Chapter- 1

Sets

Introduction to Sets and their Representation

Introduction

The concept of set is the foundation of modern Mathematics and it is fundamental in all branches of Mathematics. The modern theory of sets was developed by German Mathematician ***George Cantor*** (1845 – 1918). This concept is used in every branch of Mathematics like Geometry, Algebra, etc. Further sets are used to define relations and functions.

Sets and their representations:

In our daily life, while performing regular work, we come across a variety of things that occur in groups. $e.g.$ a team of cricket players, a group of tallboys, a pack of cards, etc. The words used above like team, group, pack, etc., convey the idea of certain collections.

Let us examine the following collections:

a) Even natural numbers less than 9, $i.e.$ 2, 4, 6, 8

b) The vowels in English alphabets, $i.e.$ a, e, i, o, u

c) The solutions of the equation $x^2 - 3x + 2 = 0$ $i.e.$ 1 and 2.

Each of the above examples is a well–defined collection of objects. By **well–defined collections,** we mean that there is a rule, with the help of which it is possible to say, whether an object belongs to the given collection or not.

Further, well-defined examine the following collections:

(a) The collection of five good doctors in our city.

(b) The collection of five good novels of Premchand.

All the above are collections but here, how do we know five good doctors of our city? Who will define what is good here? A doctor, which is good to one patient may not be the same to another patient. This collection is **not well-defined.** Similarly, a novel that is good to one person may not be liked by others. Hence, again, the collection is not well-defined.

Definition:

A **well–defined** collection of objects, is called a **set**.

Sets are usually denoted by the capital letters A, B, C, ... etc. and all its members are represented by small letters a, b, x, y, ... etc.

The statement "x is an element of a set S" is written as $x \in S$ (read as x in S or x belongs to S).

$[\in -epsilon]$

The statement " is not an element of S" is written as $x \notin S$ and is read as "x does not belong to S".

Sets are enclosed by curly brackets.

Let S be the set of vowels. So $S = \{a, e, i, o, u\}$

Example: Which of the following are sets? Justify your answer.

- The collection of all the days in a week beginning with the letter S.

Sol: $A = \{Saturday, Sunday\}$. It is a set.

- The collection of famous dancers of India.

Sol: It is not a set. What are the criteria for being famous here?

Representation of Sets

Sets are generally represented in two ways.

Roster Form or Tabular Form or Listing Method:

In this form, all the elements of a set are listed, the elements are being separated by commas and are enclosed within braces.

Example: The set of natural numbers less than 10 are represented in roster form as

$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Set – builder Form or Rule Method :

In this form, all the elements of the set possess a single common property $p(x)$, which is not possessed by any other element outside the set.

In such a case, a set is described by $= \{x : p(x)\ holds\}$.

Example: The collection of all peoples of India can be written as

$S = \{x : x \text{ is an Indian}\}$

Some Important Number Sets:

$N =$ Set of natural numbers

$W =$ Set of whole numbers

Z or $I =$ set of integers

$Q =$ Set of rational numbers

$R =$ Set of real numbers

$C =$ Set of complex numbers

Example: Describe each of the following sets in roster form:

(i) $S = \{x : x$ is a positive integer and divisor of 21$\}$.

Sol: $S = \{1, 3, 7, 21\}$

(ii) $S = \{x : x = n^2, 1 < n \leq 5, n \in N\}$.

Sol: $S = \{4, 9, 16, 25\}$

(iii) $S = \{x : x$ is a positive integer and $x^2 < 40\}$

Sol: $S = \{1, 2, 3, 4, 5, 6\}$

(iv) $S = \{x : x \in Z$ and $|x| \leq 2\}$

Sol: $S = \{-2, -1, 0, 1, 2\}$

(v) $S = \{x : x$ is a two-digit number such that the sum of its digits is 9$\}$

Sol: $S = \{18, 27, 36, 45, 54, 63, 72, 81, 90\}$

Example: Describe the following sets in set-builder form:

(i) The set of reciprocals of all natural numbers.

Sol: $A = \{x : x = \frac{1}{n}, n \in N\}$

(ii) $B = \{1, 5, 25, 125, 625\}$
Sol: $B = \{x : x = 5^n, 0 \leq n \leq 4, n \in Z\}$

(iii) $C = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}\right\}$

Sol: $C = \{x : x = \frac{n}{n+1},\ 1 \le n \le 6\ and\ n \in N\}$

$(iv)\ D = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,7\}$

Sol: $D = \{x : x \in N\ and\ x \le 7\}$

$(v)\ E = \{\ 0\ \}$

Sol: $E = \{\ x : x = 0\}$

$(vi)\ F = \{\ 1,\ 2\}$

Sol: $F = \{x : x = 1\ or\ x = 2\} = \{x : x^2 - 3x + 2 = 0\}$

Remember:

- The repetition of any element in a set does not occur in any other element.

  For example, $S = \{1,\ 1,\ 1,\ 2,\ 2,\ 3\}$ is the same as $S = \{1,\ 2,\ 3\}$.

- The order in which the elements of a set are written is immaterial.

  For example $S = \{1,\ 2,\ 3\} = \{2,\ 3,\ 1\} = \{1,\ 3,\ 2\} = \{3,\ 1,\ 2\}$

Example:

(i) Describe the set of all letters of the word "COMMITTEE" in roster form.

Sol: $S = \{\ c,\ e,\ i,\ m,\ o, t\}$

(ii) List all the elements of the set $S = \{\ x : x\ is\ a\ letter\ of\ the\ word\ MISSISSIPPI\}$

Sol: $S = \{\ M,\ I,\ S,\ P\}$

Types of Sets:

Singleton Set:

A set consisting of a single element is called a singleton set.

Example: The sets $\{5\}$, $\{x\}$ are singleton sets.

Empty Set:

A set that does not contain any element, is called an empty set or null set, or void set.

It is denoted by $\emptyset$ (phi).

Consider the following examples:

i. (i) Let $A = \{x : x^2 - 5 = 0$ and $x \in N\}$.

ii. (ii) Let $B = \{x : x$ is a point common to any two parallel lines$\}$.

iii. (iii) Let $S =$ collection of months having 32 days.

The set builder form of the empty set is $\{ x : \ x \neq x\}$

Finite Set: A set that is empty or consists of a definite number of elements, is called a finite set.

Example:

(i) Let $A = \{ x \in N : 2 < x < 3\}$. Since there is no natural number x which lies between 2 and 3. So, set A does not contain any element $i.e.$ $A = \emptyset$. So, A is a finite set.

(ii)The set $\{1,\ 3,\ 5,\ 7\}$ is a finite set since it contains a definite number of elements $i.e.$ only 4 elements.

Cardinal Number: The number of distinct elements in a finite set A is called cardinal number or **cardinality** of A and it is denoted by $n(A)$ or $|A|$

We have, the cardinal number of the empty set is 0.

The cardinal number of a singleton set is 1.

Example: Find the cardinality

$(i)\ \ A = \{-3,\ -1,\ 8,\ 10,\ 13\} \Rightarrow n(A) = 5$

$(ii)\ \ B = \{Jan,\ Feb,\ \ \dots\dots ,\ Dec\} \Rightarrow n(B) = 12$

$(iii)\ C = \{\ \emptyset\ \} \Rightarrow n(C) = 1$

Infinite Set: A set that consists of an infinite number of elements is called an infinite set.

The cardinal number of an infinite set is not defined.

Example: $S = \{1,\ 4,\ 9,\ 16,\ 25,\ \ \dots..\}$ is an infinite set.

Also, the set of lines parallel to the $x - axis$ is an infinite set.

Remark: The set of all persons in the world is finite, but the number of elements in the set is so large that it is difficult to find the number. So, one can say that the cardinal number of a set is sometimes very difficult to find, but it does not mean that cardinal number is not defined for such sets.

Example: From the sets given below, select empty set, singleton set, finite set, an infinite set.

