2

POLYNOMIALS

INTRODUCTION

In IX class we have studied little about polynomials. Let us first review some basic concepts and than we will learn about geometrical meaning of the zeroes of the polynomials and relation between zeroes and coefficient of polynomials.

An algebraic expression f (x) of the form $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$. Where $a_0, a_1, a_2, ...,a_n$ are real numbers and all the index of x are non-negative integers is called a polynomial in x and the highest index n is called the degree of the polynomial. Here $a_0 , a_1x , a_2x^2 , ... + a_nx^n$ are called the terms of the polynomial and $a_0, a_1, a_2 .... a_n$ are called various coefficients of the polynomial f (x). A polynomial in x is said to be in standard form when the terms are written either increasing order or decreasing order of the indices of x in various terms.

A symbol which takes an various numerical values is known as a variable or a literal.

Like terms : Terms having same literal coefficients are called like terms, otherwise they are called unlike terms.

If $a_0, a_1, a_2, ...a_n$ are all integers then f (x) is said to be a polynomial over integers.

If a polynomial involves two or more variables, then the sum of the powers of all the variables in each term is taken up and the highest sum so obtained is the degree of the polynomial.

Examples :

(1) $6x^7 - 5x^4 + 2x + 3$ is a polynomial of degree 7.

(2) $2 + 5x^{3/2} + 7x^2$ is an expression but not a polynomial , since it contains a term in which power of x is 3/2, which is not a non-negative integer.

(3) $3ab^2 - 4\ a\sqrt{b}\ + 5b^3$ is an expression but not a polynomial, as it contains a term in which the sum of the powers of the variables is $\frac{3}{2}$, which is not a non-negative integer.

(4) In $-9y^2$, the numerical coefficient is -9 and literal coefficient is y^2.

(5) $6a^2, -8b^2. - 4\ ab$ are unlike terms.

Different types of polynomials :

There are four types of polynomials based on degrees. These are listed below :

(i) **Linear polynomials :** A polynomial of degree one is called a linear polynomial. The general formula linear polynomial is ax + b, where a and b are any real constant and $a \neq 0$.
Example : (3 + 5x) is a linear polynomial.

(ii) **Quadratic polynomials :** A polynomial of degree two is called a quadratic polynomial. The general form of quadratic polynomial is $ax^2 + bx + c$, where $a \neq 0$.
Example : $2y^2 + 3y - 1$ is a quadratic polynomials.

(iii) **Cubic polynomials :** A polynomial of degree three is called a cubic polynomial. The general form of a cubic polynomial is $ax^3 + bx^2 + cx + d$, where $a \neq 0$.
Example : $6x^3 - 5x^2 + 2x + 1$ is a cubic polynomial.

(iv) **Biquadratic polynomials :** A polynomial of degree four is called a biquadratic polynomial. The general form of a biquadratic polynomial is $ax^4 + bx^3 + cx^2 + dx + e$ where $a \neq 0$.

A polynomial of degree five or more than five does not have any particular name. Such a polynomial usually called a polynomial of degree five or six or .... etc. There are three types of polynomials based on number of terms. These are as follows :

(a) Monomial : A polynomial is said to be a monomial if it has only one term.
For Ex., $-3x^2$, $5x^3$, $10x$ are monomials.
(b) Binomial : A polynomial is said to be a binomial if it contains two terms.
For example, $(2x^2 + 5)$, $(3x^3 - 7)$, $(6x^2 + 8x)$ are binomials.
(c) Trinomials : A polynomial is said to be a trinomial if it contains three terms.

For example, $3x^3 - 8x + \frac{5}{2}$, $\sqrt{7}x^{10} + 8x^4 - 3x^2$, $5 - 7x + 8x^9$ etc. are trinomials.

ZERO DEGREE POLYNOMIAL

Any non-zero number is regarded as a polynomial of degree zero or zero degree polynomial.
For example, $f(x) = a$, where $a \neq 0$ is a zero degree polynomial, since we can write $f(x) = a$ as $f(x) = ax^0$.
Zero polynomial : A polynomial whose coefficient are all zeros is called a zero polynomial i.e., $f(x) = 0$ but we cannot talk about the degree of a zero polynomial.

Example 1 :
Which of the following functions are polynomials ?

(a) $5x^2 - 3x + 9$ (ii) $\frac{4}{3}x^7 - 5x^4 + 3x^2 - 1$

Sol. Both are polynomials.

Example 2 :
Write down the degree of the following polynomials.
(a) $3x + 5$ (b) $3t^2 - 5t + 9t^4$. (c) $2 - y^2 - y^3 + 2y^8$

Sol. (a) The highest power term is 3x and its exponent is 1.
$\therefore$ degree $(3x + 5) = 1$
(b) The highest degree term is $9t^4$ and its exponent is 4. So, degree of the given polynomial is 4.
(c) The highest power of the variable is 8. So, the degree of the polynomial is 8.

Example 3 :
Write the following polynomials in standard form :

(i) $x^6 - 3x^4 + \sqrt{2}x + \frac{5}{2}x^2 + 7x^5 + 4$

Sol. The given polynomials in standard form are :

(i) $x^6 + 7x^5 - 3x^4 + \frac{5}{2}x^2 + \sqrt{2}x + 4$

Value of a polynomial : The value of a polynomial f (x) at $x = \alpha$ is obtained by substituting $x = \alpha$ in the given polynomial and is denoted by $f(\alpha)$.
Consider the polynomial $p(x) = 5x^3 - 2x^2 + 3x - 2$
If we replace x by 1 everywhere in p (x), we get
$p(1) = 5 \times (1)^3 - 2 \times (1)^2 + 3 \times (1) - 2$
$= 5 - 2 + 3 - 2 = -2$
So, we say that the value of p (x) at x = 1 is 4.
Similarly, $p(0) = 5(0)^3 - 2(0)^2 + 3(0) - 2 = -2$
If P(x) = 0 then we say x as root of the polynomial (zeroes of a polynomial)

Example 4 :

If x = 4/3 is a root of the polynomial $f(x) = 6x^3 - 11x^2 + kx - 20$ then find the value of k.

Sol. $f(x) = 6x^3 - 11x^2 + kx - 20$; $f\left(\frac{4}{3}\right) = 6\left(\frac{4}{3}\right)^3 - 11\left(\frac{4}{3}\right)^2 + k\left(\frac{4}{3}\right) - 20 = 0$

$\Rightarrow 6.\frac{64}{9.3} - 11.\frac{16}{9} + \frac{4k}{3} - 20 = 0 \Rightarrow 128 - 176 + 12k - 180 = 0 \Rightarrow 12k + 128 - 356 = 0 \Rightarrow 12k = 228 \Rightarrow k = 19$

Example 5 :

If x = 2 and x = 0 are roots of the polynomials $f(x) = 2x^3 - 5x^2 + ax + b$. Find the values of a and b.

