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REAL NUMBERS

INTRODUCTION

Real number represent actual physical quantities in a meaningful way. These can be represented on the number line. Number line is geometrical straight line with arbitrarily define zero (origin).

The field of all rational and irrational numbers is called the real numbers, or simply the "reals," and denoted R. The set of real numbers is also called the continuum, denoted C. The set of reals is called Reals in Mathematica, and a number can be tested to see if it is a member of the reals using the command Element [x, Reals].

The real numbers can be extended with the addition of the imaginary number i, equal to $\sqrt{-1}$.

Numbers of the form $x + i\,y$, where x and y are both real, are called complex numbers.

PROPERTIES OF REAL NUMBERS

Closure : The set of real numbers is closed under addition and multiplication. This means that the sum of two real numbers is a real number and the product of two real numbers is a real number. The set of real numbers is also closed under subtraction (the difference of two real numbers is a real number), but not under division (the quotient of two real numbers may not be a real number i.e., division by zero does not yield a real number)

Commutative : The set of real numbers is commutative under addition and multiplication. This means that the order of the terms (addition) or factors (multiplication) is irrelavent to the answer. $a + b = b + a$ and $a\,b = b\,a$. The set of real numbers is not commutative with respect to subtraction and division, however $a - b \neq b - a$ & $a/b \neq b/a$.

Associative : The set of real numbers is associative under addition and subtraction. This means that the grouping of terms (addition) or factors (multiplication) is irrelevant to the answer. $(a + b) + c = a + (b + c)$ or $(ab)\,c = a\,(bc)$. The set of real numbers is not associative with respect to subtraction and division, however.

$(a - b) - c \neq a - (b - c)$ and $(a/b) / c \neq a / (b/c)$.

Identity : There is an additive identity and a multiplicative identity. The identity is the number that you can add or multiply by and get the same answer you started with. The additive identity is zero (0) and the multiplicative identity is one (1). Subtraction and division are defined in terms of addition and multiplication and the same identities hold.

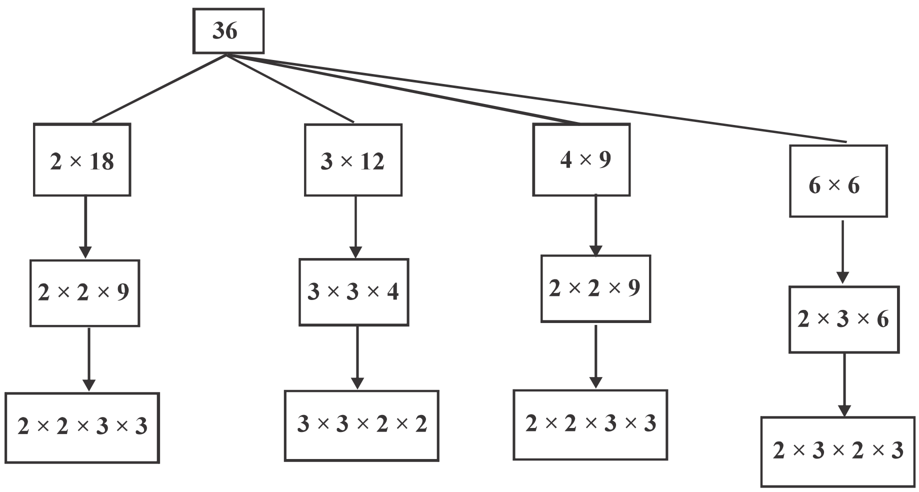
Inverse : There is an additive inverse for all real numbers, and a multiplicative inverse all real numbers except for the additive identity zero (0). The sum of a number and its additive inverse is the additive identity zero (0). Another name for additive inverse is opposite. The product of a number and its mulitplicative inverse is the multiplicative identity one (1). Another name for multiplicative inverse is reciprocal. Every number except zero (0) has a reciprocal.

Distributive : There isn't a separate distributive property for addition and multiplication like there were with the other five properties. This is because the distributive property combines addition and multiplication. Stated simply, it says that "Multiplication distributes over addition". The left distributive property is: $a\,(b + c) = ab + ac$, and the right distributive property is $(a + b)\,c = ac + bc$. With real numbers, it is not important to distinguish between the left and right distributive properties because of commutativity. When we talk about Matrices, which aren't commutative under multiplication, then we must distinguish between the left and right properties.

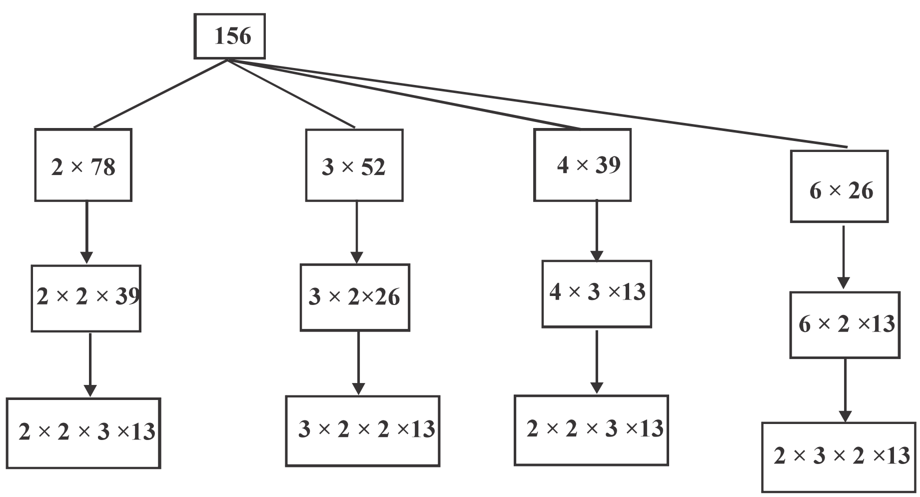
FUNDAMENTAL THEOREM OF ARITHMETIC

Fundamental theorem of arithmetic states that 'Every positive composite number can be expressed as the product of primes, uniquely except the order in which the prime factors occur. This theorem is also known as the 'Unique Factorisation Theorem' In simple words every natural number has a factorisation into primes which is unique