$(i)\quad A = \{x : x < 1\ and\ x > 3\}$ (Empty set)

(ii) $B = \{x : x^3 - 1 = 0,\ x \in R\}$ (Singleton set}

(iii) $C = \{x : x \in N$ and x is a prime number}(infinite set)

(iv) $D = \{2,\ 4,\ 6,\ 8,\ 10\}$ (Finite set)

(v) $E =$ Set of odd natural numbers divisible by 2. (Empty set)

Example: Find the cardinal number of the following sets:

$(i)\ \{\{\emptyset\}, \{\{\emptyset\}\}\}$ $(ii)\ \{\{\emptyset\}\}$ $(iii)\ \{0,\ \{5\}\}$ $(iv)\{a,\ b,\ \{a,\ b\}\}$

Ans: (i) 2 (ii) 1 (iii) 2 (iv) 3

Equal and Equivalent Sets

Equivalent Sets: Two sets A and B are said to be equivalent or **similar** (denoted as $A \sim B$) if they have the same number of elements. $i.e.$ $n(A) = n(B)$

Example: Let $A = \{a,\ b,\ c,\ d\}$ and $= \{1,\ 29,\ 53,\ 107\}$.

Here $n(A) = 4$ and $n(B) = 4$. Therefore A and B are equivalent sets.

Two infinite sets A and B are said to be equivalent if there exists ($) a one–to–one correspondence between the elements of A and B.

Let $= \{\ 1,\ 3,\ 5,\ 7,\ \ldots\ldots.\ \}$ and $B = \{2,\ 4,\ 6,\ 8,\ \ldots\ldots.\ \}$.

Here A and B are equivalent. So, $A{\sim}B$.

Equal Sets: Two sets A and B are said to be equal if they contain the same elements and we write it as $A = B$. Otherwise, two sets are said to be **unequal** and we write $A \neq B$.

Example: Let $A = \{1,\ 2\}$ and

$B = \{x : x^2 - 3x + 2 = 0\} = \{1,\ 2\}$. So $A = B$.

Mathematically, $A = B$

if $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$

$i.e.$ $x \in A \Leftrightarrow x \in B$

Remember: Two equal sets are equivalent but not conversely.

Example: In the following, state whether $A = B$ or not:

(i) $A = \{x : x$ is a letter in the word 'area'$\}$

$B = \{y : y$ is a letter in the word 'ear'$\}$

Ans: $= \{a,\ e,\ r\}$ $B = \{a,\ e,\ r\}$. So $A = B$.

(ii) $A = \{4^n - 3n - 1 : n \in N\}$

$B = \{9(n-1) : n \in N\}$

Ans: $A = \{0,\ 9,\ 54,\ \ldots\ldots\}$ $B = \{0,\ 9,\ 18,\ \ldots\ldots\}$

So, $A \neq B$

Example: Which of the following sets are equal?

$A = \{1,\ 3,\ 5\}$, $B = \{x \in N : (x-1)(x-3)(x-5) = 0\}$

$C = \{1,\ 3,\ 3,\ 3,\ 5,\ 5,\ 5,\ 5\}$, $D = \{x : x$ is an odd natural number less than 6$\}$

Sol: Here $A = \{1,\ 3,\ 5\}$, $B = \{1,\ 3,\ 5\}$, $C = \{1,\ 3,5\}$, $D = \{1,\ 3,5\}$

So $A = B = C = D$

Example: From the sets given below, select equal sets and equivalent sets.

$A = \{0,\ a\}$, $B = \{1,\ 2,3,\ 4\}$, $C = \{1,\ -1\}$ $D = \{3,\ 1,2,\ 4\}$

Sol: B and D are equal sets. A and C are equivalent sets. Also, B and D are equivalent sets.

Subsets and Super Sets

Subset: Let A and B be two sets. If every element of A is an element of B, then A is called a subset of B. If A is a subset of B, then we write $A \subseteq B$.

Example: Let $A = \{1,\ 3,\ 5\}$ and $B = \{1,\ 2,\ 3,\ 4,\ 5\}$. So $A \subseteq B$.

If $A \subseteq B$, then B is called a superset of A.

Mathematically, $A \subseteq B$ if $x \in A \Rightarrow x \in B$.

If there exists at least one element of A which is not an element of B, then we say that 'A is not a subset of B' and we denote it by writing $A \not\subset B$.

Example: Let $A = \{\ b,\ x,\ z,\ w\}$ and $B = \{a,\ b,\ c,\ x,\ y,\ u,\ v\}$

So $A \not\subset B$

Example: N = set of natural numbers and Z = set of integers

So, $N \subseteq Z$

Proper Subset: If $A \subseteq B$ and $A \neq B$ then A is called a proper subset of B and written as $A \subset$ B.

Example: Let $A = \{2,\ 5,\ 7\}$ and $B = \{2,\ 5,\ 7\}$

So, $A \subseteq B$. Also, we can say that A and B are equal sets.

Again let $A = \{2,\ 5,7\}$ and $B = \{2,\ 3,\ 5,\ 7\}$, then we can say that A is a subset of B. Also, A is a proper subset of B.

Remember:

(i) If $A \subseteq B$, then $n(A) \leq n(B)$

(ii) If $A \subset B$, then $n(A) < n(B)$

Example: If $A = \{3,\ \{4,\ 5\},\ 6\}$, then find which of the following are true or false.

(i) $\{4,\ 5\} \subset A$ (false)

(ii) $\{3,\ 6\} \subset A$ (true)

Some Important Results:

- $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
- Every set is a subset of itself. $i.e.$ $A \subseteq A$
- Every set is not a proper subset of itself.
- The set A is called a proper subset of B, if there must be an element $x \in B$ such that $x \notin A$.
- The empty set $\emptyset$ is a subset of every set. $i.e.$ $\emptyset \subseteq A$
- For a given set A, empty set $\emptyset$ and set A, itself are called improper subsets.
- $N\ \ Z\ \ Q \subset R \subset C$
- If T be the set of irrational numbers, then $N \not\subset T$ and $\not\subset T$.
- If $A \subseteq B$ and $B \subseteq C$, then $\subseteq C$.
- The total number of subsets of a finite set A containing n elements is 2^n.

  Let $A = \{1,\ 2,\ 3\}$. The subsets of A are

  $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1,\ 2\}$, $\{2,\ 3\}$, $\{1,\ 3\}$, $\{1,\ 2,\ 3\}$ $i.e.$ $2^3 = 8$.

- The total number of proper subsets of a finite set A containing n elements is $2^n - 1$.

  Let $A = \{1, 2,3\}$. The proper subsets of A are

  $\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}$ $i.e.$ $2^3 - 1 = 7$.

- The total number of non – empty proper subsets of a finite set A containing n elements is $2^n - 2$.

  Let $A = \{1, 2,3\}$. The proper subsets of A are

  $\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}$ $i.e.$ $2^3 - 2 = 6$.

Example: Consider the following sets:

$\emptyset$, $A = \{a, e\}$, $B = \{e, i\}$, $C = \{a, e, i, o, u\}$. Insert the correct symbol $\subset$ or $\not\subset$ between each of the following pair of sets.

(i) $\emptyset$ …. B Ans: $\subset$ $(iii) A$ … C Ans: $\subset$

(ii) A …. B Ans: $\not\subset$ (iv) B … C Ans: $\subset$

Example: Write down all the non – empty proper subsets of $\{3, 4, 5\}$.

Sol: $\{3\}, \{4\}, \{5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}$

Example: Find an example of three sets A, B and C such that $A \not\subset B$, $B \not\subset C$ but $\subseteq C$.

Sol: Consider $A = \{1\}$, $B = \{2,3\}$ and $C = \{1, 3\}$.

Then $A \not\subset B$, $B \not\subset C$ but $\subseteq C$.

Example: The empty set is a subset of every set.

Sol: Let A be any set and $\emptyset$ be the empty set. In order to show that $\emptyset \subseteq A$, we must show that every element $\emptyset$ is an element of A also. But $\emptyset$ contains no element. So, every element $\emptyset$ is in A. Hence $\emptyset \subseteq A$.