Sol. If $f(2) = 2(2)^3 - 5(2)^2 + a(2) + b = 0 \Rightarrow 16 - 20 + 2a + b = 0 \Rightarrow 2a + b = 4$

$\Rightarrow f(0) = 2(0)^3 - 5(0)^2 + a(0) + b = 0 \Rightarrow b = 0 \Rightarrow 2a = 4 \quad \Rightarrow a = 2, b = 0$

Basic operations with polynomials : The sum of two polynomials can be found by grouping like power terms, retaining their signs and adding the coefficients of like powers.

Also, the negative of a polynomial P (x) is a polynomial, to be denoted by P (x) and it is obtained by replacing each coefficient by its additive inverse. i.e., Change the sign of the term to be subtracted and add this new term with the first term from which subtraction is to be made.

Example 6 :

Add : $(4a^3 - 5a^2 + 6a - 3), (2 + 8a^2 - 3a^3), (9a - 3a^2 + 2a^3 + a^4), (1 - 2a - 3a^3)$

Sol. Arranging columnwise with like terms in same column and adding, we get

$$\begin{array}{l} \quad\;\; 4a^3 - 5a^2 + 6a - 3 \\ \;\; -3a^3 + 8a^2 \qquad\quad + 2 \\ a^4 + 2a^3 - 3a^2 + 9a \\ \;\; -3a^3 \qquad - 2a + 1 \\ \hline a^4 \qquad\qquad + 13a \\ \hline \end{array}$$

$\therefore$ Required term = $a^4 + 13a$

Example 7 :

If $p(y) = y^6 - 3y^4 + 2y^2 + 6$ and $q(y) = y^5 - y^3 + 2y^2 + y - 6$, find p (y) + q (y) and p (y) – q (y).

Sol.

$$p(y) = y^6 + 0y^5 - 3y^4 + 0y^3 + 2y^2 + 0y + 6$$
$$q(y) = \qquad y^5 + 0y^4 - y^3 + 2y^2 + y - 6$$

$\therefore \quad p(y) + q(y) = y^6 + y^5 - 3y^4 - y^3 + 4y^2 + y + 0$

Also, $-q(y) = -y^5 - 0y^4 + y^3 - 2y^2 - y + 6$

Now, $p(y) = y^6 + 0y^5 - 3y^4 + 0y^3 + 2y^2 + 0y + 6$

And, $-q(y) = -y^5 - 0y^4 + y^3 - 2y^2 - y + 6$

$\therefore \quad p(y) - q(y) = y^6 - y^5 - 3y^4 + y^3 + 0y^2 - y + 12$

Thus, $p(y) - q(y) = y^6 - y^5 - 3y^4 + y^3 - y + 12$

Multiplication of Monomials :

Product of monomials = (Product of their numerical coefficients) × (Product of their variable parts)

Multiplication of Two monomials :

Multiply each term of the multiplicand by each term of the multiplier and take the algebraic sum of these products.

Example 8 :

Find the product of $(x + 3)$ and $(x^2 + 4x + 5)$.

Sol. $(x + 3)(x^2 + 4x + 5) = x(x^2 + 4x + 5) + 3(x^2 + 4x + 5)$.

$= (x^3 + 4x^2 + 5x) + (3x^2 + 12x + 15) = x^3 + (4 + 3)x^2 + [5 + 12]x + 15$

$= x^3 + 7x^2 + 17x + 15$

Example 9 :

Multiply : $p(t) = t^4 - 6t^3 + 5t - 8$ and $q(t) = t^3 + 2t^2 + 7$.

Also, find the degree of p (t). q (t).

Sol.

$$
\begin{array}{r}
t^4 - 6t^3 + 5t - 8 \\
t^3 + 2t^2 + 0t + 7 \\
\hline
t^7 - 6t^6 + 0t^5 + 5t^4 - 8t^3 \qquad\qquad\qquad\qquad \\
+ 2t^6 - 12t^5 + 0t^4 + 10t^3 - 16t^2 \qquad\quad \\
+ 7t^4 - 42t^3 + 0t^2 + 35t - 56 \\
\hline
\end{array}
$$

$\therefore\ p(t).q(t) = t^7 - 4t^6 - 12t^5 + 12t^4 - 40t^3 - 16t^2 + 35t - 56$

The highest power term is t^7, and its exponent is 7.

$\therefore$ degree of p (t). q (t) is 7.

Division of polynomial by a Monomial : Let us divide = $4x^4 + 2x^3 + 2x^2$ by $2x^2$

$(4x^4 + 2x^3 + 2x^2) \div 2x^2 = 2x^2 + x + 1$

We can write $4x^4 + 2x^3 + 2x^2 = (2x^2)(2x^2 + x + 1)$

We say $2x^2$ and $2x^2 + x + 1$ are factors of $4x^4 + 2x^3 + 2x^2$

Let us consider another example : Divide $(6x^2 + 2x + 1)$ by x

$(6x^2 + 2x + 1) \div x = 3x + 2 + (1 \div x)$

We cannot divide 1 by x to get a polynomial term.

Division Alogrithm for Polynomial : On dividing a polynomial p (x) by a polynomial d (x), let the quotient be q (x) and the remainder be r(x), then $p(x) = d(x) . q(x) + r(x)$, where either $r(x) = 0$ or deg. $r(x) <$ deg.$d(x)$

Here , Dividend = p (x), Divisor = d (x), Quotient = q (x) and Remainder = r (x).

Division of a polynomial by a polynomial :

Step 1 : Arrange the terms of the dividend and the divisor in descending order of their degrees.

Step 2 : Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.

Step 3 : Multiply all the terms of the divisor by the first term of the quotient and subtract the result from the dividend.

Step 4 : Consider the remainder as new dividend and proceed as before.

Step 5 : Repeat this process till we obtain a remainder which is either 0 or a polynomial of degree less than that of the divisor.

Example 10 :

Divide $x^3 + x^2 - 2x - 30$ by $x - 3$

Sol. $(x - 3)$ $x^3 + x^2 - 2x - 30$ $x^2 + 4x + 10$

$$\begin{array}{l} x^3 + x^2 - 2x - 30 \\ x^3 - 3x^2 \\ \hline \quad 4x^2 - 2x - 30 \\ \quad 4x^2 - 12x \\ \hline \qquad 10x - 30 \\ \qquad 10x - 30 \\ \hline \qquad \times \end{array}$$

Example 11 :

Divide : $(15x^2 - 32y^2 + 38xy)$ by $(3x - 2y)$

Sol. Arranging the terms of the dividend and the divisor in descending order of powers of x and then dividing, we get

$$3x - 2y \overline{)15x^2 + 38xy - 32y^2}(5x + 16y$$

$$\begin{array}{l} 15x^2 - 10xy \\ \hline \quad 48xy - 32y^2 \\ \quad 48xy - 32y^2 \\ \hline \qquad \times \\ \hline \end{array}$$

$\therefore$ $(15x^2 - 32y^2 + 38xy) \div (3x - 2y) = (5x + 16y)$

Example 12 :

Find all the zeroes of $2x^4 - 3x^3 - 3x^2 + 6x - 2$, if you know that two of its zeroes are $\sqrt{2}$ and $-\sqrt{2}$.