except for ordering.
Primes appear many times in arithmetic, hence the reason this is the fundamental theorem.
The Least Common Multiple (LCM) is found by taking each prime factor in all the terms the most number of times it appears in any of the terms. The Greatest Common Factor (GCF) is found by taking each prime factor in all the terms the least number of times it appears in any of the terms. When reducing fractions, it is easy to see what to reduce if the numerator and denominator have been written in terms of primes. Prime numbers can be used in the divisibility tests to arrive at divisibility tests for composite numbers. Prime factorization is the technique used to find that unique factorization. For example, $65790 = 2 \times 3 \times 3 \times 5 \times 17 \times 43$, and any other factorisation of 65790 into primes would have the same factors 2, 3, 3, 5, 17 and 43, except perhaps in a different order Let us express following number as the product of prime factors. (i) 36 (ii) 156
(i) 36



In each of the cases prime factors of 36 are $2 \times 2 \times 3 \times 2$.
(ii) 156



So, prime factors of 156 are $2 \times 2 \times 3 \times 13$
So, the prime factorisation of a number is unique.
Example: Using fundamental theorem of arithmetic prove that 8^n cannot end with the digit zero for any natural number, n. Consider the prime factorisation of $(8)^n = (2)^{3n}$
So, by fundamental theorem of arithmetic 2^{3n} is the only prime factorisation of $8^{n.}$
If 8^n is to end with 0 it must contain the prime 5 which is not possible as 2 is the only prime which it contains.
Hence, 8^n will not end with zero for any value of n where $n \in N$.

EUCLID'S LEMMA

The critical element in the proof of the Fundamental Theorem is a lemma called Euclid's Lemma. (A lemma is a minor theorem which is useful only to help prove some other more important theorem. Sometimes a minor theorem is originally developed as a lemma, and then everyone decides that the lemma is actually quite important for it's own sake, but they keep on calling it a "lemma" anyway.)

Euclid's Lemma states that if a prime number p (prime is a number > 1 such that its only factors are itself and 1) divides a number N (i.e. N is a multiple of p), and N is the product of two numbers a and b, then p must divide at least one of a or b.

To make this more concrete, let us consider a particular prime, for example 43. So: if 43 divides a number N, and N is the product of two numbers a and b, then 43 must divide at least one of a or b.

We can think of this as saying that there is a property of "forty-three-ness", which is possessed by any number that is a multiple of 43, and that if a number N has this property, and it is decomposed multiplicatively into two factors a and b, then the "forty-three-ness" will be found in at least one of those factors.

Think of a cake, with at least one raisin in it, which has the property of "raisin-containing-ness". If we cut the cake into two pieces A and B, at least one of those pieces must have a raisin in it. Now you might be thinking: what if the knife cuts the raisin in half? does that still count?, and to avoid that problem, we'll replace the raisins with marbles, and use a plastic knife, so that the knife can't possibly cut the marbles in half. (If we push this analogy too far, we'll eventually discover that the cake is made up entirely of marbles, and there isn't really any edible cake. But we haven't got there yet.)

What if we cut cake into three pieces? Will there still be at least one marble in at least one piece? We can answer a definite yes by considering the division into three pieces to be a sequence of two divisions into two pieces. First there is a division into two pieces A and BC, and there must be at least one marble in at least one of A or BC. Then we cut BC into B and C, if there was a marble in BC then there must now be a marble in at least one of B or C.

The same kind of extension holds for Euclid's Lemma. If a number N is a multiple of 43, and N is factorised into $a \times b \times c$, then at least one of a, b or c must be a multiple of 43. We apply the lemma first for the decomposition of N into a and $b \times c$, and then we apply it a second time decomposing $b \times c$ into b and c.

Proof of Euclid's Lemma : Euclid's Lemma says that : If a number N is a multiple of a prime number p, and $N = a \times b$, then at least one of a and b must be a multiple of p.

Another way to express this lemma is to state that there are no divisors of zero in arithmetic modulo p.

Consider an example to understand meaning of modulo arithmetic. For example, what is 3 hours after 11 o'clock? The answer is 2 o'clock, and this implies that $11 + 3 = 2$. This equation makes sense if we qualify it by saying that it holds modulo 12, which is a short-hand for "ignoring multiples of 12".

What makes modulo arithmetic interesting is that it works consistently for addition, subtraction and multiplication. For example, consider any number equal to 11 modulo 12, such as 23 or 1727 or 120011, and add it to any number equal to 3 modulo 12, such as 15 or 144003 or even -21, then the answer will be equal to 2 modulo 12. The same applies to subtraction and multiplication: if we add multiples of 12 to the numbers being subtracted or multiplied, the answer will change by a multiple of 12, and will be the "same" modulo 12.