Example: Prove that $A \subseteq \emptyset$ implies $A = \emptyset$.

Sol: We know that two sets A and B are equal if $A \subseteq B$ and $\subseteq A$. Also, we know that $\emptyset \subseteq A$ and $A \subseteq \emptyset$.

$\therefore A = \emptyset$.

Example: Every set is a subset of itself.

Sol: Let A be any set. Then, each element of A is clearly in A itself.

Hence, $A \subseteq A$.

Example:

Two finite sets have m and n elements. The number of subsets of the first set is 112 more than that of the second set. Find the values of m and n.

Sol. Let the two sets be A and B such that $n(A) = m$ and $n(B) = n$.

Then, the number of subsets of set $A = 2^m$ and number of subsets of set $B = 2^n$.

According to the given condition, we have $2^m - 2^n = 112 = 2^7 - 2^4$.

On comparing both sides, we get $m = 7,\ n = 4$.

Intervals as Subsets of R

Let $a,\ b \in R$ and $a < b$. Then, the set of real numbers $\{x : a < x < b\}$ is called an **open interval** and is denoted by $(a,\ b)$. All the numbers between a and b belong to the open interval $(a,\ b)$ but a and b do not belong to this set(interval).

The interval which contains the endpoints a and b is called a **closed interval** and is denoted by $[a,\ b]$.

Hence $[a,\ b] = \{x : a \le x \le b\}$

Some intervals are closed at one end and open at the other, such intervals are called **semi-closed** or **semi-open intervals.**

$[a,\ b) = \{x: a \le x < b\}$ is a **semi-open interval** from a to b which includes a but excludes b.

$(a,\ b] = \{x : a < x \le b\}$ is a **semi-closed interval** from a to b which excludes a but includes b.

If all the elements of a set are greater than or less than a constant c, then the set is called an **infinite interval.**

If $x > c$ then $x \in (c,\ \infty)$

If $x \ge c$ then $x \in [c,\ \infty)$

If $x < c$ then $x \in (-\infty,\ c)$

If $x \le c$ then $x \in (-\infty,\ c]$

The set of real numbers R is given in interval form as $(-\infty,\ \infty)$.

Example: Write the following intervals in set–builder form.

$$(i)\ (-3,\ 0) = \{x : x \in R\ ,\ -3 < x < 0\}$$

$$(ii)\ (6,\ 12] = \{x:\ x \in R,\ 6 < x \le 12\}$$

$$(iii)[-23,\ 5) = \{x : x \in R,\ -23 \le x < 5\}$$

$$(iv)\ \left[\frac{1}{3}, 7\right] = \left\{x : x \in R, \frac{1}{3} \le x \le 7\right\}$$

Example: Write the following as intervals and also represent on the number line.

$(i)\{x : x \in R, -5 < x \le 6\}$ $(ii)\{x : x \in R, -11 < x < -9\}$

$(iii)\{x : x \in R, 2 \le x < 8\}$ $(iv)\{x : x \in R, 5 \le x \le 6\}$

Sol. (i) $\{x : x \in R, -5 < x \le 6\}$ is the set that does not contain -5 but contains 6. So, it can be written as a semi-closed interval whose first end point is open and last end point is closed i.e. $(-5, 6]$.

On the real line, $(-5, 6]$ can be graphed as shown in figure given below

The dark portion on the number line represent $(-5, 6]$.

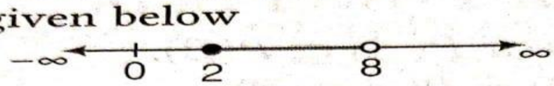
(ii) $\{x : x \in R, -11 < x < -9\}$ is the set that neither contains -11 nor -9, so it can be represented as open interval i.e. $(-11, -9)$.

On the real line, $(-11, -9)$ can be graphed as shown in figure given below

The dark portion on the number line represent $(-11, -9)$.

(iii) $\{x : x \in R, 2 \le x < 8\}$ is the set that contains 2 but not contain 8.So, it can be represented as a semi-open interval whose first end point is closed and the other end point is open i.e. $[2, 8)$.

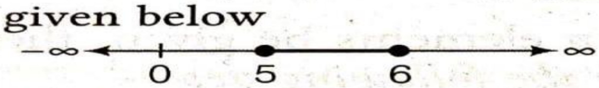
On the real line, $[2, 8)$ can be graphed as shown in figure given below



The dark portion on the number line represent $[2, \infty)$.

(iv) $\{x : x \in R, 5 \le x \le 6\}$ is the set which contains 5 and 6 both. So, it is equivalent to a closed interval i.e. $[5, 6]$.

On the real line, $[5, 6]$ can be graphed as shown in the figure given below



The dark portion on the number line represent $[5, 6]$.

Power Sets, Venn Diagrams

Power Set:

The collection of all subsets of a given set S is called the power set and it is denoted by $P(S)$.

In $P(S)$, every element is a set.

Notes:

- The number of subsets = Number of elements of power set.
- If a set S has n elements, then the total number of elements in its power set is 2^n.
- If S is an empty set $\emptyset$, then $P(S)$ has just one element namely $\emptyset$. Thus $P(S) = \{\emptyset\}$
- The power set of a given set is always non – void.
- If $a \in S$ then $\{ a \} \in P(S)$. Also $\{ a \} \subset S$

Example: Let $S = \{x\}$. Then subsets of S are $\emptyset$ and $\{x\}$.

So $P(S) = \{ \emptyset, \{x\}\}$

Example: If $S = \{x, y\}$, then find the power set of S.

Sol: All possible subsets of S having single element are $\{x\}, \{y\}$.

All possible subsets of S having two elements (all) at a time are $\{x, y\}$. Since $\emptyset$ is a subset of every set so

$P(S) = \{\emptyset , \{x\}, \{y\}, \{x, y\}\}$

Example: If $A = \{1, 2, 3\}$, then find $P(A)$.

Sol: All possible subsets of A having single element are $\{1\}, \{2\}, \{3\}$. All possible subsets of A having two elements at a time are $\{1, 2\}, \{2, 3\}, \{1, 3\}$. All possible subsets of A having three(all) elements at a time are $\{1, 2, 3\}$. Also $\emptyset \subset A$.

Thus the required power set is

$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1,3\}, \{1, 2, 3\}\}$

Example: If a set A has k elements, then how many elements has $P\left(P(A)\right)$?

Sol: We have $n(A) = k$

$\Rightarrow n\left(P(A)\right) = 2^k = m(say)$

$\Rightarrow n\left(P\left(P(A)\right)\right) = 2^m = 2^{2^k}$

Example: How many elements present in $P\left(P\left(P\left(P(\emptyset)\right)\right)\right)$?

Sol: We know that $n(\emptyset) = 0$

$\Rightarrow n(P(\emptyset)) = 2^0 = 1 \Rightarrow n\left(P(P(\emptyset))\right) = 2^1 = 2$

$\Rightarrow n\left(P\left(P(P(\emptyset))\right)\right) = 2^2 = 4$

$$n\left(P\left(P\left(P(P(\emptyset))\right)\right)\right) = 2^4 = 16$$

Comparable Sets: Two sets A and B are said to be comparable if either

$A \subseteq B$ or A.

Universal Set: If there are some sets under consideration, then a set can be chosen arbitrarily which is a superset of each one of the given sets. Such a set is known as the universal set and it is denoted by U.

$e.g.$ (i) Let $= \{2, 4, 6\}$, $B = \{1, 3, 5\}$, $C = \{0, 7\}$.

Then $U = \{0, 1, 2, 3, 4, 5, 6, 7\}$ is a universal set.

(ii) For the set of integers, set of rational numbers, set of irrational numbers, the universal set can be the set of real numbers.

(iii) When we study 2D coordinate geometry, then the set of all points in $xy - plane$ is the universal set.

Venn Diagrams:

Venn diagrams are named after the English Mathematician John Venn (1834–1883). These diagrams represent most of the relationships between sets.