Sol. Since two zeroes are $\sqrt{2}$ and $-\sqrt{2}$, $(x - \sqrt{2}), (x + \sqrt{2}) = x^2 - 2$ is a factor of the given polynomial.

Now, we divide the given polynomial by $x^2 - 2$

$$\begin{array}{r} 2x^2 - 3x + 1 \\ x^2 - 2 \overline{)2x^4 - 3x^3 - 3x^2 + 6x - 2} \\ \underset{-}{2x^4} \qquad \underset{+}{-4x^2} \qquad\qquad \\ \hline -3x^3 + x^2 + 6x - 2 \\ \underset{+}{-3x^3} \qquad \underset{-}{+6x} \qquad \\ \hline x^2 \qquad -2 \\ \underset{-}{x^2} \qquad \underset{+}{-2} \\ \hline 0 \\ \hline \end{array}$$

First term of quotient is $\frac{2x^4}{x^2} = 2x^2$

Second term of quotient is $\frac{-3x^3}{x^2} = -3x$

Third term of quotient is $\frac{x^2}{x^2} = 1$

So, $2x^4 - 3x^3 - 3x^2 + 6x - 2 = (x^2 - 2)(2x^2 - 3x + 1)$.

Now, by splitting $-3x$, we factorise $2x^2 - 3x + 1$ as $(2x - 1)(x - 1)$.

So, its zeroes are given by $x = \frac{1}{2}$ and $x = 1$. Therefore, the zeroes of the given polynomial are $\sqrt{2}, -\sqrt{2}, \frac{1}{2}$ & 1.

ZEROES/ROOTS OF POLYNOMIALS

x = r is a root or zero of a polynomial, P (x), if P (r) = 0.

In other words, x = r is a root or zero of a polynomial if it is a solution to the equation P (x) = 0

The process of finding the zeros of P (x) really amount to nothing more than solving the equation P (x) = 0 and you will lean how to do that for second degree (quadratic) polynomials quadratic equation chapter. So, to help illustrate some of the ideas we are going to look at the zeroes of a couple of second degree polynomials.

Let's first find the zeroes for $P(x) = x^2 + 2x - 15$. To do this we simply solve the following equation.

$$x^2 + 2x - 15 = (x + 5)(x - 3) = 0 \Rightarrow x = -5, x = 3$$

So, this second degree polynomial has two zeroes or roots.

Now, let's find the zeroes for $P(x) = x^2 - 14x + 49$. That will mean solving,

$$x^2 - 14x + 49 = (x - 7)^2 = 0 \Rightarrow x = 7$$

So, this second degree polynomial has a single zero or root. Also, recall that when we first looked at these we called a root like this a double root.

We solved each of these by first factoring the polynomial and then using the zero factor property on the factored form. When we first looked at the zero factor property we saw that it said that if the product of two terms was zero then one of the terms had to be zero to start off with.

The zero factor property can be extended out to as many terms as we need. In other words, if we've got a product of n terms that is equal to zero, then at least one of them had to be zero to start off with. So, if we could factor higher degree polynomials we could then solve these as well.

Let's take a look at a couple of these.

Example 13 :

Find the zeroes of each of the following polynomials.

(a) $P(x) = 5x^5 - 20x^4 + 5x^3 + 50x^2 - 20x - 40 = 5(x+1)^2(x-2)^3$

(b) $Q(x) = x^8 - 4x^7 - 18x^6 + 108x^5 - 135x^4 = x^4(x-3)^3(x+5)$

(c) $R(x) = x^7 + 10x^6 + 27x^5 - 57x^3 - 30x^2 + 29x + 20 = (x+1)^3(x-1)^2(x+5)(x-4)$

Sol. (a) $P(x) = 5x^5 - 20x^4 + 5x^3 + 50x^2 - 20x - 40 = 5(x+1)^2(x-2)^3$

In this case we do have a product of 3 terms however the first is a constant and will not make the polynomial zero. So, from the final two terms it looks like the polynomial will be zero for x = – 1 and x = 2. Therefore, the zeroes of this polynomial are x = – 1 and x = 2

(b) $Q(x) = x^8 - 4x^7 - 18x^6 + 108x^5 - 135x^4 = x^4(x-3)^3(x+5)$

We've also got a product of three terms in this polynomial. However since the first is now an x this will introduce a third zero. The zeroes for this polynomial are, x = – 5, x = 0 and x = 3

because each of these will make one of the terms, and hence the whole polynomial, zero.

(c) $R(x) = x^7 + 10x^6 + 27x^5 - 57x^3 - 30x^2 + 29x + 20 = (x+1)^3(x-1)^2(x+5)(x-4)$

With this polynomial we have four terms and the zeroes here are,

$$x = -5, x = -1, x = 1 \text{ and } x = 4$$

Graphs of polynomials : In algebraic language the graph of a polynomial f (x) is the collection (or set) of all points (x, y), where y = f (x). In geometrical or in graphical language the graph of a polynomial f (x) is a smooth free hand curve passing through points (x_1, y_1), (x_2, y_2), (x_3, y_3), ... etc. where y_1, y_2, y_3, ... are the values of the polynomial f (x) at x_1, x_2, x_3, ... respectively. In order to draw the graph of a polynomial f (x), we may follow the following alogrithm.

Alogrithm :

Step 1 : Find the values $y_1, y_2, ...y_n, ...$ of polynomial f(x) on different points $x_1, x_2, ...x_n, ...$ and prepare a table that gives values of y or f(x) for vareious values of x.

x :	x_1	x_2	.....	x_n	x_{n+1}	....
y = f(x) :	$y_1 = f(x_1)$	$y_2 = f(x_2)$	.....	$y_n = f(x_n)$	$y_{n+1} = f(x_{n+1})$	....

Step 2 : Plot that points $x_1, y_1), (x_2, y_2), (x_3, y_3), ...(x_n, y_n),.........$ on rectangular coordinate system. In plotting these points you may use different scales on the x and y-axis.

Step 3 : Draw a free hand smooth curve passing through points plotted in step II to get the graph of the polynomial f(x).