Addition, subtraction and multiplication are quite a large part of arithmetic, but what about division? The answer turns out to be that: Division works properly for arithmetic modulo N, if and only if N is a prime number.

Showing that it does work properly for a prime number will turn out to be equivalent to Euclid's Lemma.

Showing that it doesn't work properly for a non-prime greater than 1 (i.e. a composite number) is actually quite easy. For example, consider arithmetic modulo 10. Let's try dividing 6 by 2 modulo 10. It seems easy, because:

$6 \div 2 = 3$ We also have: $16 \div 2 = 8$

which is a different answer modulo 10, even though the numbers being divided by (i.e. 6 and 16), were the same modulo 10. The cause of this "problem" is not hard to find: the difference between 16 and 6 is 10, which is the modulus, and $10 \div 2 = 5$ because $2 \times 5 = 10$. This accounts for the two answers 3 and 8 differing by 5 even though

they would have been the same if division modulo 10 worked properly. We call 2 and 5 zero divisors, because they divide into "zero" (modulo 10) and this happens because 2 and 5 are factors of the modulus.
So we can see that division cannot work in modulo arithmetic if the modulus isn't prime. But is it guaranteed to work if the modulus is a prime? If a number a is a zero divisor in arithmetic modulo a prime number p, then $a \times b$ must be a multiple of p for some number b, where neither a nor b are equal to zero modulo p (i.e. neither is a multiple of p).
We want to show that this is impossible.
This first thing to do is to simplify the situation which we are trying to prove is impossible. We can subtract multiples of p from both a and b until they are both less than p (because this won't alter the fact that they multiply to make a multiple of p).
The next thing to do is to find a smaller a. Now if a was equal to 1, then we couldn't find a smaller a. But we already know that a can't be 1, because $1 \times b = b$, which wouldn't be a multiple of p.
So suppose that a is not equal to 1, but it has the property that $a \times b = 0$ modulo p for some non-zero b. The smaller value of a that we want is the remainder after p is divided by a. This number is smaller than a, because remainders are always smaller than the number you are dividing by. And, it must have the property of being a zero-divisor. To show this, suppose that $p = a \times x + r$, where r is the remainder.
Then we can multiply this equation by b: $b \times p = b \times a \times x + b \times r$
Now $b \times p$ is a multiple of p, and $b \times a \times x$ must be a multiple of p because $b \times a$ is a multiple of p. Which means that $b \times r$ must be a multiple of p. We can think of r as inheriting the property of "being a multiple of p when multiplied by b" from a, where the inheritance occurs via the process of dividing the modulus by the divisor and keeping the remainder.
r is our "new" value for a. And we can repeat the process over and over again, each time inheriting the property of being a zero divisor of p. Since each new value is smaller than the previous value, we must eventually get a value of 1. Which we have already shown is impossible. So it must have been impossible for a to be a divisor of zero in our arithmetic modulo p.
Which means that division does work in arithmetic modulo p.
Which means that Euclid's Lemma does hold true.

Euclid's Division Lemma :
Euclid 's Division Lemma states that given positive integer a and b, there exist unique integers q and r satisfying

$$a = bq + r \; ; \; 0 \leq r < b.$$

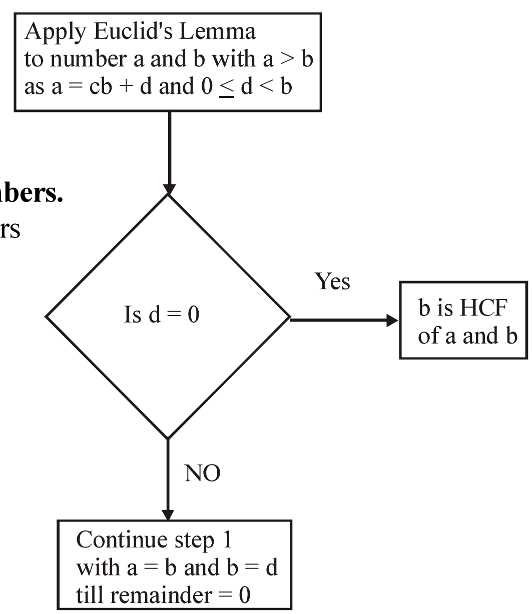
Euclid 's Division algorithm for finding the HCF of two numbers.
Euclid 's division algorithm is used to find the HCF of two numbers by the successive use of Euclid 's division lemma.
Let us find the HCF of 60 and 108 using this method.
Step 1: Since 108 > 60 applying Euclid's Lemma to 60 and 108, we have, $108 = 60 \;\; 1 + 48$ where $0 \leq 48 < 60$
Step 2: Since, remainder $48 \neq 0$
So, again applying the division lemma to 60 and 48, we have,

$$60 = 48 \times 1 + 12 \text{ where } 0 \leq 12 < 48.$$

Step 3: Again remainder $12 \neq 0$ so applying division lemma to 48 and 12, we get,

$$48 = 12 \times 4 + 0,$$ Here remainder is zero.

Therefore, 12 is the required HCF.
So, Euclid 's algorithm can be summarised as shown in the chart.



Example 1 :

To find the HCF of 1071 and 1029, using Euclid 's division algorithm.

Sol. Since, 1071 > 1029, we apply the division lemma to 1071 and 1029, to get

$1071 = 1029 \times 1 + 42$

Since, remainder $42 \neq 0$ so again applying division lemma in 1029 and 42, we get,

$1029 = 42 \times 24 + 21$ again $21 \neq 0$

Applying Euclid's Lemma again in 42 and 21, we get, $42 = 21 \times 2 + 0$

Since, remainder is zero so HCF is 21.

