).** $f : R \to R$ and f is differentiable function

$f(2) = 6$ and $f'(2) = \frac{1}{48}\ \therefore\ \lim_{x\to 2} \int_6^{f(x)} \frac{4t^3}{x-2} dt$

$= \lim_{x\to 2} \left[\frac{4t^4}{4(x-2)}\right]_6^{f(x)} = \lim_{x\to 2} \frac{(f(x))^4 - 6^4}{x-2}$

$= \lim_{x\to 2} \frac{(f(x)-6)}{x-2} (f(x))^2 + (6)^2)\ (f(x)+6)$

$= \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} \lim_{x\to 2} [(f(x))^2 + 6^2] \lim_{x\to 2} (f(x)+6)$

$\{\because\ f(2) = 6\}$

$= f'(2) \times [f(2)^2 + 36] \times [f(2)+6]$

$= \frac{1}{48} \times (6^2 + 36) \times (6+6) = \frac{1}{48} \times 72 \times 12 = 18$

(26) **(B).** $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx,\ a > 0$

Let $I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx$ ...... (1)

$I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^{-x}} dx$ ...... (2)

Adding eq. (1) and eq. (2)

$2I = \int_{-\pi}^{\pi} \frac{(1+a^x)\cos^2 x}{1+a^x} dx = \int_{-\pi}^{\pi} \cos^2 x\, dx$

$\left\{\begin{array}{l}\because \cos^2 x \text{ is even function} \\ \therefore \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx\end{array}\right.$

$= 2\int_0^{\pi} \cos^2 x\, dx = 2\int_0^{\pi} \left[\frac{1+\cos 2x}{2}\right] dx$

$= \left[x + \frac{\sin 2x}{2}\right]_0^{\pi} = \pi + \frac{\sin 2\pi}{2} - \left[0 + \frac{\sin 0}{2}\right] = 2I = \pi$

$I = \pi/2$

(27) **(A).** $I = \int_3^6 \frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}} dx$ .... (1)

$I = \int_3^6 \frac{\sqrt{6+3-x}}{\sqrt{9-(6+3-x)}+\sqrt{6+3-x}} dx$

$\left\{\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right.$

$\int_3^6 \frac{\sqrt{9-x}}{\sqrt{x}+\sqrt{9-x}} dx$ .... (2)

$2I = \int_3^6 1\,.\,dx = [x]_3^6 = 6-3;\ 2I = 3 \Rightarrow I = 3/2$

(28) **(B).** $\int_{-3\pi/2}^{-\pi/2} [(x+\pi)^3 + \cos^2(x+3\pi)]\,dx$

$= \int_{-3\pi/2}^{-\pi/2} [(x+\pi)^3 + \cos^2 x]\,dx$

$= \int_{-3\pi/2}^{-\pi/2} \left[(x+\pi)^3 + \frac{1+\cos 2x}{2}\right] dx$

$= \left[\frac{(x+\pi)^4}{4}\right]_{-3\pi/2}^{-\pi/2} + \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)_{-3\pi/2}^{-\pi/2}$

$= \left[\frac{(-\pi/2+\pi)^4}{4} - \frac{(-3\pi/2+\pi)^4}{4}\right]$

$+ \frac{1}{2}\left[-\frac{\pi}{2} + \frac{\sin(-\pi)}{4} - \left(-\frac{3\pi}{2} + \frac{\sin(-6\pi)}{2}\right)\right]$

$= \left[\frac{(\pi/2)^4}{4} - \frac{(-\pi/2)^4}{4}\right] + \frac{1}{2}\left(-\frac{\pi}{2} + \frac{3\pi}{2}\right)$

$= 0 + \frac{1}{2} \times \pi = \frac{\pi}{2}$

(29) (C). $\int_0^{\pi} x\, f(\sin x)\, dx$

Let $I = \int_0^{\pi} x\, f(\sin x)\, dx$ .... (1)

$\left\{\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right.$

$I = \int_0^{\pi} (\pi - x)\, f(\sin(\pi - x))\, dx$

$= \int_0^{\pi} (\pi - x)\, f(\sin x)\, dx$ .... (2)

Adding eq. (1) and eq. (2)

$\left\{\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x) \text{ if } f(2a-x) = f(x)\right.$

$2I = \int_0^{\pi} \pi\, f(\sin x)\, dx = \pi \int_0^{\pi} f(\sin x)\, dx$

$2I = 2\pi \int_0^{\pi/2} f(\sin x)\, dx \Rightarrow I = \pi \int_0^{\pi/2} f(\sin x)\, dx$

$= \pi \int_0^{\pi/2} f(\cos x)\, dx$ $\quad \left\{\because \int_a^a f(x)\,dx = \int_a^0 f(a-x)\,dx\right.$

(30) (A). $\int_1^a [x]\, f'(x)\, dx$,

Let $m \le a < m+1 \Rightarrow [a] = m$ .... (1)

Now, $\int_1^a [x]\, f'(x)\, dx$,

$= \int_1^2 1.\, f'(x)\,dx + \int_2^3 2.\, f'(x)\,dx + \int_3^4 3.\, f'(x)\,dx + \ldots\ldots +$