(a) Graph on a linear polynomial :

Consider a linear polynomial f(x) = ax + b, $a \neq 0$. In class IX we have studied that the graph of y = ax + b is a straight line. That is why f(x) = ax + b is called a linear polynomial. Since two points determine a straight line, so only two points need to plotted to draw the line y = ax + b. The line represented by y = ax + b crosses the x-axis at exactly one point, namely (–b/a, 0).

Example 14 :

Draw the graph of the polynomial f(x) = 2x – 5. Also, the find the coordinates of the point where it crosses X-axis.

Sol. Let y = 2x – 5

The following table list the values of y corresponding to different values of x.

x	1	4
y	–3	3

The points A (1, –3) and B (4, 3) are plotted on the graph paper on a suitable scale. A line is drawn passing through these points to obtain the graphs of the given polynomial.

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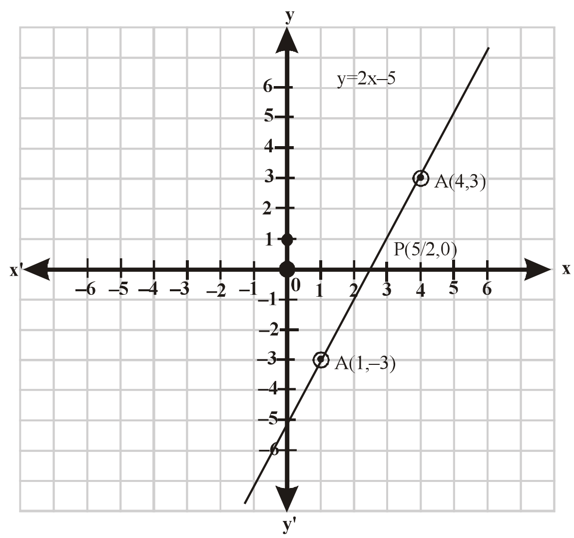


(b) Graph of quadratic polynomial :

Let a, b, c be real numbers and $a \neq 0$. Then the $f(x) = ax^2 + bx + c$ is known as quadratic polynomial in x. We shall discuss the graph of quadratic polynomial i.e. the curve whose equation is $y = ax^2 + bx + c, a \neq 0$. First of all we shall now that the graph of a quadratic polynomial is always a parabola.

Let $y = ax^2 + bx + c$ where $a \neq 0$

$\Rightarrow 4ay = 4a^2x^2 + 4abx + 4ac$

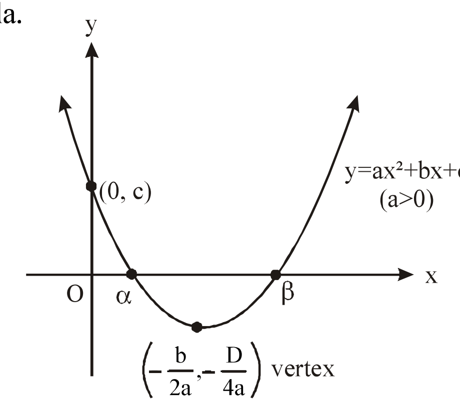
$\Rightarrow 4ay = 4a^2x^2 + 4abx + b^2 - b^2 + 4ac$

$\Rightarrow 4ay = (2ax + b)^2 - (b^2 - 4ac)$

$\Rightarrow 4ay + (b^2 - 4ac) = (2ax + b)^2$

$\Rightarrow 4ay + (b^2 - 4ac) = 4a^2 (x + b/2a)^2$

$$\Rightarrow 4a\left\{y + \frac{b^2 - 4ac}{4a}\right\} = 4a^2\left(x + \frac{b}{2a}\right)^2$$



$$\Rightarrow \left(y+\frac{D}{4a}\right)=a\left(x+\frac{b}{2a}\right)^2 \qquad .... (1)$$

where $D \equiv b^2 - 4ac$ is the discriminant of the quadratic equation.

Sign of quadratic expressions : Let α be a real root of $ax^2 + bx + c = 0$.

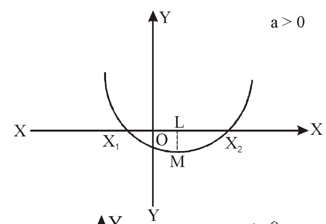
Then $a\alpha^2 + b\alpha + c = 0 \Rightarrow$ Point $(\alpha, 0)$ lies on $y = ax^2 + bx + c$. Thus, every real root of $ax^2 + bx + c = 0$ represents a point of intersection of the parabola with the x-axis. Conversely, if the parabola $y = ax^2 + bx + c$ intersects the x-axis at a point $(\alpha, 0)$, then $(\alpha, 0)$ satisfies the equation $y = ax^2 + bx + c \Rightarrow 0 = a\alpha^2 + b\alpha + c \Rightarrow$ α is a real root of $ax^2 + bx + c = 0$

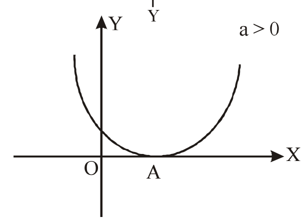
Thus, the intersection of the parabola $y = ax^2 + bx + c$ with x-axis gives us all the real roots of $ax^2 + bx + c = 0$.

Consequently, we may draw the following conclusions.

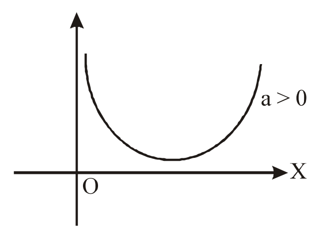
When a > 0

If roots are real and different (x_1 and x_2) then grap is like





And if Roots are equal (OA) then graph is like

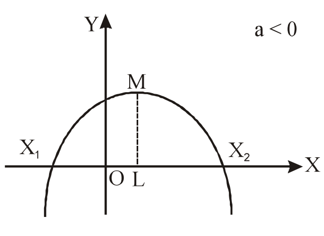


And if no real root exist graph looks like

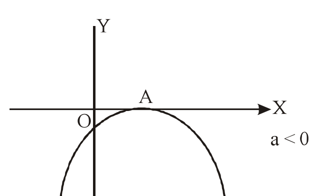
When a < 0

If roots are real and different (x_1 and x_2) then graphs (1) look like.