Example 2 :

Find the quotient and remainder q and r for the pairs of positive integers a and b given below:

(i) 23, 4 (ii) 81, 3 (iii) 12, 5 (i) 23, 4

When 23 is divided by 4 quotient is 5 remainder is 3.

Therefore, $23 = 5 \times 4 + 3$ $q = 5$; $r = 3$ and $0 \leq r < 5$

(ii) 81, 3 : When 81 is divided by 3 quotient is 27 and remainder is 0.

Therefore, $81 = 27 \times 3 + 0$. So, $q = 27$; $r = 0$ and $0 \leq r < 27$

(iii) 12, 5 : On dividing 12 by 5, we have quotient is 2 and remainder 2.

Therefore, $12 = 5 \times 2 + 2$.

So, $q = 2$; $r = 2$ and $0 \leq r < 5$.

RATIONAL NUMBERS

These are real numbers which can be expressed in the form of p/q where p and q are integers and $q \neq 0$ eg. 2/3, 37/15, –17/19.

All natural numbers, whole numbers and integers are rational.

Fractions :

Common fraction	:	Fractions whose denominator is not 10.
Decimal fraction	:	Fractions whose denominator is 10 or any power of 10.
Proper fraction	:	Numerator < Denominator
Improper fraction	:	Numerator > Denominator, mixed fraction.

Examples :

(i) 0 can be written as $\frac{0}{1}$, which is rational. $\therefore$ 0 is a rational number.

(ii) Every integer a can be written as $\frac{a}{1}$, which is rational. $\therefore$ Every integer is a rational number.

(iii) The square root of every perfect square number is rational.

e.g., $\sqrt{4} = 2$, which is rational similarly, $\sqrt{9}, \sqrt{16}, \sqrt{25}$ etc. are all rational.

(iv) Every terminating decimal is a rational number.

e.g., $0.7 = \frac{7}{10}$, which is rational, $0.375 = \frac{375}{1000}$, which is rational

(v) Every recurring decimal is a rational number

Let us consider the recurring decimal 0.333 ....

Let $x = 0.3333$ .... (1)

Then, $10x = 3.3333$ .... (2)

On subtracting (1) from (2) , we get, $9x = 3 \Leftrightarrow x = \frac{3}{9} = \frac{1}{3}$ $\therefore$ $0.333.......... = \frac{1}{3}$, which is rational.

Irrational number :

Every non-terminating and non-repeating decimal number is known as an irrational number e.g. 0.101001000100001.... **Example :** $\sqrt{2}, \sqrt{5}, \pi$ etc.

Properties of Rational numbers : If a, b, c are three rational numbers.

(i) Communicative property of addition. $a + b = b + a$

(ii) Associative property of addition $(a + b) + c = a + (b + c)$

(iii) Additive inverse $a + (-a) = 0$

0 is identity element, – a is called inverse of a.

(iv) Communicative property of multiplication $a . b = b . a$

(v) Associative property of multiplication $(a.b).c = a.(b.c)$

(vi) Multiplication inverse $a. 1/a = 1$

1 is called multiplication identity & $\frac{1}{a}$ is called multiplicative inverse of a or reciprocal of a.

(vii) Distributive property $a.(b + c) = a.b + a.c$

Operations on rational numbers : For any rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, we have

(i) Addition : $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$

(ii) Subtraction : $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$

(iii) Multiplication : $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$

(iv) Division : $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$, when $c \neq 0$

Density property of Rational numbers :

Between any two different rational numbers, there are infinitely many rational numbers.

Theorem 1 :

Let p be a prime number. If p divides a^2, then p divides a, where a is a positive integer.

Proof : Let the prime factorisation of a be as follows :

$a = p_1p_2 \ldots p_n$, where $p_1, p_2, \ldots, p_n$ are primes, not necessarily distinct.

Therefore, $a^2 = (p_1p_2 \ldots p_n)(p_1p_2 \ldots p_n) = p_1^2p_2^2 \ldots\ldots p_n^2$

Now, we are given that p divides a^2. Therefore, from the Fundamental Theorem of Arithmetic, it follows that p is one of the prime factors of a^2. However, using the uniqueness part of the Fundamental Theorem of Arithmetic, we realise that the only prime factors of a^2 are $p_1, p_2, \ldots, p_n$. So p is one of $p_1, p_2, \ldots, p_n$.

Now, since $a = p_1 p_2 \ldots p_n$, p divides a.

We are now ready to give a proof that $\sqrt{3}$ is irrational.

The proof is based on a technique called 'proof by contradiction'.

Example 3 :

Prove that $\sqrt{3}$ is irrational.

Sol. Let us assume, to the contrary, that $\sqrt{3}$ is rational. That is, we can find integers a and b ($\neq 0$) such that $\sqrt{3} = \frac{a}{b}$.

Suppose a and b have a common factor other than 1, then we can divide by the common factor, and assume that a and b are coprime. So, $b\sqrt{3} = a$

Squaring on both sides, and rearranging, we get $3b^2 = a^2$.

Therefore, a^2 is divisible by 3, it follows that a is also divisible by 3.
So, we can write a = 3c for some integer c.
Substituting for a, we get $3b^2 = 9c^2$, that is, $b^2 = 3c^2$.
This means that b^2 is divisible by 3, and so b is also divisible by 3.
Therefore, a and b have at least 3 as a common factor.
But this contradicts the fact that a and b are coprime.
This contradiction has arisen because of our incorrect assumption that $\sqrt{3}$ is rational.
So, we conclude that $\sqrt{3}$ is irrational.