$\int_{m-1}^{m} (m-1)\, f'(x)\,dx + \int_m^a m\, f'(x)\,dx$

$= [f(x)]_1^2 + 2.[f(x)]_2^3 + 3[f(x)]_3^4 + \ldots + m\,[f(x)]_m^a$

$= [f(2) - f(1)] + 2.\,[f(3) - f(2)] + 3\,[f(4) - f(3)]$

$+ \ldots\ldots + m\,[f(a) - f(m)]$

$= m\, f(a) - [f(1) + f(2) + f(3) + \ldots + f(m)]$

$= [a]\, f(a) - [f(1) + f(2) + f(3) + \ldots + f(a)]$

$\{\because m = [a]$ from (1)

(31) (A). $f(x) = \int_1^x \frac{\log t}{1+t}\, dt$ .... (1)

then $f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\log t}{1+t}\, dt$

Now put $t = 1/z \Rightarrow dt = -1/z^2\, dz$

if $t = 1$ then $z = 1$

and if $t = 1/x$ then $z = x$

$\therefore f(1/x) = \int_1^x \frac{\log 1/z}{1+1/z} \times \left(-\frac{1}{z^2}\right) dz$

$= \int_1^x \frac{-\log z}{z+1} \times \left(-\frac{1}{z}\right) dz = \int_1^x \frac{1}{z}\left(\frac{\log z}{1+z}\right) dz$

Replacing z with t we get

$f(1/x) = \int_1^x \frac{1}{t}\, \frac{\log t}{(1+t)}\, dt$ .... (2)

Adding eq. (1) and (2),

$f(x) + f(1/x) = \int_1^x \frac{\log t}{1+t}\left(t + \frac{1}{t}\right) dt$

$= \int_1^x \frac{\log t}{1+t} \times \frac{1+t}{t}\, dt = \int_1^x \frac{\log t}{t}\, dt$

$= \left|\frac{(\log t)^2}{2}\right|_1^x = \frac{(\log x)^2}{2} - \frac{(\log 1)^2}{2}$

$f(x) + f(1/x) = \frac{(\log x)^2}{2}$ $\quad \because F(x) = f(x) + f(1/x)$

$\therefore F(x) = \frac{(\log x)^2}{2} \Rightarrow F(e) = \frac{(\log e)^2}{2} = \frac{1}{2}$

(32) (A). $\int_{\sqrt{2}}^{x} \frac{dt}{t\sqrt{t^2 - 1}} = \frac{\pi}{12}$

$\Rightarrow [\sec^{-1} t]_{\sqrt{2}}^{x} = \frac{\pi}{12} \Rightarrow \sec^{-1} x - \sec^{-1}\sqrt{2} = \frac{\pi}{12}$

$\Rightarrow \sec^{-1} x - \frac{\pi}{4} = \frac{\pi}{12}$

$\Rightarrow \sec^{-1} x = \frac{\pi}{12} + \frac{\pi}{4} = \frac{4\pi}{12} \Rightarrow \sec^{-1} x = \frac{\pi}{3} \Rightarrow x = 2$

(33) **(A).** $\int \frac{dx}{\cos x + \sqrt{3}\sin x}$

$= \frac{1}{2}\int \frac{dx}{\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x} = \frac{1}{2}\int \frac{dx}{\cos\frac{\pi}{3}\cos x + \sin\frac{\pi}{3}\sin x}$

$= \frac{1}{2}\int \frac{dx}{\cos\left(x - \frac{\pi}{3}\right)} = \frac{1}{2}\int \sec\left(x - \frac{\pi}{3}\right) dx$

$= \frac{1}{2}\log\left|\tan\left(\frac{\pi}{4} + \frac{x - \pi/3}{2}\right)\right| + C = \frac{1}{2}\log\left|\tan\left(\frac{x}{2} + \frac{\pi}{12}\right)\right| + C$

(34) **(B).** $\sqrt{2}\int \frac{\sin x\, dx}{\sin\left(x - \frac{\pi}{4}\right)} = \frac{1}{2}\int \frac{\sin\left(x - \frac{\pi}{4} + \frac{\pi}{4}\right)}{\sin\left(x - \frac{\pi}{4}\right)} dx$

$= \sqrt{2}\int \frac{\sin\left(x - \frac{\pi}{4}\right)\cos\frac{\pi}{4} + \cos\left(x - \frac{\pi}{4}\right)\sin\frac{\pi}{4}}{\sin\left(x - \frac{\pi}{4}\right)} dx$

$= \sqrt{2}\int \left[\cos\frac{\pi}{4} + \sin\frac{\pi}{4} + \cot\left(x - \frac{\pi}{4}\right)\right]$

$= \sqrt{2}\left[x\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\log\left|\sin\left(x - \frac{\pi}{4}\right)\right|\right] + C$

$= x + \log\left|\sin\left(x - \frac{\pi}{4}\right)\right| + C$

(35) **(A).** $I = \int_0^1 \frac{\sin x}{\sqrt{x}} dx \quad \because \sin x < x \text{ for } 0 < x < 1$

$\frac{\sin x}{\sqrt{x}} < \frac{x}{\sqrt{x}} \Rightarrow \int_0^1 \frac{\sin x}{\sqrt{x}} dx < \int_0^1 \sqrt{x}$