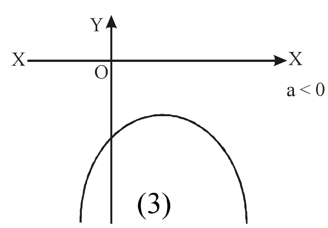
If Roots are equal =(OA) then graphs (2) look like



(1)



(2)



(3)

And if no real rot exists graph (3) look likes

Example 15 :

Draw the graph of the polynomial $f(x) = x^2 - 2x - 8$

Sol. Let $y = x^2 - 2x - 8$

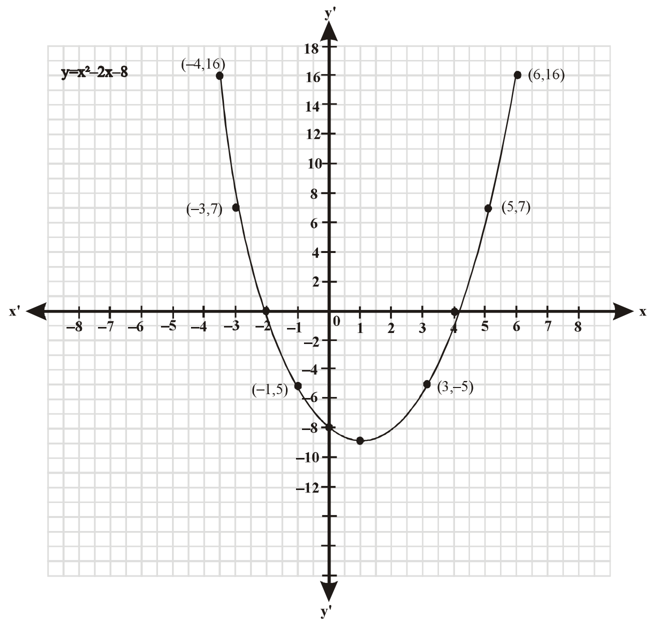
The following table gives the values of y or f (x) for various values of x.

x	– 4	– 3	– 2	– 1	0	1	2	3	4	5	6
$y = x^2 - 2x - 8$	16	7	0	– 5	–8	–9	–8	–5	0	7	16

Let us now plot the points (– 4, 16), (–3, 7), (–2, 0), (–1, –5), (0, –8), (1, –9), (2, –8), (3, –5), (4, 0), (5, 7) and (6, 16) on a graphs paper and draw a smooth free hand curve passing through these points.

The curve thus obtained represents the graphs of the polynomial $f(x) = x^2 - 2x - 8$. This is called parabola.

The lowest point P, called a minimum point, is the vertex of the parabola. Vertical line passing through P is called the axis of the parabola. Parabola is symmetric about the axis. So, it is also called the line of symmetry.



Observations :

From the graphs of the polynomial $f(x) = x^2 - 2x - 8$, we make the following observations :

(i) The coefficient of x^2 in $f(x) = x^2 - 2x - 8$ is 1 (a positive real number) and so the parabola opens upwards.

(ii) The polynomial $f(x) = x^2 - 2x - 8 = (x - 4)(x + 2)$ is factorizable into two distinct linear factors $(x - 4)$ and $(x + 2)$. So, the parabola cuts X-axis at two distinct points (4, 0) and (–2, 0). The x-coordinates of these points are zeroes of f (x).

(iii) The polynomial $f(x) = x^2 - 2x - 8$ has two distinct zeroes namely 4 and –2. So, the parabola cuts X-axis at (4, 0) and (–2, 0).

(iv) On comparing the polynomial $x^2 - 2x - 8$ with $ax^2 + bx + c$, we get a = 1, b = – 2 and c = – 8. The vertex of the parabola has coordinates (1, –9) i.e., $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$, where $D \equiv b^2 - 4ac$

(v) $D = b^2 - 4ac = 4 + 32 = 6 > 0$. So, the parabola cuts X-axis at two distinct points.

Example 16 :

Draw the graphs of the quadratic polynomial $f(x) = 3 - 2x - x^2$

Sol. Let y = f (x) or , $y = 3 - 2x - x^2$

Let us list a few values of $y = 3 - 2x - x^2$ corresponding to a few values of x as follows :

x	– 5	– 4	– 2	– 1	0	1	2	3	4
$y = 3 - 2x - x^2$	–12	– 5	3	4	3	0	–5	–12	–21

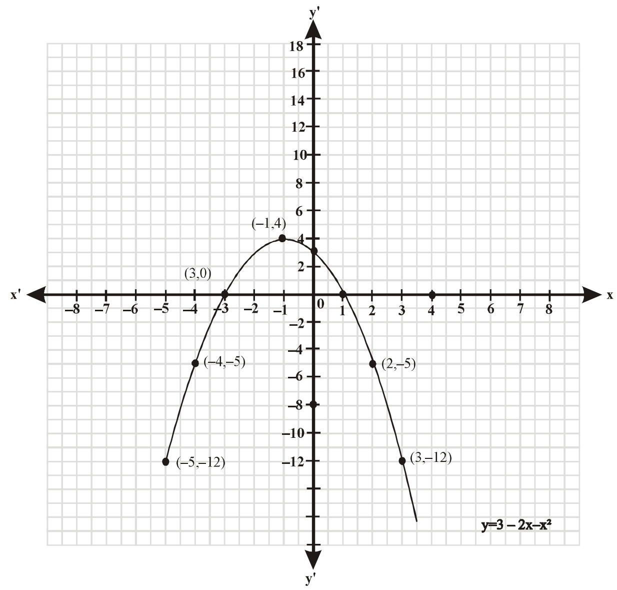
Thus, the following points lie on the graph of polynomial $y = 3 - 2x - x^2$:

(–5, –12), (–4, –5), (–3, 0), (–2, 3), (–1, 4), (0, 3), (1, 0), (2, –5), (3, – 12) and (4, –12)

Let us plot these points on a graph paper and draw a smooth free hand curve passing through these points to obtain the graphs of $y = 3 - 2x - x^2$. The curve thus obtained represents a parabola, as shown in figure. The highest point P (–1, 4) is called a maximum points, is the vertex of the parabola. Vertical line through P is the axis of the parabola. Clearly, parabola is symmetric about the axis.

Observations :

We make the following observations from the graph of the polynomial $f(x) = 3 - 2x - x^2$



(i) The coefficient of x^2 in $f(x) = 3 - 2x - x^2$ is -1 i.e. a negative real number and so the parabola opens downwards.

(ii) The polynomial $f(x) = 3 - 2x - x^2 = (1 - x)(x + 3)$ is facyorizable into two distinct linear factors $(1 - x)$ and $(x + 3)$. So, the parabola cuts X-axis at two distinct points (1, 0) and (–3, 0). The coordinate of these points are zeroes of $f(x)$.

(iii) The polynomial $f(x) = 3 - 2x - x^2$ has two distinct roots namely -3 and 1. So, the parabola $y = 3 - 2x - x^2$ cuts X-axis at two distinct points.

(iv) On comparing the polynomial $3 - 2x - x^2$ with $ax^2 + bx + c$, we have $a = -1$, $b = -2$ and $c = 3$. The vertex of the parabola is at the point (–1, 4) i.e. at $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$, while $D = b^2 - 4ac$.