In Venn diagrams, the universal set is represented by a rectangular region, and its subsets are represented by circles or closed geometrical figures inside the universal set. Also, an element of a set is represented by a point within the circle of a set.

If a set A is a subset of a set B, then the circle representing A is drawn inside the circle representing B.

But suppose $U = \{1, 2, 3, 4\}$.

$A = \{1, 2\}$ and $B = \{3, 4\}$.

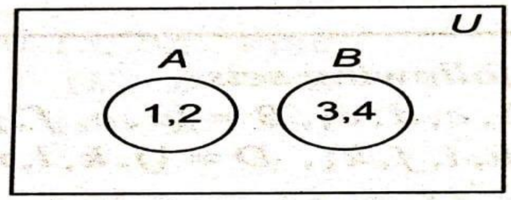


Fig. 1.10

Let $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 2\}$ and $B = \{3, 4\}$

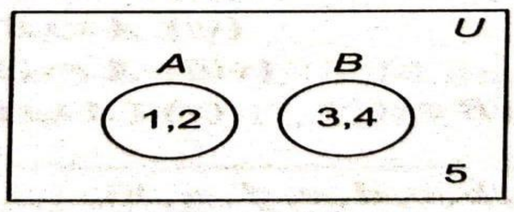
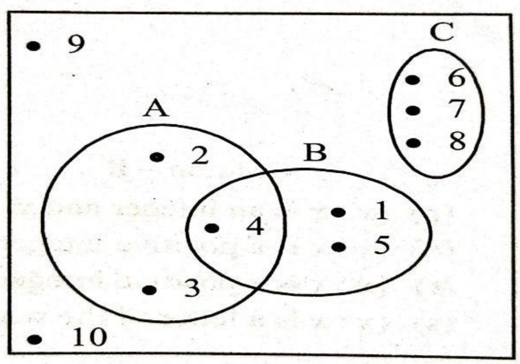


Fig. 1.11

Example: Draw a Venn diagram to represent the sets,

$$U = \{1, 2, 3, \ldots, 10\}, A = \{2, 3, 4\}, B = \{1, 4, 5\}, C = \{6, 7, 8\}$$



Here A is denoted by the circle, B by an ellipse, and C is another closed region. The points 9 and 10 do not lie in any of A, B or C.

Operation on Sets

Union of Sets:

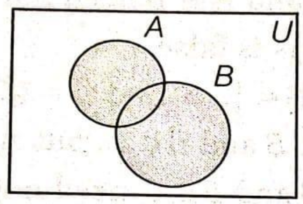
The union of two given sets A and B, denoted by $A \cup B$, is the set containing all those elements which belong either to A or to B or both A and B.

Mathematically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Ex: Let $= \{a,\ e,\ i\}, B = \{i,\ o, u\}$. Then $A \cup B = \{a,\ e,\ i,\ o,\ u\}$

Venn Diagram:

The union of sets A and B is represented by the following Venn diagram



The shaded portion represents $A \cup B$.

Laws of Union of Sets:

- Commutative Law $i.e.\ A \cup B = B \cup A$
- Associative Law $i.e.\ A \cup (B \cup C) = (A \cup B) \cup C$
- Idempotent Law $i.e.\ A \cup A = A$
- Identity Law $i.e.\ A \cup \emptyset = A$
- Universal Law $i.e.\ A \cup U = U$
- $\emptyset \cup \emptyset = \emptyset$
- $A \subseteq A \cup B$ and $B \subseteq A \cup B$
- If $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cup C$
- If $A \subseteq B$, then $A \cup B = B$
- The union of finite sets $A_1,\ A_2,\ \ldots \ldots\ A_n$ is represented by $A_1 \cup\ A_2 \cup \ldots \ldots\ \cup A_n = \cup_{i=1}^{n} A_i$

Example: Find $N \cup Z$

Sol: Since $N\ Z$, so $N \cup Z = Z$

Example: If $A = \{x : x \leq 10,\ x \in N\}$ and $B = \{y : 6 < y < 20,\ y \in N\}$, find $A \cup B$.

Sol: We have $A = \{1,\ 2,\ 3,\ \ldots,\ 10\}$ $B = \{7,\ 8,\ 9,\ \ldots,\ 19\}$

$$\therefore A \cup B = \{1,\ 2,\ 3,\ \ldots,\ 19\}$$

Example: If $A = \{4,\ 5,7,\ 8,\ 10\}$, $B = \{4,\ 5,\ 9\}$ and $C = \{1,\ 4,\ 6,\ 9\}$

Then verify that $(A \cup B) \cup C = A \cup (B \cup C)$.

Sol: $A \cup B = \{4,\ 5,7,8,9,\ 10\}$

$(A \cup B) \cup C = \{1,4,5,\ 6,7,\ 8,\ 9,\ 10\}$ ------ (i)

$B \cup C = \{1,\ 4,5,6,9\}$

$A \cup (B \cup C) = \{1,4,5,\ 6,7,\ 8,\ 9,\ 10\}$ ------ (ii)

From (i) and (ii), we get $(A \cup B) \cup C = A \cup (B \cup C)$.

The intersection of two sets

Let A and B be any two sets. The intersection of A and B is the set of those elements which belong to both A and B. It is denoted by $A \cap B$.

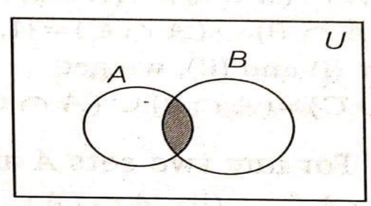
Mathematically, $A \cap B = \{x : x \in A \text{ and } x \in B\}$

Example: Let $A = \{2,\ 3,\ 4,\ 5\}$ and $B = \{1,\ 3,\ 5,\ 6\}$

Then, $A \cap B = \{\ 3,\ 5\}$

Venn diagram:

The intersection of sets A and B is represented by the following Venn diagram



The shaded portion represents $A \cap B$.

Laws of Intersection of Sets

- Commutative Law $i.e.$ $A \cap B = B \cap A$
- Associative Law $i.e$ $A \cap (B \cap C) = (A \cap B) \cap C$
- Idempotent Law $i.e.$ $A \cap A = A$
- Identity Law $i.e.$ $A \cap \emptyset = A$
- Universal Law $i.e.$ $U \cap \emptyset = A$

- $\emptyset \cap \emptyset = \emptyset$
- $A \cap B \subseteq A$ and $A \cap B \subseteq B$
- If $A \subseteq C$ and $B \subseteq C$ then $A \cap B \subseteq C$
- The intersection of finite sets $A_1, A_2 \ldots\ldots\ldots A_n$ is represented by $A_1 \cap A_2 \cap \ldots\ldots\ldots\ldots \cap A_n = \cap_{i=1}^{n} A_i$

$$\textbf{\textit{Example}}: Let\ P = \left\{\frac{1}{x} : x \in N\ ,\ x < 7\right\}\ and\ Q = \left\{\frac{1}{2x} : x \in N\ ,\ x \leq 7\right\}\ Find\ P \cap Q$$

Sol: Here P $= \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\right\}$ and Q= $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}\right\}$

Then, $P \cap Q = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}\right\}$

Example: If $A = \{3, 5, 7, 9, 11\}$, B= $\{7, 9, 11, 13\}$, $C = \{11, 13, 15\}$ Then find

$(i)\ A \cap B \cap C$

Ans: $\{11\}$

$(ii)\ A \cap (B \cup C)$

Ans: $B \cup C = \{7, 9, 11, 13, 15\}$

So, $A \cap B \cap C = \{3, 5, 7, 9, 11\} \cap \{7, 9, 11, 13, 15\} = \{7, 9, 11\}$

Example: If A $= \{a, b, c, d, e\}$, B $= \{a, c, e, g\}$, and C $= \{b, e, f, g\}$, verify that $A \cap (B \cap C) = (A \cap B) \cap C$

Sol: $(B \cap C) = \{e, g\}$ $A \cap (B \cap C) = \{e\}$

$(A \cap B) = \{a, c, e\}$ $(A \cap B) \cap C = \{e\}$

So, $A \cap (B \cap C) = (A \cap B) \cap C$

Example: If A $= \{2, 17, 29\}$ and $B = \emptyset$, find $A \cap B$

Sol: $A \cap B = \emptyset$

Example: If $a \in N$ such that $aN = \{an : n \in N\}$. Describe the set $3N \cap 7N$.