Example 4 :

Prove that $2+\sqrt{3}$ is an irrational number.

Sol. If possible, let $2+\sqrt{3}$ be a rational number

$(2+\sqrt{3})^2$ is rational

$(4+3+2\times 2\times\sqrt{3})$ is rational ; $(7+4\sqrt{3})$ is rational.

But $\sqrt{3}$ is irrational $4\sqrt{3}$ is irrational ; $(7+4\sqrt{3})$ is irrational which is a contradiction.
So, our supposition is wrong.
Hence $(2+\sqrt{3})$ is an irrational number.

Decimal Expansion of Rational Numbers :

Theorem 2 : Let x be a rational number whose decimal expansion terminates. Then x can be expressed in the form $\frac{p}{q}$ where p and q are coprime, and the prime factorisation of q is of the form 2^n5^m, where n, m are non-negative integers. If we have a rational number of the form $\frac{p}{q}$ and the prime factorisation of q is of the form 2^n5^m, where n, m are non negative integers, then does $\frac{p}{q}$ have a terminating decimal expansion?

Let us see if there is some obvious reason why this is true. You will surely agree that any rational number of the form $\frac{a}{b}$ where b is a power of 10, will have a terminating decimal expansion. So it seems to make sense to convert a rational number of the

form $\frac{p}{q}$, where q is of the form 2^n5^m, to an equivalent rational number of the form $\frac{a}{b}$, where b is a power of 10.

Examples : (i) $\frac{3}{8}=\frac{3}{2^3}=\frac{3\times 5^3}{2^3\times 5^3}=\frac{375}{10^3}=0.375$ (ii) $\frac{13}{125}=\frac{13}{5^3}=\frac{13\times 2^3}{2^3\times 5^3}=\frac{104}{10^3}=0.104$

(iii) $\frac{7}{80}=\frac{7}{2^4\times 5}=\frac{7\times 5^3}{2^4\times 5^4}=\frac{875}{10^4}=0.0875$ (iv) $\frac{14588}{625}=\frac{2^2\times 7\times 521}{5^4}=\frac{2^6\times 7\times 521}{2^4\times 5^4}=\frac{233408}{10^4}=23.3408$

So, these examples show us how we can convert a rational number of the form $\frac{p}{q}$, where q is of the form 2^n5^m, to an equivalent rational number of the form $\frac{a}{b}$, where b is a power of 10. Therefore, the decimal expansion of such a rational number terminates.

Theorem 3 : Let $x = \frac{p}{q}$ be a rational number, such that the prime factorisation of q is of the form 2^n5^m, where n, m are non-negative integers. Then x has a decimal expansion which terminates.

We are now ready to move on to the rational numbers whose decimal expansions are non-terminating and recurring.
Once again, let us look at an example to see what is going on.
Here, remainders are 3, 2, 6, 4, 5, 1, 3, 2, 6, 4, 5, 1, . . . and divisor is 7.
Notice that the denominator here, i.e., 7 is clearly not of the form 2^n5^m.
Therefore, from Theorems 2 and 3, we know that

$\frac{1}{7}$ will not have a terminating decimal expansion. Hence, 0 will not show up as a remainder and the remainders will start repeating after a certain stage. So, we will have a block of digits, namely, 142857, repeating in the quotient of $\frac{1}{7}$.

```
  0.1428571
7)10
  7
  30
  28
   20
   14
   60
   56
   40
   35
   50
   49
   10
    7
   3 0
```

What we have seen, in the case of $\frac{1}{7}$, is true for any rational number not covered by Theorems 3 and 4. For such numbers we have:

Theorem 4 : Let $x = \frac{p}{q}$ be a rational number, such that the prime factorisation of q is not of the form 2^n5^m, where n, m are non-negative integers. Then, x has a decimal expansion which is non-terminating repeating (recurring).
From the discussion above, we can conclude that the decimal expansion of every rational number is either terminating or non-terminating repeating.

Example 5 :
Without performing the long division, state whether the following rational numbers will have a terminating or non-terminating repeating decimal expansion:

(i) $\frac{13}{64}$ (ii) $\frac{7}{80}$ (iii) $\frac{25}{2^3 \times 5^7 \times 7}$ (iv) $\frac{75}{1230}$

Sol. (i) $\frac{13}{64}$. Here denominator q = 64, Prime factors of 64 = 26, which is of the form 2^n5^m with n = 6 and m = 0
Therefore, decimal expansion will terminate.

(ii) $\frac{7}{80}$. Here denominator = 80, Prime factors of 80 , which is given of the form 2^n5^m with n = 4 and m = 0.
Therefore, decimal expansion will terminate.

(iii) $\frac{25}{2^3 \times 5^7 \times 7} = \frac{5^2}{2^3 \times 5^7 \times 7} = \frac{1}{2^3 \times 5^5 \times 7}$
Denominator of the above rational number is not of the form 2^n5^m hence the number is repeating.

(iv) $\frac{75}{1230} = \frac{3 \times 5 \times 5}{2 \times 5^4} = \frac{3}{2 \times 5^2}$
Since, the prime factorisation of denominator is of form 2^n5^m with n = 1, m = 2.
So, the decimal expansion will terminate.

Divisibility Tests : A positive integer is divisible by the given integer if the condition is met.