$I < \left(\frac{x^{2/3}}{3/2}\right)_0^1 \; ; \; I < \frac{2}{3}[1 - 0]$

$I < 2/3 \text{ and } J = \int_0^1 \frac{\cos x}{\sqrt{x}} dx \quad \because \cos x < 1 \text{ for } 0 < x < 1$

$\Rightarrow \frac{\cos x}{\sqrt{x}} < \frac{1}{\sqrt{x}} \Rightarrow \int_0^1 \frac{\cos x}{\sqrt{x}} dx < \int_0^0 \frac{1}{\sqrt{x}} dx$

$J < \left[2\sqrt{x}\right]_0^1 \; ; \; J < 2$

(36) **(D).** $I = \int_0^\pi [\cot x]\, dx$

$I = \int_0^\pi [-\cot x]\, dx \Rightarrow 2I = \int_0^\pi ([\cot x] + [-\cot x])\, dx = -\pi$

$\Rightarrow I = -\frac{\pi}{2}$

(37) **(A).**
$p'(x) = p'(1-x) \Rightarrow p(x) = -p(1-x) + c$
at $x = 0$
$p(0) = -p(1) + c \Rightarrow 42 = c$
now $p(x) = -p(1-x) + 42$
$\Rightarrow p(x) + p(1-x) = 42$

$I = \int_0^1 p(x)\, dx = \int_0^1 p(1-x)\, dx \; ; \; 2I = \int_0^1 (42)\, dx \Rightarrow I = 21$

(38) **(A).** $x = \tan\theta$
$dx = \sec^2\theta\, d\theta$

$I = \int_0^{\pi/4} \frac{8\ln(1+\tan\theta)}{\sec^2\theta}\sec^2\theta d\theta = 8\int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta$

$\Rightarrow I = 8\int_0^{\pi/4} \ln(1 + \tan(\pi/4 - \theta))\, d\theta = 8\int_0^{\pi/4} \ln\left(\frac{2}{1+\tan\theta}\right) d\theta$

$2I = 8\int_0^{\pi/4} \ln(2)\, d\theta \Rightarrow 2I = 8\frac{\pi}{4}\ln 2 = 2\pi\ln 2$

$\Rightarrow I = \pi\ln 2$

(39) **(A).** $\frac{dy}{dx} = y + 3 \; ; \; \frac{dy}{y+3} = dx$

$\ln(y+3) = x + c$; Given at $x = 0, y = 2$
$\ln 5 = c \;\therefore\; \ln(y+3) = x + \ln 5$

$\ln\left(\frac{y+3}{5}\right) = x \; ; \; y + 3 = 5e^x$

$y = 5e^x - 3 \;\therefore\; y(\ln 2) = 5e^{\ln 2} - 3 = 7$

(40) **(D).** $\int \frac{5\tan x}{\tan x - 2} dx = \int \frac{5\tan x}{\sin x - 2\cos x} dx$

$= \int \frac{(\sin x - 2\cos x) + 2(\cos x + 2\sin x)}{(\sin x - 2\cos x)} dx$

$= \int dx + \int \frac{\cos x + 2\sin x}{\sin x - 2\cos x} dx = x + 2\ln|(\sin x - 2\cos x)| + k$

$\Rightarrow a = 2$

(41) **(B or C).**

$g(x+\pi) = \int_0^{x+\pi} \cos 4t\, dt = g(x) + \int_0^\pi \cos 4t\, dt = g(x) + g(\pi)$

Here, $g(\pi) = \int_0^{\pi} \cos 4t \, dt = 0$

(42) **(C).** $\int f(x)\,dx = \psi(x)$; $I = \int x^5 f(x^3)\,dx$

Put $x^3 = t \Rightarrow x^2\,dx = dt/3$

$= \frac{1}{3}\int t\,f(t)\,dt = \frac{1}{3}\left[t\,\psi(t) - \int \psi(t)\,dt\right]$

$= \frac{1}{3}\left[x^3\,\psi(x^3) - 3\int x^2\,\psi(x^3)\,dx\right] + C$

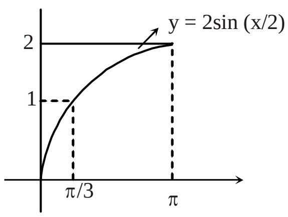
$= \frac{1}{3}x^3\,\psi(x^3) - \int x^2\,\psi(x^3)\,dx + C$

(43) **(D).** $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\,dx}{1+\sqrt{\tan x}}$

$\Rightarrow 2I = \int_{\pi/6}^{\pi/3} dx \Rightarrow I = \frac{1}{2}\left[\frac{\pi}{3} - \frac{\pi}{6}\right] = \frac{\pi}{12}$, Statement-1 is false.

$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ it is property.