Sol: Given $aN = \{an : n \in N\}$

$3N = \{3n : n \in N\} = \{3, 6, 9,12, \ldots\ldots\ldots\}$

and $7N = \{7n : n \in N\} = \{7, 14, 21, 28, \ldots\ldots\ldots\}$

Hence, $3N \cap 7N = \{21, 42, \ldots\ldots\ldots\} = \{21n : n \in N\} = 21N$

Note: $aN \cap bN = cN$, *where c is the LCM of a and b.*

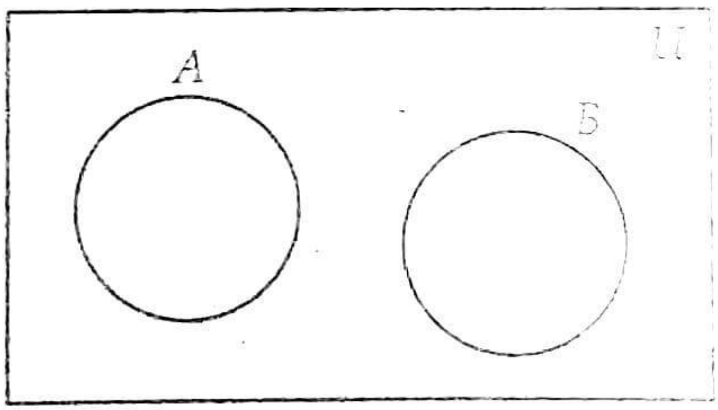
Disjoint Sets:

Two sets A and B are said to be disjoint or non-overlapping if they have no common element $i.e.$ $A \cap B = \emptyset$.

Let A $= \{1, 2, 3, 4, 5, 6\}$ and $B = \{7, 8, 9, 10\}$. Then $A \cap B = \emptyset$.

Hence, A and B are disjoint sets.

The disjoint of two sets A and B can be represented by the Venn diagram



Example: Which of the following pairs of sets are disjoint?

(i) A= $\{1, 2, 3, 4, 5, 6\}$ and $B = \{x : x \text{ is a natural number and } 4 \leq x \leq 8\}$

Sol: Here A= $\{1, 2, 3, 4, 5, 6\}$ and $B = \{4, 5, 6, 7, 8\}$

$\therefore A \cap B = \{4, 5, 6\} \neq \emptyset$

Hence, this pair of sets are not disjoint.

(ii) $A = \{x : x \text{ is the boys in your school}\}$

$$B = \{x : x \text{ is the girls in your school}\}$$

Sol: Here$A \cap B = \emptyset$

Hence, this pair of sets are disjoint.

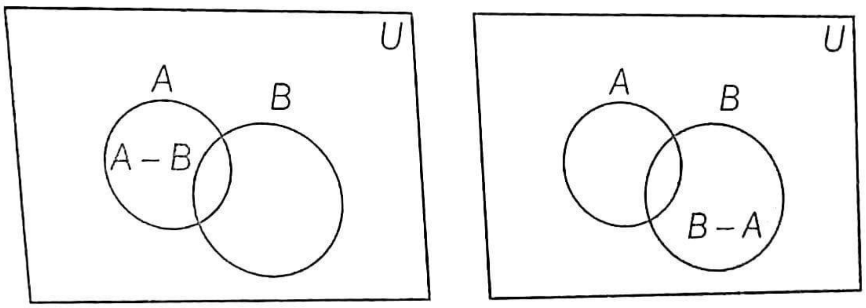
Difference of Sets

Let A and B be any two sets. The difference of sets A and B in this order is the set of all those elements of A which do not belong to B. It is denoted by $A - B$ *or* $A \setminus B$.

Mathematically, $A - B = \{x : x \in A \; and \; x \notin B\}$

Similarly, $B - A = \{x : x \in B \; and \; x \notin A\}$

The difference between two sets A and B can be represented by the following Venn diagram.



Laws of Difference of Sets

- The difference is not commutative $i.e. A - B \neq B - A$
- $A - A = \emptyset$
- $A - \emptyset = A$
- $A - B = A - (A \cap B)$
- $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
- $(A - B) \cup B = A \cup B \; and \; (A - B) \cap B = \emptyset$
- $A - (B \cup C) = (A - B) \cap (A - C)$
- $A - (B \cap C) = (A - B) \cup (A - C)$
- $(A \cup B) - C = (A - C) \cup (B - C)$
- $(A \cap B) - C = (A - C) \cap (B - C)$

Example: If A = $\{1,2, 3, 4\}$, B = $\{3, 4, 5, 6\}$ and C= $\{1, 4, 6, 7, 8\}$ then verify the following.

$(i) \; A \cap (B - C) = (A \cap B) - (A \cap C)$

Sol: Now $B - C = \{3, 5\} \Rightarrow A \cap (B - C) = \{3\}$ .....(I)

Again $A \cap B = \{3, 4\}$, $A \cap C = \{1, 4\}$

So, $(A \cap B) - (A \cap C) = \{3\}$ .....(II)

From, (I) and (II), we get $\cap (B - C) = (A \cap B) - (A \cap C)$.

$(ii)\ A - (B \cup C) = (A - B) \cap (A - C)$

Sol: $B \cup C = \{1, 3, 4, 5, 6, 7, 8\} \Rightarrow A - (B \cup C) = \{2\}$ .....(I)

Again $A - B = \{1, 2\}$ and $A - C = \{2, 3\}$

$$(A - B) \cap (A - C) = \{2\} \quad \ldots..(II)$$

From (I) and (II), we get $A - (B \cup C) = (A - B) \cap (A - C)$.

Example: If A $= \{1, 3, 5, 7\}, B = \{3, 7, 11, 13\}$ and $C = \{1, 7, 9, 11\}$, then verify the following.

$(i)\ (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

Sol: $A - B = \{1, 5\}$ and $B - A = \{11, 13\}$

L.H.S. $= (A - B) \cup (B - A) = \{1, 5, 11, 13\}$

Again, $A \cup B = \{1, 3, 5, 7, 11, 13\}$ and $A \cap B = \{3, 7\}$

R.H.S. $= (A \cup B) - (A \cap B) = \{1, 5, 11, 13\}$

So, L.H.S. = R.H.S.

$(ii)\ (A \cap B) - C = (A - C) \cap (B - C)$

Sol: $(A \cap B) = \{3, 7]$

L.H.S $= (A \cap B) - C = \{3\}$

Again, $A - C = \{3, 5\}$ and $B - C == \{3, 13\}$

R.H.S.$= (A - C) \cap (B - C) = \{3\}$

So, L.H.S.=R.H.S.

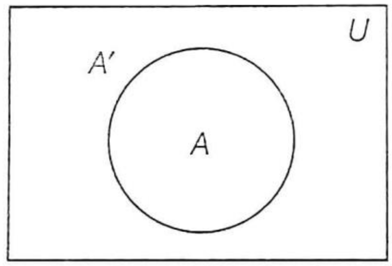
Complement of a Set and its Properties

Complement of a Set

Let U be the universal set and A be any subset of U. The compliment of A with respect to U is the set of all those elements of U which are not in A. It denoted by A'.

Mathematically, $A' = \{x : x \in U\ and\ x \notin A\} = U - A$

The complement of set A is represented by the following Venn diagram.