1. Every positive integer is divisible by 1.

2. If the last digit is a 0, 2, 4, 6, or 8

3. If the sum of the digits is divisible by 3

4. If the last two digits are divisible by 4

5. If the last digit is a 0 or 5

6. If the number is divisible by both 2 and 3.

7. If the last three digits are divisible by 8

8. If the sum of the digits is divisible by 9

9. If the last digit is a 0

10. Subtract the sum of the digits in the even positions (2nd digit, 4th digit, etc) from the sum of the digits in the odd positions (1st digit, 3rd digit, etc). If this difference is divisible by 11, then the number is divisible by 11.

A test may be constructed for numbers such as 12, 15, and 18 according to the following rule. If a number can be factored so that the factors are relatively prime (that is, they have no common factors besides one), then the test for divisibility for that number the requirement that the number be divisible by the factors. 12 is a factor if 3 and 4 are both factors, but not necessarily if 2 and 6 are factors. 18 is a factor if 2 and 9 both are, but not necessarily if 3 and 6 both are. 14 is a factor if both 2 and 7 are, but there isn't an easy test for 7.

Divisibility by 7, 11, and 13

Gustavo Toja from Brasil came up with an interesting method of divisibility by 7, the method also works to verify the divisibility by 11 and 13. Accordingly, let F be one of these numbers, 7, 11, or 13.

Any given number A written in the decimal system admits a representation into two digit numbers:

For example, $1234567 = 10^6.1 + 10^4.23 + 10^2.45 + 67$.

As we see, the leftmost member of the representation may be a one-digit number.

The procedure is as follows: Find the remainders d_i of division of a_i by F. Alternately, replace d_i by $(F - d_i)$ and write the result in the reverse order. The resulting number will be divisible by F depending whether A is itself divisible or not.

If the divisibility of A is all we need to determine, it does not matter where you start replacing d_i with its additive inverse

$(F - d_i)$. One can begin with the first digit as well as the second. However, it is quite advantageous to make sure that the first digit of the number written in reverse has not been modified. Thus given a sequence of remainders, replace them with their additive inverses starting from the second remainder on the right and move leftwards. This will guarantee that the rightmost remainder has not been changed.

If you prefer to think in terms of "moving rightwards", then the rule is this: Let n be the number of the 2-digit parts of A. For n even, replace di starting with dn-1, the leftmost remainder; for n odd, start with dn-2, the second remainder.

Example: check that A = 38391787 is divisible by 7.

First split A into two digit numbers: 38 39 17 87

Next find the remainders of division of these numbers by 7: 3 4 3 3

Replace every other remainder with its additive inverse modulo 7: 4 4 4 3

Consider the number having the above digits but written in the reverse order: 3444. Apply the same procedure to 3444: $34\ 44 \Rightarrow 6\ 2 \Rightarrow 1\ 2 \Rightarrow 21$. Since 21 is divisible by 7, so is 3444 and also 38391787

Example: F = 11 and A = 4711927.

$4\ 71\ 19\ 27 \Rightarrow 4\ 5\ 8\ 5 \Rightarrow 7\ 5\ 3\ 5 \Rightarrow 5357 \Rightarrow 53\ 57 \Rightarrow 9\ 2 \Rightarrow 2\ 2 \Rightarrow 22$.

Conclude: since 22 is divisible by 11, so is 5357, and so is 4711927.

Example: F = 13, A = 61255051.

$61\ 25\ 50\ 51 \Rightarrow 9\ 12\ 11\ 12 \Rightarrow 4\ 12\ 2\ 12$.

In this case, the remainders written in the reverse order give the sequence: 12 2 12 4. To form a number, the left digit of the two digit "parts" must be carried over:

$12\ 2\ 12\ 4 \Rightarrow 12324 \Rightarrow 1\ 23\ 24 \Rightarrow 1\ 10\ 11 \Rightarrow 1\ 3\ 11 \Rightarrow 11\ 3\ 1 \Rightarrow 1131 \Rightarrow 11\ 31 \Rightarrow 11\ 5 \Rightarrow 2\ 5 = 52$.

since 52 is divisible by 13, so is 12324 and, consequently, 61255051.

ADDITIONAL EXAMPLES

Example 1 :

If p is prime and p | ab, then show that p | a or p | b.

Sol. $\because$ p | ab

Therefore, there exist an integer c such that

ab = pc .... (1) $\because$ p is prime

$\therefore$ either p | a or (p, a) = 1

If p | a, we are done. If (p, a) = 1

$\therefore$ There exist integers m and n such that pm + an = 1

Multiplying both sides by b, pmb + (ab) n = b

on putting ab = pc from (1) $\therefore$ pmb + pcn = b or p (mb + cn) = b $\therefore$ p | b

Example 2 :

Show that $5\sqrt{3}$ is an irrational number.

Sol. If possible, let $5\sqrt{3}$ be a rational number.

So, $5\sqrt{3}$ = p/q where p and q are co prime integers and q ≠ 0 So, $\sqrt{3} = \frac{p}{5q}$

So, RHS is a rational number and hence $\sqrt{3}$ is also rational which is a contradiction.

So our supposition is wrong. Hence, 5 is an irrational number.