(v) $D \equiv b^2 - 4ac = 4 + 12 = 16 > 0$. So, the parabola cuts X-axis at two distinct points.

Graphs of a Cubic Functions

A cubic is a function of the form $y = ax^3 + bx^2 + cx + d$

where $a \neq 0$, and a, b, c and d are constants. If $a > 0$ the graph of a cubic looks similar to one of the graphs in Figures 1, 2, and 3.

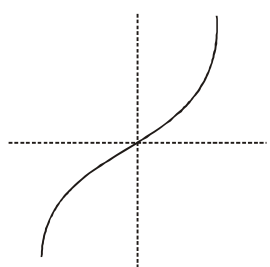


fig. 1

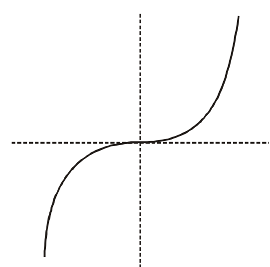


fig. 2

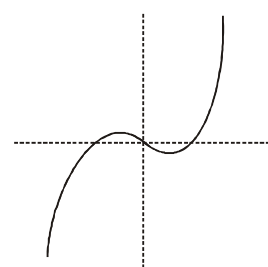


fig. 3

If $a < 0$, then the above curves will be reflected in the X axis as illustrated in Figures 4, 5, and 6.

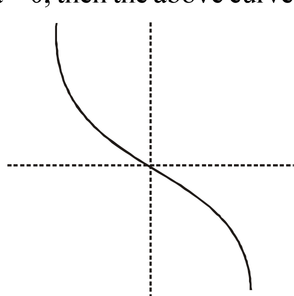


fig. 4

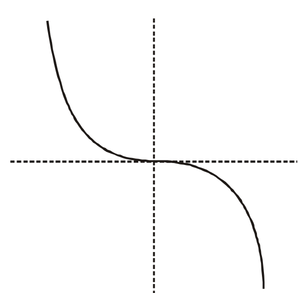


fig. 5

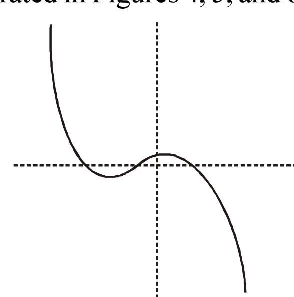


fig. 6

Let us consider a function of the form $y = ax^3 + bx^2 + cx + d$, where a is positive. All we know at the moment is that its graph looks like one of Figures 1, 2 or 3. We know that when x is a large positive number y will also be large and positive, while if x is a large negative number y will also be large and negative. This is because the term ax^3 in the equation of the polynomial is positive if $x > 0$ and negative if $x < 0$, and this term dominates the others for large values of x.

Example 17 :

Draw the graphs of the polynomial $f(x) = x^3 - 4x$.

Sol. Let $y = f(x) = x^3 - 4x$.

The values of y for variable value of x are listed in the following table :

x	– 4	– 3	– 2	– 1	0	1	2	3	4
$y = x^3 - 4x$	– 48	–15	0	3	0	–3	0	15	48

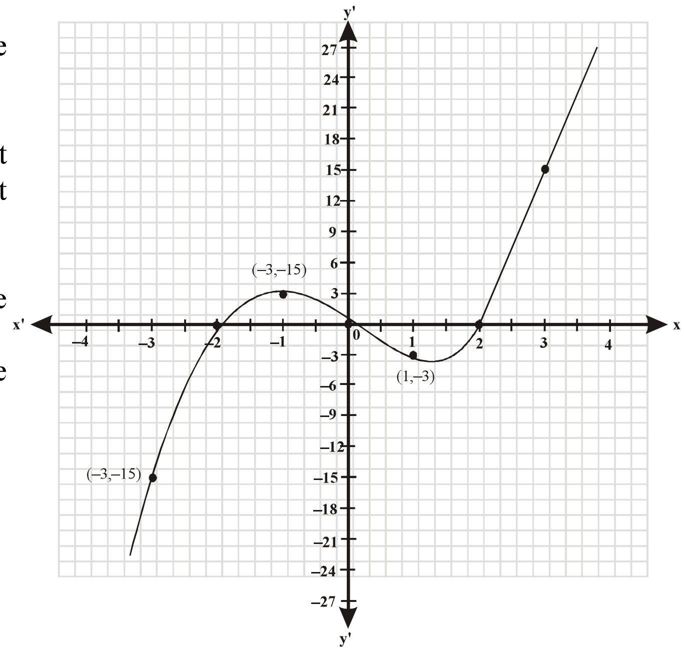
Observations :

From the graphs of the polynomial $f(x) = x^3 - 4x$, we make the following observations :

(i) The polynomial $f(x) = x^3 - 4x = x(x^2 - 4)$ $= x(x-2)(x+2)$ is a factorizable into three distinct linear factors. The curve $y = f(x)$ also cuts X-axis at three distinct points.

(ii) We have $f(x) = x(x-2)(x+2)$

Therefore, 0, 2 and –2 are three zeroes of f (x). The curve $y = f(x)$ cuts X-axis at three points O (0, 0), P (2, 0) and Q (–2, 0) whose x-coordinates are the zeroes of the polynomial f (x).



RELATIONSHIP BETWEEN ZEROES AND COEFFICIENT OF A POLYNOMIAL

Fundamental Theorem of Algebra

If P(x) is a polynomial function of degree n ($n > 0$) with complex coefficients, then the equation $P(x) = 0$ has n roots assuming you count double roots as 2, triple roots as 3, etc.

If P(x) is a polynomial with rational coefficients and a and b are also rational such that the square root of b is irrational, then if $a + \sqrt{b}$ is a root of the equation $P(x) = 0$, then $a - \sqrt{b}$ is also a root.

If P(x) is a polynomial of odd degree with real coefficients, then the equation $P(x) = 0$ has at least one real solution.

For a polynomial equation with an a_n the leading coefficient and a_0 as the constant then the following is true:

(a) the sum of the roots is $-\frac{a_{n-1}}{a_n}$ (b) the product of the roots is:

a_0/a_n if n is even ; $-a_0/a_n$ if n is odd

In a quadratic whose leading coefficient is a :

the sum of the roots is the (negative of the coefficient of x / coefficient of x^2)

and the product of the roots is the (constant term / coefficient of x^2)

That is, if $ax^2 + bx + c = 0$, and the roots are r and s, then $r + s = -b/a$, $rs = c/a$

For, if the roots are r and s, then the quadratic is $(x – r)(x – s) = x^2 – rx – sx + rs = x^2 – (r + s)x + rs$
The coefficient of x is $–(r + s)$, which is the negative of the sum of the roots. The constant term is rs, which is their product. To find equations given the root is: x^2 – (sum of the roots)x + (product of the roots) = 0

EXTRA EDGE

Complex Conjugates Theorem : If P(x) is a polynomial function with real coefficients, and a + bi is a solution of the equation P(x) = 0, then a - bi is also a solution.