Example: Let $U = \{1, 2, 3, 4, 5\}$ and $= \{2, 4\}$, Then $A' = \{1, 3, 5\}$

Note: If A is a subset of the universal set U , then its complement A' is also a subset of U

Properties:

❖ **Involution Law :** $(A')' = A$

❖ **Complement Laws:**

(i) $A \cup A' = U$

(ii) $A \cap A' = \emptyset$

❖ **Law of empty set and universal set:**

(i) $\emptyset' = U$

(ii) $U' = \emptyset$

❖ **De – Morgan's Laws:**

(i) $(A \cup B)' = A' \cap B'$

(ii) $(A \cap B)' = A' \cup B'$

Example: Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5,6\}$. Find

(i) B'

Sol: $B' = \{ 1, 3, 5, 7, 9\}$

(ii) $(A \cup B)'$

Sol: $A \cup B = \{1, 2, 3, 4, 6, 8\} \Rightarrow (A \cup B)' = \{5, 7, 9\}$

$(iii)(A \cap C)'$

Sol: $A \cap C = \{3, 4\} \Rightarrow (A \cap C)' = \{1, 2, 5, 6, 7, 8, 9\}$

$(iv)(B - C)'$

Sol: $B - C = \{2, 8\} \Rightarrow (B - C)' = \{1, 3, 4, 5, 6, 7, 9\}$

$(v)\ (A')'$

Sol: $A' = \{5, 6, 7, 8, 9\} \Rightarrow (A')' = \{1, 2, 3, 4\}$

So $(A')' = A$

Example: If $U = \{1, 2,3, 4, 5, 6, 7, 8\}$, $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, then verify that

$(i)\ (A \cap B)' = A' \cup B'$

Sol: $A \cap B = \{3,4\} \Rightarrow (A \cap B)' = \{1, 2, 5, 6, 7, 8\}$ … (1)

Again $A' = \{5, 6, 7, 8\}$ and $B' = \{1, 2, 7, 8\}$

So, $A' \cup B' = \{1, 2, 5, 6, 7, 8\}$ … (2)

From (1) and (2), we get $(A \cap B)' = A' \cup B'$.

$(ii)(A \cup B)' = A' \cap B'$

Sol: $A \cup B = \{1, 2,3, 4, 5,6\} \Rightarrow (A \cup B)' = \{7, 8\}$… (3)

Again $A' = \{5, 6, 7, 8\}$ and $B' = \{1, 2, 7, 8\}$

So, $A' \cap B' = \{7, 8\}$ … (4)

From (3) and (4), we get $(A \cup B)' = A' \cap B'$.

Law of Algebra of Sets

1. **Idempotent Laws:** For any set A

$(i)\ A \cup A = A$ $(ii)\ A \cap A = A$

Proof:

$$(i)\ A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$$

$(ii)\ A \cap A = \{x : x \in A \text{ and } x \in A\} = \{x : x \in A\} = A$

2. **Identity Laws:** For any set A,

$(i)\ A \cup \emptyset = A$ $(ii)\ A \cap U = A$

Proof:

$$(i)\ A \cup \emptyset = \{x : x \in A \text{ or } x \in \emptyset\} = \{x : x \in A\} = A$$

$(ii)\ A \cap U = \{x : x \in A \text{ and } x \in U\} = \{x : x \in A\} = A$

3. **Commutative Laws:**

$$(i)\ A \cup B = B \cup A \quad (ii)\ A \cap B = B \cap A$$

Proof:

(i) Let x be an arbitrary element of $A \cup B$.

Then $x \in A \cup B \Rightarrow x \in A \text{ or } x \in B$

$\Rightarrow\ x \in B \text{ or } x \in A \Rightarrow x \in B \cup A$

$\therefore\ A \cup B \subseteq B \cup A$

$\therefore$ Similarly, $B \cup A \subseteq A \cup B$

Hence, $A \cup B = B \cup A$

(ii) Same as (i)

4. Associative Laws:

$$(i)(A \cup B) \cup C = A \cup (B \cup C)$$

$$(ii)(A \cap B) \cap C = A \cap (B \cap C)$$

Proof: (i) Let x be an arbitrary element of $(A \cup B) \cup C$.

Then, $x \in (A \cup B) \cup C \Rightarrow x \in (A \cup B)$ or $x \in C$

$\Rightarrow\ (x \in A \text{ or } x \in B) \text{ or } x \in C \Rightarrow x \in A \text{ or } (x \in B \text{ or } x \in C)$

$\Rightarrow\ x \in A \text{ or } x \in (B \cup C) \Rightarrow x \in A \cup (B \cup C)$

$\therefore (A \cup B) \cup C \subseteq A \cup (B \cup C)$

Similarly, $A \cup (B \cup C) \subseteq (A \cup B) \cup C$. Hence $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) Same as (i)

5. Distributive Laws:

$$(i)\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$(ii)\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof: (i) Let x be an arbitrary element of $A \cup (B \cap C)$

Then, $x \in A \cup (B \cap C) \Rightarrow x \in A\ or\ x \in (B \cap C) \Rightarrow x \in A\ or\ (x \in B\ and\ x \in C)$

$\Rightarrow\ (x \in A\ or\ x \in B)\ and\ (x \in A\ or\ x \in C)$

$\Rightarrow\ x \in (A \cup B)\ and\ x \in (A \cup C) \Rightarrow x \in ((A \cup B) \cap (A \cup C))$

$\therefore A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Similarly, $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Hence, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) Same as (i)

6. De – Morgan's Laws:

(i) The complement of union is equal to the intersection of complements. $i.e.,\ (A \cup B)' = A' \cap B'$

(ii) The complement of intersection is equal to the union of complements. $i.e.,\ (A \cap B)' = A' \cup B'$

Proof: (i) Let x be an arbitrary element of $(A \cup B)'$.

Then $x \in (A \cup B)' \Rightarrow x \notin (A \cup B) \Rightarrow x \notin A\ and\ x \notin B$

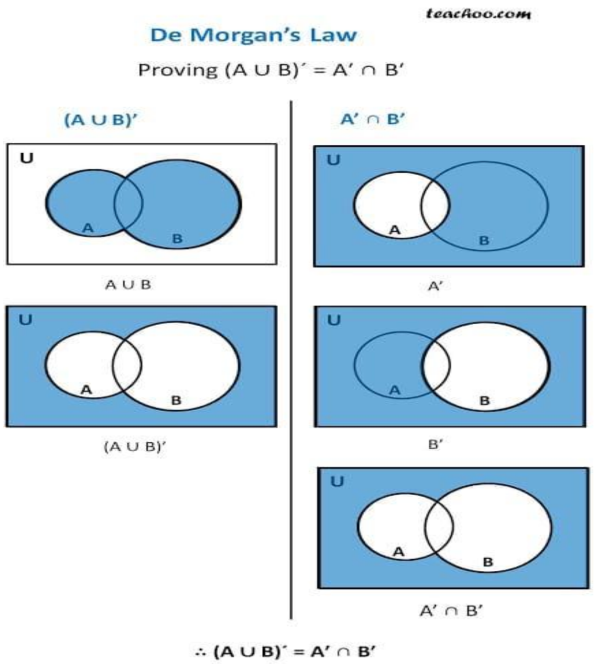
$\Rightarrow x\ \in A' and\ x \in B' \Rightarrow x \in A' \cap B'$

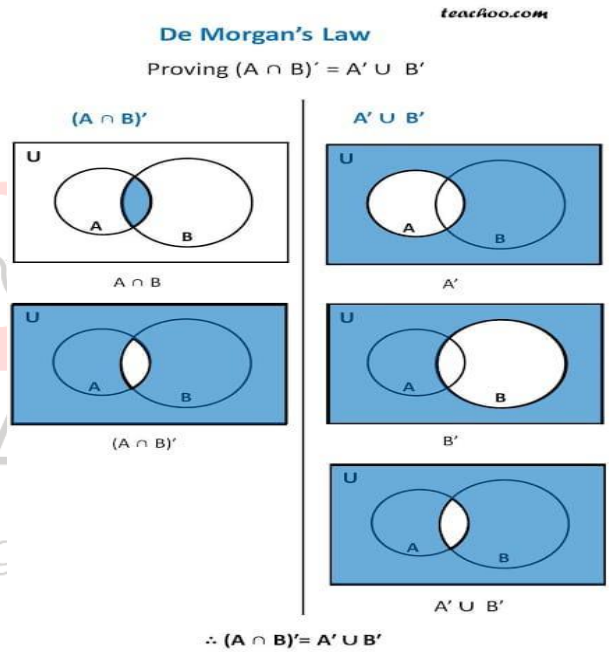
$\therefore (A \cup B)' \subseteq A' \cap B'$

Similarly, $A' \cap B' \subseteq (A \cup B)'$. Hence $(A \cup B)' = A' \cap B'$.