Example 3 :

When 2^{256} is divided by 17 the remainder would be –

(A) 1 (B) 16 (C) 14 (D) None of these

Sol. **(A).** When 2^{256} is divided by 17 then $\Rightarrow \frac{2^{256}}{2^4+1} \Rightarrow \frac{(2^2)^{64}}{(2^4+1)}$

By remainder theorem when f (x) is divided by x + a the remainder = f (– a)

Here f (a) = $(2^2)^{64}$ and x = 2^4 and a = 1

$\therefore$ Remainder = f (–1) = $(-1)^{64}$ = 1

Example 4 :

Find the LCM of $2x^4 - 32$, $2x^4 - 4x^3 + 8x^2 - 16x$

Sol. $p(x) = 2x^4 - 32 = 2(x^4 - 16)$

$= 2[(x^2)^2 - 4^2] = 2(x^2 + 4)(x^2 - 4) = 2(x^2 + 4)(x + 2)(x - 2)$

$q(x) = 2x^4 - 4x^3 + 8x^2 - 16x$

$= 2x(x^3 - 2x^2 + 4x - 8) = 2x[x^2(x - 2) + 4(x - 2)] = 2x(x - 2)(x^2 + 4)$

$\therefore$ LCM of p (x) and q (x) $= 2x(x - 2)(x + 2)(x^2 + 4) = 2x(x^2 - 4)(x^2 + 4) = 2x(x^4 - 16) = 2x^5 - 32x$

Example 5 :

The HCF of two polynomials is $x^2 - 1$ and their LCM is $x^4 - 10x^2 + 9$. If one of the polynomials is $x^3 - 3x^2 - x + 3$, find the other.

Sol. Given that HCF of p (x) and q (x) $= x^2 - 1 = (x + 1)(x - 1)$

Also, LCM of p (x) and q (x) $= x^4 - 10x^2 + 9 = x^4 - 9x^2 - x^2 + 9$

$= x^2(x^2 - 9) - (x^2 - 9) = (x^2 - 9)(x^2 - 1) = (x + 3)(x - 3)(x + 1)(x - 1)$

and $p(x) = x^3 - 3x^2 - x + 3 = x^2(x-3) - (x-3) = (x-3)(x^2-1) = (x-3)(x+1)(x-1)$

$p(x).q(x) = (HCF).(LCM)$

$$\therefore \quad q(x) = \frac{(HCF)\ (LCM)}{p(x)} = \frac{(x+1)(x-1)(x+3)(x-3)(x+1)(x-1)}{(x-3)(x+1)(x-1)}$$

$= (x+3)(x+1)(x-1) = (x+3)(x^2-1) = x^3 + 3x^2 - x - 3$

Example 6 :

Find the HCF and LCM of 6, 72 and 120, using the prime factorisation method.

Sol. We have : $6 = 2 \times 3$, $72 = 2^3 \times 3^2$, $120 = 2^3 \times 3 \times 5$

Here, 2^1 and 3^1 are the smallest powers of the common factors 2 and 3 respectively.

So, HCF (6, 72, 120) $= 2^1 \times 3^1 = 2 \times 3 = 6$

2^3, 3^2 and 5^1 are the greatest powers of the prime factors 2, 3 and 5 respectively involved in the three numbers.

So, LCM (6, 72, 120) $= 2^3 \times 3^2 \times 5^1 = 360$.

Example 7 :

Two bills of Rs 6075 and Rs 8505 respectively are to be paid separately by cheques of same amount. Find the largest possible amount of each cheque.

Sol. Largest possible amount of cheque will be the HCF (6075, 8505).

Applying Euclid's division lemma to 8505 and 6075, we have, $8505 = 6075 \times 1 + 2430$

Since, remainder $2430 \neq 0$ again applying division lemma to 6075 and 2430

$6075 = 2430 \times 2 + 1215$. Again remainder $1215 \neq 0$

So, again applying the division lemma to2430 and 1215, $2430 = 1215 \times 2 + 0$

Here the remainder is zero. So, HCF = 1215

Therefore, the largest possible amount of each cheque will be 1215.

Example 8 :

Using Euclid 's division lemma show that square of any positive integer is either of the form 3m or 3m + 1 for some integer m.

Sol. Let a, b = 3 be any positive integers

Therefore, using Euclid's lemma, we have, $a = 3q + r$ where $0 \leq r < 3$

So r can take any of the values 0, 1, .2

Therefore, a can take either of the values 3q, 3q + 1 or 3q + 2

Now, $(3q)^2 = 9q^2 = 3 \, . \, 3q^2$; $(3q+1)^2 = 9q^2 + 6q + 1 = 3q(3q+2) + 1$

$(3q+2)^2 = 9q^2 + 12q + 4 = 9q^2 + 12q + 3 + 1 = 3(3q^2 + 4q + 1) + 1$

Now, if any of the above numbers is divided by 3 then remainder is 0 or 1.

So, it is of the form 3m or 3m + 1 for some integer m.

Example 9 :

Write the decimal expansion using prime factorisation : **(i)** $\frac{35}{16}$ **(ii)** $\frac{17}{8}$ **(iii)** $\frac{327}{500}$

Sol. **(i)** $\frac{35}{16} = \frac{35 \times 5^4}{2 \times 5^4} = \frac{35 \times 625}{(10)^4} = \frac{21875}{10000} = 2.1875$ **(ii)** $\frac{17}{8} = \frac{17 \times 5^3}{2^3 \times 5^3} = \frac{17 \times 125}{(10)^3} = \frac{2125}{1000} = 2.125$

(iii) $\frac{327}{500} = \frac{327}{5 \times 5 \times 5 \times 2 \times 2} = \frac{327}{5^3 \times 2^2} = \frac{327 \times 2}{5^3 \times 2^3} = \frac{654}{(10)^3} = 0.654$

Example 10 :

Show that any positive odd integer is of the $8q + 1$, $8q + 3$, $8q + 5$, $8q + 7$, where q is some integer.