(44) **(D).**



$I = \int_0^{\pi} \sqrt{1 + 4\sin^2\frac{x}{2} - 4\sin\frac{x}{2}}\,dx = \int_0^{\pi} \left|1 - 2\sin\frac{x}{2}\right| dx$

$= \int_0^{\pi/3} \left(1 - 2\sin\frac{x}{2}\right) dx + \int_{\pi/3}^{\pi} \left(2\sin\frac{x}{2} - 1\right) dx$

$= \left[x + 4\cos\frac{x}{2}\right]_0^{\pi/3} + \left[-4\cos\frac{x}{2} - x\right]_{\pi/3}^{\pi} = 4\sqrt{3} - 4 - \frac{\pi}{3}$

(45) **(B).** $\int \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}}\,dx$

$= \int e^{\left(x+\frac{1}{x}\right)} dx + \int x\left(1 - \frac{1}{x^2}\right) e^{\left(x+\frac{1}{x}\right)} dx$

$= \int e^{\left(x+\frac{1}{x}\right)} dx + xe^{\left(x+\frac{1}{x}\right)} - \int e^{\left(x+\frac{1}{x}\right)} dx = xe^{\left(x+\frac{1}{x}\right)} + c$

(46) **(C).** $\int \frac{dx}{x^2(x^4+1)^{3/4}} = \int \frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}$

Let $1 + \frac{1}{x^4} = t \Rightarrow \frac{-4}{x^5}dx = dt$

So, $I = \frac{-1}{4}\int \frac{dt}{t^{3/4}} = \frac{-1}{4}\int t^{-3/4}dt$

$= \frac{-1}{4}\left(\frac{t^{3/4}}{1/4}\right) + c = -\left(1 + \frac{1}{x^4}\right)^{1/4} + c$

(47) **(B).** $I = \int_2^4 \frac{\log x^2}{\log x^2 + \log(36 - 12x + x^2)} dx$

$I = \int_2^4 \frac{\log(6-x)^2\,dx}{\log x^2 + \log(6-x)^2}$

$2I = \int_2^4 1\,dx$; $2I = 2$; $I = 1$

(48) **(A).** $\int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3} dx$

$= \int \frac{2x^{12} + 5x^9}{x^{15}(1 + x^{-2} + x^{-5})^3} dx = \int \frac{2x^{-3} + 5x^{-6}}{(x^{-5} + x^{-2} + 1)^3} dx$

$x^{-5} + x^{-2} + 1 = t$

$(+5x^{-6} + 2x^{-3})\,dx = -\,dt$

$-\int \frac{dt}{t^3} = -\left(\frac{t^{-2}}{-2}\right) + C = \frac{1}{2t^2} + C = \frac{x^{10}}{2(x^5 + x^3 + 1)^2} + C$

(49) **(A).** $\ell = \left(\frac{n+1}{n} \cdot \frac{n+2}{n} \cdot \frac{n+3}{n} \ldots\ldots \frac{n+2n}{n}\right)^{1/n}$

$\log \ell = \frac{1}{n}\left[\log\left(\frac{n+1}{n}\right) + \log\left(\frac{n+2}{n}\right) + \ldots\ldots + \log\left(\frac{n+2n}{n}\right)\right]$

$\log \ell = \int_0^2 \log(1+x)\,dx \quad [1 + x = t]$

$\log \ell = \int_1^3 \log t\,dt$; $\log \ell = t \log t - \int \frac{1}{t} \cdot t\,dt$

$\log \ell = t(\log t - 1)$

$\log \ell = 3(\log 3 - 1) - 1(\log 1 - 1) = 3\log 3 - 2$

$\log \ell = \log 27 - \log e^2 = \log\frac{27}{e^2}$; $\ell = \frac{27}{e^2}$

(50) **(D).** $I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$

$I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1-\cos x}$ Using [a + b – x] property

$2I = \int_{\pi/4}^{3\pi/4} \frac{2}{1-\cos^2 x} dx$

$I = \int_{\pi/4}^{3\pi/4} \operatorname{cosec}^2 x\, dx = \cot x\Big|_{3\pi/4}^{\pi/4} = 1-(-1) = 2$

(51) **(D).** $\int \tan^4 x\, dx + \int \tan^6 x\, dx = a \tan^5 x + bx^5 + c$

Differentiating both sides

$\tan^4 x + \tan^6 x = 5a \tan^4 x \sec^2 x + 5bx^4$

$= 5a \tan^4 x (\tan^2 x + 1) + 5bx^4$

$= 5a \tan^4 x + 5a \tan^6 x + 5bx^4 \Rightarrow a = 1/5, b = 0$

Alt. $I_4 + I_6 = \int (\tan^4 x + \tan^6 x)\, dx$

$= \int \tan^4 x \sec^2 x\, dx = \frac{1}{5}\tan^5 x + c$

(52) **(B).** Given $\int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx$

$f(x) + f(-x) = \frac{\sin^2 x}{1+2^x} + \frac{2^x(\sin^2 x)}{1+2^x} = \sin^2 x$

$\int_0^{\pi/2} \sin^2 x\, dx = \frac{\pi}{4}$

(53) **(D).** $\int \frac{\sin^2 \cos^2 x}{(\sin^5 x + \cos^3 x \sin^2 x + \sin^3 x \cos^2 x + \cos^5 x)^2} dx$