ADDITIONAL EXAMPLES

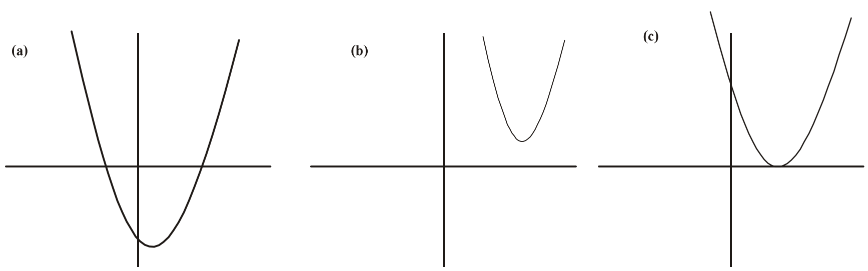
Example 1 :

If x = 2 and x = 0 are roots of the polynomials $f(x) = 2x^3 – 5x^2 + ax + b$. Find the values of a and b.

Sol. $f(2) = 2(2)^3 – 5(2)^2 + a(2) + b = 0 \Rightarrow 16 – 20 + 2a + b = 0 \Rightarrow 2a + b = 4$
$f(0) = 2(0)^3 – 5(0)^2 + a(0) + b = 0 \Rightarrow b = 0$
$\Rightarrow 2a = 4 \Rightarrow a = 2, b = 0$

Example 2 :

How many real roots, i.e. roots that are real numbers, has the quadratic of each graph ?



Sol. Graph (a) has two real roots. It has two x-intercepts.
Graph (b) has no real roots. It has no x-intercepts. Both roots are complex.
Graph (c) has two real roots. But they are a double root.

Example 3 :

If $ax^3 + bx + c$ has a factor of the form $x^2 + px + 1$, show that $a^2 – c^2 = ab$.

Sol. By actual division, we get $\frac{ax^3 + bx + c}{x^2 + px + 1} = ax – ap + \frac{R(x)}{x^2 + px + 1}$
Remainder polynomial $R(x) = (b – a + ap^2)x + c + ap$
If $ax^3 + bx + c$ has a factor of the form $x^2 + px + 1$, then R (x) must be identically zero,
if $b – a + ap^2 = 0$ and $c + ap = 0$

Eliminating p between these equations, we get $b – a + a\left(-\frac{c}{a}\right)^2 = 0$ or $a^2 – c^2 = ab$

Example 4 :

Show that x = 2 is a root of $2x^3 + x^2 – 7x – 6$

Sol. $p(x) = 2x^3 + x^2 – 7x – 6$; $p(2) = 2(2)^2 + (2)^2 – 7(2) – 6 = 16 + 4 – 14 – 16 = 0$.
Hence x = 2 is a root of p (x).

Example 5 :

If $x = 4/3$ is a root of the polynomial $f(x) = 6x^3 - 11x^2 + kx - 20$ then find the value of k.

Sol. $f(x) = 6x^3 - 11x^2 + kx - 20$

$$f\left(\frac{4}{3}\right) = 6\left(\frac{4}{3}\right)^3 - 11\left(\frac{4}{3}\right)^2 + k\left(\frac{4}{3}\right) - 20 = 0 \Rightarrow 6.\frac{64}{9.3} - 11.\frac{16}{9} + \frac{4k}{3} - 20 = 0$$

$$\Rightarrow 128 - 176 + 12k - 180 = 0 \Rightarrow 12k + 128 - 356 = 0 \Rightarrow 12\,k = 228 \Rightarrow k = 19$$

Example 6 :

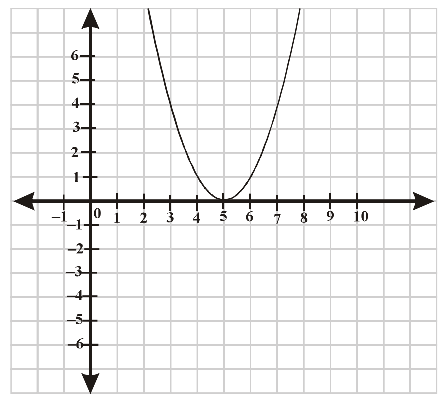
$f(x) = x^2 - 0x + 25$. Find the roots of $f(x)$, and sketch the graph of $y = f(x)$.

Sol. $x^2 - 10x + 25 = (x - 5)(x - 5) = (x - 5)^2$.

The "two" roots are 5, 5.

5 is called a double root. At a double root, the graph does not cross the x-axis. It just touches it.

A double root occurs when the quadratic is a perfect square trinomial: $x^2 \pm 2ax + a^2$; that is, when it is the square of a binomial: $(x \pm a)^2$.



Example 7 :

Find the zeroes of the polynomial $x^2 - 3$ and verify the relationship between the zeroes and the coefficients.

Sol. Recall the identity $a^2 - b^2 = (a - b)(a + b)$. Using it, we can write: $x^2 - 3 = (x - \sqrt{3})\,(x + \sqrt{3})$

So, the value of $x^2 - 3$ is zero when $x = \sqrt{3}$ or $x = -\sqrt{3}$. Therefore, the zeroes of $x^2 - 3$ are $\sqrt{3}$ and $-\sqrt{3}$

Now, sum of zeroes $= \sqrt{3} - \sqrt{3} = 0 = \dfrac{-\text{(Coefficient of x)}}{\text{Coefficient of } x^2}$

product of zeroes $= (\sqrt{3})(-\sqrt{3}) = -3 = \dfrac{-3}{1} = \dfrac{\text{Constant term}}{\text{Coefficient of } x^2}$

Example 8 :

Find the remainder when $f(x) = x^3 - 6x^2 + 2x - 4$ is divided by $g(x) = 1 - 2x$.

Sol. $1 - 2x = 0 \Rightarrow 2x = 1 \Rightarrow x = 1/2$

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) - 4 = \frac{1}{8} - \frac{3}{2} + 1 - 4 = \frac{1 - 12 + 8 - 32}{8} = -\frac{35}{8}$$

Example 9 :

Show that $x + 1$ and $2x - 3$ are factors of $2x^3 - 9x^2 + x + 12$.

Sol. To prove that $(x + 1)\,(2x - 3)$ are factors of $2x^3 - 9x^2 + x + 12$ it is sufficient to show tha p (–1) and p (3/2) both are equal to zero.