Sol. Let a and $b = 8$ be two positive integers where a is odd.

Applying division lemma $a = 8q + r$ where $0 \leq r < 8$

So, r can take any of the values 0, 1, 2, 3, 4, 5, 6, 7

Therefore, $a = 8q, 8q + 1, 8q + 2, 8q + 3, 8q + 4, 8q + 5, 8q + 6, 8q + 7, 8q + 8$

Since, a is odd.

Therefore, a cannot take values $8q$, $8q + 2$, $8q + 4$, $8q + 8$ since they can expressed as multiples of 2.

So, a will take values $8q + 1$, $8q + 3$, $8q + 5$, $8q + 7$.

Also, $8q + 5 = 8q + 8 - 3 = 8(q + 1) - 3 = 8q' - 3$, where $q' = q + 8q + 7 = 8q + 8 - 3 = 8q' - 1$

So, every positive odd integer is of the form $8q \pm 1$, $8q \pm 3$.

Example 11 :

A garden consists of 135 rose plants planted in certain number of columns. There are another set of 225 marigold plantlets which is to be planted in the same number of columns. What is the maximum number of columns in which they can be planted?

Sol. To find the maximum number of columns we need to find the HCF(135, 225)

Using Euclid's algorithm, we have, $225 = 135 \times 1 + 90$

Since, remainder $90 \neq 0$

So, again applying division lemma, we have, $135 = 90 \times 1 + 45$

Remainder $45 \neq 0$ again using Euclid 's division lemma, we have, $90 = 45 \times 2 + 0$. Since, remainder is 0

So, HCF = 45 . Therefore, 45 is the maximum number of columns in which the plants can be planted.

Example 12 :

Find the GCD of : $14x^3 + 14$, $42(x^2 + 4x + 3)(x^2 - x + 1)$

Sol. $p(x) = 14x^3 + 14 = 14(x^3 + 1) = 2 \times 7(x + 1)(x^2 - x + 1)$

$q(x) = 42(x^2 + 4x + 3)(x^2 - x + 1)$

$= 42(x^2 + 3x + x + 3)(x^2 - x + 1) = 42[x(x + 3) + (x + 3)](x^2 - x + 1)$

$= 2 \times 3 \times 7(x + 3)(x + 1)(x^2 - x + 1)$

$\therefore$ GCD of p (x) and q (x) $= 14(x + 1)(x^2 - x + 1) = 14(x^3 + 1)$

CONCEPT MAP

REAL NUMBERS

* **Euclid's division lemma:**
  Given positive integers a and b, there exist whole numbers q and r satisfying $a = bq + r$, $0 \le r < b$.
* **Euclid's division algorithm :** According to this, the **HCF** of any two positive integers **a and b**, with $a > b$, is obtained as follows:
  Step-I : Apply the division lemma to find q and r where $a = bq + r$, $0 \le r < b$.
  Step-II : If $r = 0$, the HCF is b. If $r \neq 0$, apply Euclid's lemma to b and r.
  Step-III : Continue the process till the remainder is zero. The divisor at this stage will be HCF (a, b).

* **The Fundamental Theorem of Arithmetic:**
  Every composite number can be expressed (factorised) as a product of prime numbers, and this factorization is unique, apart from the order in which the prime factors occur.
* **Applications :**
  - to study the certain properties of integers
  - to find the H.C.F and L.C.M (prime factorization method) of positive integers.
* **Properties of Real Numbers**

1. If p is a prime and p divides a^2, then p divides a, where a is a positive integer.

2. If for a rational number of the form p / q, where p and q are co prime, the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers, then the decimal expansion of this number is terminating.

3. If decimal expansion of a rational number is terminating the prime factorization of q (if the number is in the form p/q) is of the form 2^n5^m, where n, m are non-negative integers.

4. If for a rational number of the form p / q, where p and q are co prime, the prime factorization of q is not of the form 2^n5^m, where n, m are non-negative integers, then the decimal expansion of this number is non-terminating recurring.

* If HCF or LCM of positive integers say a, b, c and d are to be obtained, then

1. Express each of a, b, c and d as the product of positive integral powers of prime numbers.

2. (i) HCF (a, b, c, d) = product of least positive integral powers of common prime numbers involved in the factorisation of a, b, c and d.
(ii) LCM (a, b, c, d) = product of greatest positive integral powers of all prime numbers involved in the factorisation of a, b, c and d.

3. Relation between LCM and HCF of two positive integers a and b and their product is : LCM (a, b) . HCF (a, b) = a . b
But this is not true for three or more positive integers.

* To prove that a given number (not in decimal form) is irrational
  (i) Let the given number be x.
  (ii) Let x = p/q, where p and q are integers. Cancel out the common factors of p and q. Then x = m/n, where m and n are coprimes.
  (iii) Cross multiply and bring expression of the form $\sqrt{r}$ where r is not the square of an integer, on left side and other terms on right side and then square. If $\sqrt[3]{r}$ occurs cube both sides.
  (iv) Now use the fundamental theorem of arithmetic according to which if prime number p divides a^2, a being an integer, then p divides a.
  (v) Finally try to get a common factor other than 1 in m and n which contracticts our assumption.
  (vi) In some cases, we can arrive a contradiction by getting an equality which has a rational number on one side and irrational number on the other side.