$\int \frac{\tan^2 x \sec^6 x}{(\tan^5 x + \tan^2 x + \tan^3 x + 1)^2} dx$

Put $\tan x = t \Rightarrow \sec^2 x = \frac{dt}{dx}$

$\int \frac{t^2(1+t^2)^2}{(t^3+1)^2(t^2+1)^2} dt$

$t^3 + 1 = y$

$3t^2 = \frac{dy}{dt}$

$\frac{1}{3}\int \frac{dy}{y^2} = -\frac{1}{3(y)} + C = -\frac{1}{3(\tan^3 x + 1)} + C$

(54) **(A).** Put $(x^2 - 1) = 1 \Rightarrow 2x\, dx = dt$

$\therefore\ I = \frac{1}{2}\int \sqrt{\frac{1-\cos t}{1+\cos t}}\, dt = \frac{1}{2}\int \tan\left(\frac{t}{2}\right) dt = \ln\left|\sec\frac{t}{2}\right| + c$

$I = \ln\left|\sec\left(\frac{x^2-1}{2}\right)\right| + c$

(55) **(D).** $\int_0^{\pi} |\cos x|^3\, dx = \int_0^{\pi/2} \cos^3 x\, dx - \int_{\pi/2}^{\pi} \cos^3 x\, dx$

$= \int_0^{\pi/2}\left(\frac{\cos 3x + 3\cos x}{4}\right) dx - \int_{\pi/2}^{\pi}\left(\frac{\cos 3x + 3\cos x}{4}\right) dx$

$= \frac{1}{4}\left[\left(\frac{\sin 3x}{3} + 3\sin x\right)_0^{\pi/2} - \left(\frac{\sin 3x}{3} + 3\sin x\right)_{\pi/2}^{\pi}\right]$

$= \frac{1}{4}\left[\left(\frac{-1}{3}+3\right) - (0+0) - \left\{(0+0) - \left(\frac{-1}{3}+3\right)\right\}\right] = \frac{4}{3}$

(56) **(D).** $g(f(x)) = \ln(f(x)) = \ln\left(\frac{2 - x\cos x}{2 + x\cos x}\right)$

$I = \int_{-\pi/4}^{\pi/4} \ln\left(\frac{2 - x\cos x}{2 + x\cos x}\right) dx$

$= \int_{-\pi/4}^{\pi/4}\left(\ln\left(\frac{2 - x\cos x}{2 + x\cos x}\right) + \ln\left(\frac{2 + x\cos x}{2 - x\cos x}\right)\right) dx$

$= \int_0^{\pi/2} (0)\, dx = 0 = \log_e(1)$

(57) **(C).**

$\int \frac{\sin\frac{5x}{2}}{\sin\frac{x}{2}} dx = \int \frac{2\sin\frac{5x}{2}\cos\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx = \int \frac{\sin 3x + \sin 2x}{\sin x} dx$

$= \int \frac{3\sin x - 4\sin^3 x - 2\sin x \cos x}{\sin x} dx$

$= \int (3 - 4\sin^2 x + 2\cos x)\, dx$

$= \int (3 - 2(1 - \cos 2x) + 2\cos x)\, dx$

$= \int (1 + 2\cos 2x + 2\cos x)\, dx = x + \sin 2x + 2\sin x + c$

(58) **(A).** $f(x)=\int_0^x g(t)\,dt$; $f(-x)=\int_0^{-x} g(t)\,dt$

put $t=-u$

$$=-\int_0^x g(-u)\,du=-\int_0^x g(u)\,d(u)=-f(x)$$

$f(-x)=-f(x)$

$\Rightarrow$ $f(x)$ is an odd function

Also $f(5+x)=g(x)$

$f(5-x)=g(-x)=g(x)=f(5+x)$

$\Rightarrow$ $f(5-x)=f(5+x)$

Now, $I=\int_0^x f(t)\,dt$; $t=u+5$

$$I=\int_{-5}^{x-5} f(u+5)\,du=\int_{-5}^{x-5} g(u)\,du=\int_{-5}^{x-5} f'(u)\,du$$

$$=f(x-5)-f(-5)=-f(5-x)+f(5)=f(5)-f(5+x)$$

$$=\int_{5+x}^{5} f'(t)\,dt=\int_{5+x}^{5} g(t)\,dt$$

(59) **(D).** $\int \frac{dx}{x^3(1+x^6)^{2/3}}=xf(x)(1+x^6)^{1/3}+C$

$$\int \frac{dx}{x^7\left(\frac{1}{x^6}+1\right)^{2/3}}=xf(x)(1+x^6)^{1/3}+C$$

Let $t=\frac{1}{x^6}+1$; $dt=\frac{-6}{x^7}dx$

$$-\frac{1}{6}\int\frac{dt}{t^{2/3}}=-\frac{1}{2}t^{1/3}$$

$$=-\frac{1}{2}\left(\frac{1}{x^6}+1\right)^{1/3}=-\frac{1}{2}\frac{(1+x^6)^{1/3}}{x^2}\quad\therefore\ f(x)=-\frac{1}{2x^3}$$