(ii) Same as (i)

Proof of De – Morgan's Law by using Venn - Diagram





Maximum Set:

Let A and B be subsets of the universal set U. Then the sets $A \cup B$, $A \cup B'$, $A' \cup B$ and $A' \cup B$ are called maximum sets with respect to A and B.

Minimum Set:

Let A and B be subsets of the universal set U. The sets $A \cap B$, $A \cap B'$, $A' \cap B$ and $A' \cap B'$ are called minimum sets with respect to A and B.

Example: If $n(A) = p$, $n(B) = q$ and $p < q$, then find the maximum and the minimum number of elements present in $A \cup B$ and $A \cap B$.

Sol: Maximum $n(A \cup B) = p + q$

Minimum $n(A \cup B) = q$

Maximum $n(A \cap B) = p$

Minimum $n(A \cap B) = 0$

EXAMPLE 5 *For any two sets A and B, prove that* $A \cup B = A \cap B \Leftrightarrow A = B$.

SOLUTION First let $A = B$. Then,

$$A \cup B = A \text{ and } A \cap B = A \Rightarrow A \cup B = A \cap B$$

Thus, $A = B \Rightarrow A \cup B = A \cap B$ ...(i)

Conversely, let $A \cup B = A \cap B$. Then, we have to prove that $A = B$. For this, let

$x \in A \Rightarrow x \in A \cup B$

$\Rightarrow x \in A \cap B$ $[\because A \cup B = A \cap B]$

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in B$

$\therefore$ $A \subset B$ ...(ii)

Now, let

$y \in B \Rightarrow y \in A \cup B$

$\Rightarrow y \in A \cap B$ $[\because A \cup B = A \cap B]$

$\Rightarrow y \in A$ and $y \in B$

$\Rightarrow y \in A$

$\therefore$ $B \subset A$ ...(iii)

From (ii) and (iii), we get $A = B$.

Thus, $A \cup B = A \cap B \Rightarrow A = B$ ...(iv)

From (i) and (iv), we obtain

$$A \cup B = A \cap B \Leftrightarrow A = B.$$

EXAMPLE 9 *For any two sets A and B prove that:* $P(A \cap B) = P(A) \cap P(B)$.

SOLUTION In order to prove that $P(A \cap B) = P(A) \cap P(B)$, it is sufficient to prove that

$$P(A \cap B) \subset P(A) \cap P(B) \text{ and } P(A) \cap P(B) \subset P(A \cap B).$$

First let

$X \in P(A \cap B)$

$\Rightarrow$ $X \subset A \cap B$

$\Rightarrow$ $X \subset A$ and $X \subset B$

$\Rightarrow$ $X \in P(A)$ and $X \in P(B)$

$\Rightarrow$ $X \in P(A) \cap P(B)$

$\therefore$ $P(A \cap B) \subset P(A) \cap P(B)$

Now, let

$Y \in P(A) \cap P(B)$. Then,

$Y \in P(A) \cap P(B)$

$\Rightarrow$ $Y \in P(A)$ and $Y \in P(B)$

$\Rightarrow$ $Y \subset A$ and $Y \subset B$

$\Rightarrow$ $Y \subset A \cap B$

$\Rightarrow$ $Y \in P(A \cap B)$

$\therefore$ $P(A) \cap P(B) \subset P(A \cap B)$ ...

From (i) and (ii), we get: $P(A \cap B) = P(A) \cap P(B)$.

Practical Problems based on Operation on Sets

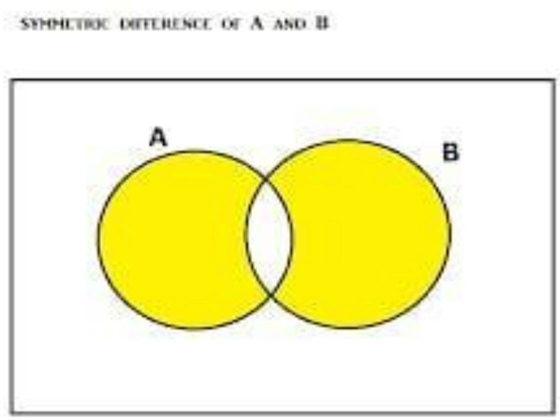
Some Important Results on Number of Elements in Sets

If A, B, and C are finite sets and U be the finite universal set. Then

$(i)\ n(A \cup B) = n(A) + n(B) \Leftrightarrow A,\ B$ are disjoint sets

$(ii)\ n(A \cup B) = n(A) + n(B) + n(A \cap B)$

Proof: Given that A and B are two finite sets. From the Venn diagram, it is clear that $A - B,\ A \cap B$ and $B - A$ are three disjoint sets. Also, the union of these three sets is equal to $A \cup B$.



$\therefore n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$

$$= n(A - B) + n(A \cap B) + n(B - A) + n(A \cap B) - n(A \cap B)$$

$$= [n(A - B) + n(A \cap B)] + [n(B - A) + n(A \cap B)] - n(A \cap B)$$

$$= n(A) + n(B) - n(A \cap B)$$

$$(iii) n(A - B) = n(A) - n(A \cap B)$$

(iv) The number of elements that belongs to exactly one of A or B is

$$n(A - B) + n(B - A) = n(A) + n(B) - 2n(A \cap B)$$

$(v)\ n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

(vi) The number of elements present exactly in A is

$$n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$$

(vii) The number of elements in exactly one of the sets $A,\ B,\ C$ is

$$n(A) + n(B) + n(C) - 2\,n(A \cap B) - 2n(B \cap C) - 2n(C \cap A) + 3\,n(A \cap B \cap C)$$

$(viii)$ The number of elements in exactly two of the sets $A, B,\ C$ is

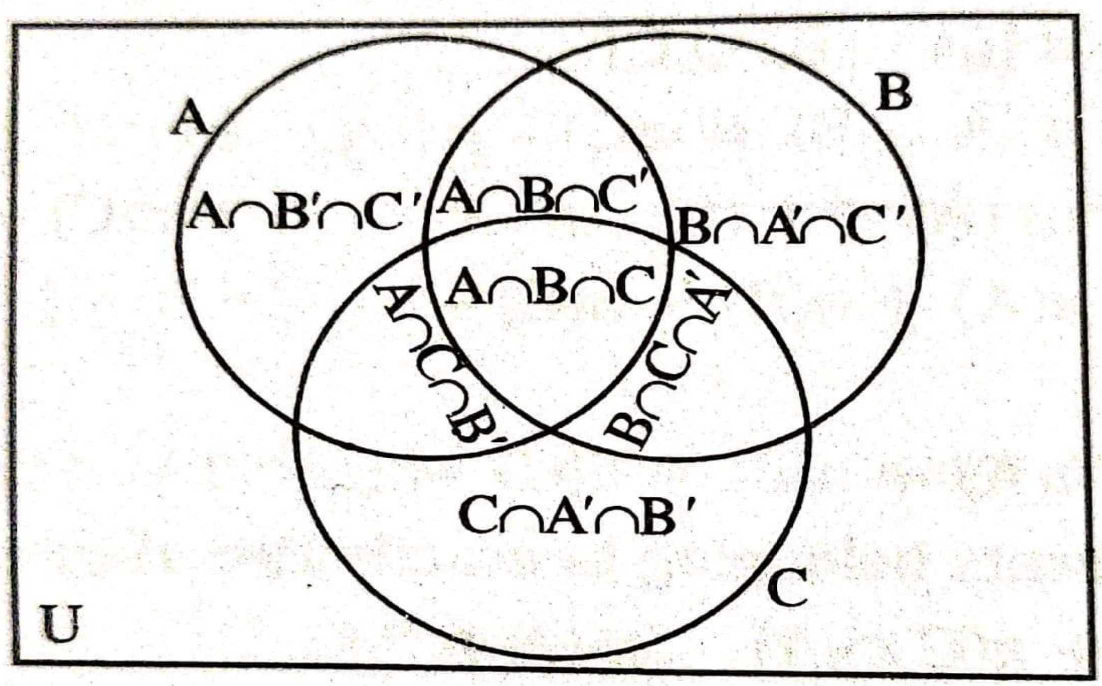
$$n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$$

$(ix) n(A' \cup B') = n((A \cap B)') = n(U) - n(A \cap B)$

$(x)\ n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$

In the following table, we give some verbal descriptions with their equivalent set theoretical notations (involving two sets A and B).