(60) **(C).** $I=\int_0^{\pi/2}\frac{\sin^3 x}{\sin x+\cos x}dx$

$$I=\int_0^{\pi/4}\frac{\sin^3 x+\cos^3 x}{\sin x+\cos x}dx=\int_0^{\pi/4}(1-\sin x\cos x)\,dx$$

$$=\left(x-\frac{\sin^2 x}{2}\right)_0^{\pi/4}=\frac{\pi}{4}-\frac{1}{4}=\frac{\pi-1}{4}$$

(61) **(D).** $I=\int\frac{dx}{(\sin x)^{4/3}\cdot(\cos x)^{2/3}}$

$$\Rightarrow I=\int\frac{dx}{\left(\frac{\sin x}{\cos x}\right)^{4/3}\cdot\cos^2 x}\Rightarrow I=\int\frac{\sec^2 x}{(\tan x)^{4/3}}dx$$

Put $\tan x=t\Rightarrow \sec^2 x\,dx=dt$

$$\therefore\ I=\int\frac{dt}{t^{4/3}}\Rightarrow I=\frac{-3}{t^{1/3}}+C\Rightarrow I=\frac{-3}{(\tan x)^{1/3}}+C$$

(62) **(A).** $I=\int_0^1 x\tan\left(\frac{1}{1+x^2(x^2-1)}\right)dx$

$$I=\int_0^1 x\tan(\tan^{-1}x^2-\tan^{-1}(x^2-1))\,dx$$

$x^2=t\Rightarrow 2x\,dx=dt$

$$I=\frac{1}{2}\int_0^1(\tan^{-1}t-\tan^{-1}(t-1))\,dx$$

$$=\frac{1}{2}\int_0^1\tan^{-1}t\,dt-\frac{1}{2}\int_0^1\tan^{-1}(t-1)\,dt$$

$$=\frac{1}{2}\int_0^1\tan^{-1}t\,dt-\frac{1}{2}\int_0^1\tan^{-1}dt=\int_0^1\tan^{-1}dt$$

$\tan^{-1}t=\theta\Rightarrow t=\tan\theta$; $dt=\sec^2\theta\,d\theta$

$$\int_0^{\pi/4}\theta\cdot\sec^2\theta\,d\theta$$

$$1=(\theta\cdot\tan\theta)|_0^{\pi/4}-\int_0^{\pi/4}\tan\theta\,d\theta=\left(\frac{\pi}{4}-0\right)-\ln(\sec\theta)\Big|_0^{\pi/4}$$

$$=\frac{\pi}{4}-(\ln\sqrt{2}-0)=\frac{\pi}{4}-\frac{1}{2}\ln 2$$

(63) **(D).** $\int e^{\sec x}(\sec x\tan x\,f(x)+(\sec x\tan x+\sec^2 x)dx$

$$=e^{\sec x}f(x)+C$$

Diff. both sides w.r.t. 'x'

$e^{\sec x}(\sec x\tan x\,f(x)+(\sec x\tan x+\sec^2 x))$

$=e^{\sec x}.\sec x\tan x\,f(x)+e^{\sec x}f'(x)$

$f'(x)=\sec^2 x+\tan x\sec x$

$\Rightarrow$ $f(x)=\tan x+\sec x+c$

(64) **(C).** $I=\int_0^{2\pi}[\sin 2x\,(1+\cos 3x)]\,dx$

$$=\int_0^{2\pi}([\sin 2x+\sin 2x\cos 3x]\,dx$$

$$+[-\sin 2x-\sin 2x\cos 3x])\,dx=\int_0^{\pi}-dx=-\pi$$

(65) (D). $\int \frac{dx}{((x-1)^2+9)^2} = \frac{1}{27}\int \cos^2\theta\, d\theta$

Put $x-1 = 3\tan\theta$

$= \frac{1}{54}\int (1+\cos 2\theta)\, d\theta = \frac{1}{54}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$

$= \frac{1}{54}\left(\tan^{-1}\left(\frac{x-1}{3}\right) + \frac{3(x-1)}{x^2-2x+10}\right) + C$

(66) (A). Let $x^2 = t$

$2x\, dx = dt$

$\frac{1}{2}\int t^2 \cdot e^{-t} dt = \frac{1}{2}\left[-t^2 e^{-t} + \int 2t \cdot e^{-t} dt\right]$

$= \frac{1}{2}(-t^2 \cdot e^{-t}) + (-t \cdot e^{-t} + \int 1 \cdot e^{-t} dt)$

$= -\frac{t^2 \cdot e^{-t}}{2} - te^{-t} - e^{-t} = \left(-\frac{t^2}{2} - t - 1\right)e^{-t}$

$= \left(-\frac{x^4}{2} - x^2 - 1\right)e^{-x^2} + C$

$g(x) = -1 - x^2 - \frac{x^4}{2} + ke^{x^2}$

For $k = 0$, $g(-1) = -1 - 1 - \frac{1}{2} = -\frac{5}{2}$

(67) (A). $I = \int \frac{1}{\cos^{2/3} x \sin^{1/3} x \sin x} dx = \int \frac{\tan^{2/3} x}{\tan^2 x} \cdot \sec^2 x\, dx$

$= \int \frac{\sec^2 x}{\tan^{4/3} x} dx$ $\{\tan x = t, \sec^2 x\, dx = dt\}$

$= \int \frac{dt}{t^{4/3}} = \frac{t^{-1/3}}{-1/3} = -3(t^{-1/3})$

$I = -3\tan(x)^{-1/3}$

$I = \left.\frac{3}{(\tan x)^{1/3}}\right|_{\pi/6}^{\pi/3} = -3\left(\frac{1}{(\sqrt{3})^{1/3}} - (\sqrt{3})^{1/3}\right)$

$= 3\left(3^{1/3} - \frac{1}{3^{1/6}}\right) = 3^{7/6} - 3^{5/6}$

(68) (A). $\int_0^{\pi/2} \frac{\cot x}{\cot x + \operatorname{cosec} x} dx$

$\int_0^{\pi/2} \frac{\cot x}{\cot x + 1} dx = \int_0^{\pi/2} \frac{2\cos^2 \frac{x}{2} - 1}{2\cos^2 \frac{x}{2}} dx$

$\int_0^{\pi/2} \left(1 - \frac{1}{2}\sec^2 \frac{x}{2}\right) dx = \left[x - \tan\frac{x}{2}\right]_0^{\pi/2}$

$= \frac{1}{2}[\pi - 2]$; $m = 1/2, n = -2$; $mn = -1$

(69) (A). $\int \frac{2x^3 - 1}{x^4 + x} dx$

$\int \frac{2x - \frac{1}{x^2}}{x^2 + \frac{1}{x}} dx$; $x^2 + \frac{1}{x} = t$; $\left(2x - \frac{1}{x^2}\right) dx = dt$

$\int \frac{dt}{t} = \ln(t) + C = \ln\left(x^2 + \frac{1}{x}\right) + C$

(70) (D). $\int_\alpha^{\alpha+1} \frac{(x+\alpha+1)-(x+\alpha)}{(x+\alpha)(x+\alpha+1)} dx$

$= (\ln|x+\alpha| - \ln|x+\alpha+1|)|_\alpha^{\alpha+1}$

$= \ln\left|\frac{2\alpha+1}{2\alpha+2} \times \frac{2\alpha+1}{2\alpha}\right| = \ln\frac{9}{8} \Rightarrow \alpha = -2, 1$

(71) (C). $\int \frac{\tan x + \tan\alpha}{\tan x - \tan\alpha} dx = \int \frac{\sin(x+\alpha)}{\sin(x-\alpha)} dx$

Let $x - \alpha = t$

$\int \frac{\sin(t+2\alpha)}{\sin t} dt = \int \cos 2\alpha\, dt + \int \cot(t) \sin 2\alpha\, dt$

$= t\cos 2\alpha + \ln|\sin t| . \sin 2\alpha + C$

$= (x-\alpha)\cos 2\alpha + \ln|\sin(x-\alpha)|. \sin 2\alpha + C$

(72) (D). $I = \frac{1}{(a+b)}\int_a^b x[f(x) + f(x+1)]\, dx$ ......(1)

$x \to a + b - x$

$$I=\frac{1}{(a+b)}\int_a^b (a+b-x)[f(a+b-x)+f(a+b+1-x)]\,dx$$

$$I=\frac{1}{(a+b)}\int_a^b (a+b-x)[f(x+1)+f(x)]\,dx$$

.....(2)

[Put $x \to x+1$ in given equation]

Eq. (1) + Eq. (2)

$$2I=\int_a^b [f(x+1)+f(x)]\,dx\;;\; 2I=\int_a^b f(x+1)\,dx+\int_a^b f(x)\,dx$$

$$\int_a^b f(a+b+1-x)\,dx+\int_a^b f(x)\,dx\;;\; 2I=2\int_a^b f(x)\,dx$$

(73) **(A).** $2\cot^2\theta-\frac{5}{\sin\theta}+4=0$; $\frac{2\cos^2\theta}{\sin^2\theta}-\frac{5}{\sin\theta}+4=0$

$2\cos^2\theta - 5\sin\theta + 4\sin^2\theta = 0$, $\sin\theta \neq 0$

$2\sin^2\theta - 5\sin\theta + 2 = 0$

$(2\sin\theta - 1)(\sin\theta - 2) = 0$

$\sin\theta = 1/2$; $\theta=\frac{\pi}{6},\frac{5\pi}{6}$

$$\int_{\pi/6}^{5\pi/6}\cos^2 3\theta\,d\theta=\int_{\pi/6}^{5\pi/6}\frac{1+\cos 6\theta}{2}\,d\theta$$

$$=\frac{1}{2}\left[\theta+\frac{\sin 6\theta}{6}\right]_{\pi/6}^{5\pi/6}=\frac{1}{2}\left[\frac{5\pi}{6}-\frac{\pi}{6}+\frac{1}{6}(0-0)\right]$$

$$=\frac{1}{2}\times\frac{4\pi}{6}=\frac{\pi}{3}$$

(74) **(A).** $4\alpha\left\{\int_{-1}^{0} e^{\alpha x}dx+\int_0^2 e^{-\alpha x}dx\right\}=5$

$$4\alpha\left\{\left(\frac{e^{\alpha x}}{\alpha}\right)_{-1}^{0}+\left.\frac{e^{-\alpha x}}{-\alpha}\right|_0^2\right\}=5$$

$$4\alpha\left\{\left(\frac{1-e^{-\alpha}}{\alpha}\right)-\left(\frac{e^{-2\alpha}-1}{\alpha}\right)\right\}=5$$