Verbal Description	Set Theoretical Notation
not A	A'
only A	$A \cap B'$
A but not B	$A \cap B'$
only B	$A' \cap B$
B but not A	$A' \cap B$
either A or B	$A \cup B$
At least one of A and B	$A \cup B$
A and B	$A \cap B$
neither A nor B	$A' \cap B'$



Example: If X and Y are any two sets such that $n(X) = 45,\ n(Y) = 43$ and $n(X \cup Y) = 76,$ find $n(X \cap Y)$.

Sol: We know that $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$

$\Rightarrow\ \ 76 = 45 + 43 - n(X \cap Y)$

$\Rightarrow\ \ n(X \cap Y) = 88 - 76 = 12$

Example: If A and B are two sets such that $n(A) = 35,\ n(A \cap B) = 11$ and $n((A \cup B)') = 17$. If $n(U) = 57,$ find $n(B - A)$.

Sol: We have $n((A \cup B)') = 17$

$\Rightarrow\ \ n(U) - n(A \cup B) = \ = 17$

$\Rightarrow\ \ n(A \cup B) = 57 - 17 = 40$

We know that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow\ \ 40 = 35 + n(B) - 11 \Rightarrow n(B) = 16$

Again, $n(B - A) = n(B) - n(A \cap B) = 16 - 11 = 5$

Example: In a school, 20 teachers teach mathematics or physics. Of these, 12 teach mathematics and 4 teach both physics and mathematics. How many teach physics?

Sol: Let M denote the set of teachers who teach mathematics and P denote the set of teachers who teach physics.

We, therefore have $n(M \cup P) = 20,\ n(M) = 12$ and $n(M \cap P) = 4$

Using the result, $n(M \cup P) = n(M) + n(P) - n(M \cap P)$,

We obtain $20 = 12 + n(P) - 4$

$\Rightarrow\ \ n(P) = 12$

Hence 12 teachers teach physics.

Example: In a class of 35 students, 24 like to play cricket and 16 like to play football. Also, each student likes to play at least one of the two games. How many students like to play both cricket and football?

Sol: Let X be the set of students who like to play cricket and Y be the set of students who like to play football.

Then $X \cup Y$ is the set of students who like to play at least one game and $X \cap Y$ is the st of students who like to play both the games.

Given, $n(X) = 24,\ n(Y) = 16,\ n(X \cup Y) = 35$

Using the formula, $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$, We get $35 = 24 + 16 - n(X \cap Y)$

Thus $n(X \cap Y) = 5$ $i.e.,$ 5 students like to play both games.

Example: In a group of 50 people, 35 speak Hindi, 25 speak both English and Hindi and all people speak at least one of two languages. How many people speak only English and not Hindi? How many people speak English?

Sol: Let H denote the set of people speaking Hindi and E, the set of people speaking English. Then, it is given that $n(H \cup E) = 50$, $n(H) = 35$, $n(H \cap E) = 25$

Now, $n(E - H) = n(H \cup E) - n(H) = 50 - 35 = 15$

Thus, the number of people speaking English but not Hindi is 15.

Now, $n(H \cup E) = n(H) + n(E) - n(H \cap E)$

$\Rightarrow \quad 50 = 35 + n(E) - 25$

$\Rightarrow \quad n(E) = 40$

Hence, the number of people who speak English is 40.

Example: In a survey of 400 students in a school, 100 were listed as taking apple juice, 150 as taking the orange juice, and 75 were listed as taking both apples as well as orange juice. Find how many students were taking neither apple juice nor orange juice.

Sol: Let U denote the set of surveyed students and A denote the set of students taking apple juice and B denote the set of students taking orange juice. Then

$n(U) = 400$, $n(A) = 100$, $n(B) = 150$ and $n(A \cap B) = 75$.

Now $n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$

$$= n(U) - [n(A) + n(B) - n(A \cap B)]$$

$$= 400 \; - [100 + 150 - 75]$$

$$= 225$$

Hence 225 students were taking neither apple juice nor orange juice.

Example: In a survey of 25 students, it was found that 15 had taken Mathematics, 12 had taken Physics and 11 had taken Chemistry, 5 had taken Mathematics and Chemistry, 9 had taken Mathematics and Physics, 4 had taken Physics and Chemistry and 3 had taken all the three subjects. Find the number of students that had taken

(i) only Chemistry.

(ii) only Mathematics.

(iii) only Physics.

(iv) Physics and Chemistry but not Mathematics.

(v) Mathematics and Physics but not Chemistry.

(vi) only one of the subjects.

(vii) at least one of the three subjects.

$(viii)$ none of the subjects.

Sol: Let M denote the set of students who had taken Mathematics, P the set of students who had taken Physics, and C the set students who had taken Chemistry. It is given that

$$n(U) = 25,\ n(M) = 15,\ n(P) = 12,\ n(C) = 11,\ n(M \cap C) = 5$$

$n(M \cap P) = 9$, $n(P \cap C) = 4$ and $n(M \cap P \cap C) = 3$

(i) Number of students who had opted Chemistry only

$$= n(M' \cap P' \cap C) = n((M \cup P)' \cap C) = n(C) - n((M \cup P) \cap C)$$

$$= n(C) - n((M \cap C) \cup (P \cap C))$$

$$= n(C) - [\,n(M \cap C) + n(P \cap C) - n(M \cap P \cap C)] = 11 - (5 + 4 - 3) = 5$$

(ii) The number of students who had opted for Mathematics only

$$= n(M) - [n(M \cap P) + n(M \cap C) - n(M \cap P \cap C)]$$

$= 15 - (9 + 5 - 3) = 4$

(iii) The number of students who had opted for Physics only

$$= n(P) - [n(P \cap M) + n(P \cap C) - n(P \cap M \cap C)]$$

$$= 12 - [9 + 4 - 3] = 2$$

(iv) Required number of students $= n(P \cap C \cap M')$

$$= n(P \cap C) - n(P \cap C \cap M) = 4 - 3 = 1$$

(v) Required number of students $= n(M \cap P \cap C')$

$$= n(M \cap P) - n(M \cap P \cap C) = 9 - 3 = 6$$

(vi) Required number of students

$$= n(M) + n(P) + n(C) - 2[n(M \cap P) + n(P \cap C) + n(M \cap C)] + 3\,n(M \cap P \cap C)$$

$$= 15 + 12 + 11 - 2(9 + 4 + 5) + 3 \times 3 = 38 - 36 + 9 = 11$$

(vii) Required number of students $= n(M \cup P \cup C)$

$$= n(M) + n(P) + n(C) - n(M \cap P) - n(P \cap C) - n(M \cap C) + n(M \cap P \cap C)$$

$$= 15 + 12 + 11 - 9 - 4 - 5 + 3 = 23$$

$(viii)$ Required number of students $= n(U) - n(M \cup P \cup C) = 25 - 23 = 2$