$\Rightarrow$ $4(2-e^{-\alpha}-e^{-2\alpha})=5$ Put $e^{-\alpha}=t$

$\Rightarrow$ $4t^2+4t-3=0 \Rightarrow (2t+3)(2t-1)=0$

$\Rightarrow$ $e^{-\alpha}=1/2 \Rightarrow \alpha=\ln 2$

(75) **(B).** $\sin x = t$; $\cos x\,dx = dt$

$$I=\int\frac{dt}{t^3(1+t^6)^{2/3}}=\int\frac{dt}{t^7\left(1+\frac{1}{t^6}\right)^{2/3}}$$

Put $1+\frac{1}{t^6}=r^3 \Rightarrow \frac{dt}{t^7}=\frac{-1}{2}r^2dr$

$$-\frac{1}{2}\int\frac{r^2dr}{r^2}=-\frac{1}{2}r+c=-\frac{1}{2}\left(\frac{\sin^6 x+1}{\sin^6 x}\right)^{1/3}+c$$

$$=-\frac{1}{2\sin^2 x}(1+\sin^6 x)^{1/3}+c$$

$f(x)=-\frac{1}{2}\operatorname{cosec}^2 x$ and $\lambda=3$; $\lambda f\left(\frac{\pi}{3}\right)=-2$

(76) **(A).** $f(x)=\frac{1}{\sqrt{2x^3-9x^2+12x+4}}$

$$f'(x)=\frac{-1}{2}\frac{6x^2-18x+12}{(2x^3-9x^2+12x+4)^{3/2}}$$

$$=\frac{-6(x-1)(x-2)}{2(2x^3-9x^2+12x+4)^{3/2}}$$

$f(1)=\frac{1}{3}$, $f(2)=\frac{1}{\sqrt{8}}$; $\frac{1}{3}<1<\frac{1}{\sqrt{8}}$

(77) **(D).** $I=\int_0^{2\pi}\frac{x\sin^8 x}{\sin^8 x+\cos^8 x}dx$ .... (1)

$$=\left[\int_0^{\pi}\frac{x\sin^8 x}{\sin^8 x\cos^8 x}dx+\int_0^{\pi}\frac{(2\pi-x)\sin^8 x}{\sin^8 x+\cos^8 x}dx\right]$$

$$=2\pi\int_0^{\pi}\frac{\sin^8 x}{\sin^8 x+\cos^8 x}dx$$

$$I=2\pi\left[\int_0^{\pi/2}\frac{\sin^8 x}{\sin^8 x+\cos^8 x}dx+\int_0^{\pi/2}\frac{\cos^8 x}{\sin^8 x+\cos^8 x}dx\right]$$

$$=2\pi\int_0^{\pi/2}1\,dx=2\pi\frac{\pi}{2}=\pi^2$$

(78) **(A).** $I=\int \frac{dx}{(x+4)^{8/7}(x-3)^{6/7}}=\int \frac{dx}{\left(\frac{x+4}{x-3}\right)^{8/7}(x-3)^2}$

Let $\frac{x+4}{x-3}=t \Rightarrow \frac{dx}{(x-3)^2}=-\frac{1}{7}dt$

$I=-\frac{1}{7}\int \frac{dt}{t^{8/7}}=-\frac{1}{7}\int t^{-8/7}dt$

$=t^{-1/7}+C=+\left(\frac{x+4}{x-3}\right)^{-1/7}+C=\left(\frac{x-3}{x+4}\right)^{1/7}+C$

(79) **(C).** $f(x)=a+bx+cx^2$

$\int_0^1 f(x)\,dx=\left[ax+\frac{bx^2}{2}+\frac{cx^3}{3}\right]_0^1$

$=a+\frac{b}{2}+\frac{c}{3}=\frac{1}{6}[6a+3b+c]=\frac{1}{6}\left[f(0)+f(1)+4f\left(\frac{1}{2}\right)\right]$

(80) **(A).** $I=\int \frac{d\theta}{\cos^2\theta(\tan 2\theta+\sec 2\theta)}$

$=\int \frac{\sec^2\theta\, d\theta}{\frac{2\tan\theta}{1-\tan^2\theta}+\frac{1+\tan^2\theta}{1-\tan^2\theta}}=\int \frac{(1-\tan^2\theta)\sec^2\theta\, d\theta}{(1+\tan\theta)^2}$

$\tan\theta=t \Rightarrow \sec^2\theta\, d\theta=dt$

$I=\int \frac{1-t^2}{(1+t)^2}dt=\int \frac{(1-t)(1+t)}{(1+t)^2}dt=\int \frac{1}{1+t}-\frac{t}{1+t}dt$

$=\ln|1+t|-\int\left(\frac{1+t}{1+t}-\frac{1}{1+t}\right)dt$

$=\ln|1+t|-t+\ln|1+t|=2\ln|1+t|-t+C$

$=2\ln|1+\tan\theta|-\tan\theta+C$

$\lambda=-1,\ f(\theta)=1+\tan\theta$