$p(-1) = 2(-1)^3 - 9(-1)^2 + (-1) + 12 = -2 - 9 - 1 + 12 = -12 + 12 = 0$

and $p\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 9\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right) + 12 = \frac{27}{4} - \frac{81}{4} + \frac{3}{2} + 12 = \frac{27 - 81 + 6 + 48}{4} = \frac{-81 + 81}{4} = 0$

Example 10 :

Factorise : $x^2 - 31x + 220$

Sol. $x^2 - 31x + 220 = x^2 - 2.\frac{31}{2}x + \left(\frac{31}{2}\right)^2 - \left(\frac{31}{2}\right)^2 + 220 = \left(x - \frac{31}{2}\right)^2 - \frac{964}{4} + 220 = \left(x - \frac{31}{2}\right)^2 - \frac{81}{4}$

$= \left(x - \frac{31}{2}\right)^2 - \left(\frac{9}{2}\right) = \left(x - \frac{31}{2} + \frac{9}{2}\right)\left(x - \frac{31}{2} - \frac{9}{2}\right) = (x - 11)(x - 20)$

Example 11 :

Factorise : $2x^2 + 12\sqrt{2}x + 35$

Sol. Product ac = 70 and $b = 12\sqrt{2}$

$\therefore$ Split the middle term as $7\sqrt{2}, 5\sqrt{2} \Rightarrow 2x^2 + 12\sqrt{2}x + 35 = 2x^2 + 7\sqrt{2}x + 5\sqrt{2}x + 35$

$= \sqrt{2}x\,[\sqrt{2}x + 7] + 5\,[\sqrt{2}x + 7] = [\sqrt{2}x + 5][\sqrt{2}x + 7]$

Example 12 :

Factorise : $x^2 - 14x + 24$

Sol. Product ac = 24 and b = – 14

$\therefore$ Split the middle term as -12 and -2

$\Rightarrow x^2 - 14x + 24 = x^2 - 12x - 2x + 24 = x(x - 12) - 2(x - 12) = (x - 12)(x - 2)$

Example 13 :

Factorise : $x^2 - \frac{13}{24}x - \frac{1}{12}$

Sol. $x^2 - \frac{13}{24}x - \frac{1}{12} = \frac{1}{14}[24x^2 - 13x - 2]$. Product ac = – 48 and b = – 13

$\therefore$ We split the middle term as

$-16x + 3x = \frac{1}{24}[24x^2 - 16x + 3x - 2] = \frac{1}{24}[8x(3x - 2) + 1(3x - 2)] = \frac{1}{24}(3x - 2)(8x + 1)$

Example 14:

Factorise : $\frac{3}{2}x^2 - 8x - \frac{35}{2}$

Sol. $\frac{3}{2}x^2 - 8x - \frac{35}{2} = \frac{1}{2}(3x^2 - 16x - 35) = \frac{1}{2}(3x^2 - 21x + 5x - 35) = \frac{1}{2}[3x(x - 7) + 5(x - 7)] = \frac{1}{2}(x - 7)(3x + 5)$

Example 15 :

If $f(x) = 2x^3 - 13x^2 + 17x + 12$ then find out the value of $f(-2)$ and $f(3)$

Sol. $f(x) = 2x^3 - 13x^2 + 17x + 12$

$f(-2) = 2(-2)^3 - 13(-2)^2 + 17(-2) + 12 = -16 - 52 - 34 + 12 = -90$

$f(3) = 2(3)^3 - 13(3)^2 + 17(3) + 12 = 54 - 117 + 51 + 12 = 0$

Example 16 :

The polynomials $ax^3 + 3x^2 - 13$ and $2x^3 - 5x + a$ are divided by $x + 2$ if the remainder in each case is the same, find the value of a.

Sol. $p(x) = ax^3 + 3x^2 - 13$ and $q(x) = 2x^3 - 5x + a$

When p (x) and q (x) are divided by $x + 2 = 0 \Rightarrow x = -2$

$p(-2) = q(-2) \Rightarrow a(-2)^3 + 3(-2)^2 - 13 = 2(-2)^3 - 5(-2) + a$

$\Rightarrow -8a + 12 - 13 = -16 + 10 + a$

$\Rightarrow -9a = -5 \Rightarrow a = 5/9$

Example 17 :

Factorise : $6x^2 - 5xy - 4y^2 + x + 17y - 15$

Sol. $6x^2 + x[1 - 5y] - [4y^2 - 17y + 15]$

$= 6x^2 + x[1 - 5y] - [4y^2 - 12y - 5y + 15]$

$= 6x^2 + x[1 - 5y] - [4y(y - 3) - 5(y - 3)]$

$= 6x^2 + x[1 - 5y] - (4y - 5)(y - 3)$

$= 6x^2 + 3(y - 3)x - 2(4y - 5)x - (4y - 5)(y - 3)$

$= 3x[2x + y - 3] - (4y - 5)(2x + y - 3)] = (2x + y - 3)(3x - 4y + 5)$

Example 18 :

Find α and β if $x + 1$ and $x + 2$ are factors of $p(x) = x^3 + 3x^2 - 2\alpha x + \beta$

Sol. When we put $x + 1 = 0$ or $x = -1$ but $x + 2 = 0$ or $x = -2$ in p (x)

Then, $p(-1) = 0$ and $p(-2) = 0$

Therefore $p(-1) = (-1)^3 + 3(-1)^2 - 2\alpha(-1) + \beta = 0$

$\Rightarrow -1 + 3 + 2\alpha + \beta = 0 \Rightarrow \beta = -2\alpha - 2$ .... (1)

$p(-2) = (-2)^3 + 3(-2)^2 - 2\alpha(-2) + \beta = 0$

$\Rightarrow -8 + 12 + 4\alpha + \beta = 0 \Rightarrow \beta = -4\alpha - 4$ .... (2)

By equalising both of the above question

$-2\alpha - 2 = -4\alpha - 4 \Rightarrow 2\alpha = -2 \Rightarrow \alpha = -1$

$\alpha = -1$ put in eq. (1)

$\Rightarrow \beta = -2(-1) - 2 = 2 - 2 = 0$. Hence, $\alpha = -1, \beta = 0$

CONCEPT MAP

POLYNOMIALS

* "A **polynomial** is an **algebraic expression** in which the **variables** have **non-negative integral exponents** only".

(i) Linear Polynomial: It is a polynomial of **degree one**. It is of the form $ax + b$, where a & b are real number with $a \neq 0$.

(ii) Quadratic Polynomial: It is a polynomial of **degree two**. It is of the form $ax^2 + bx + c$, where a, b, c are real number with $a \neq 0$.

(iii) Cubic polynomial: It is a polynomial of **degree three**. It is of the form $ax^3 + bx^2 + cx + d$, where a, b, c& d are real number with $a \neq 0$.

* **Zeroes or roots of a polynomial :**
It is that value of a variable at which polynomial P(x) becomes zero.
Ex- If polynomial $P(x) = x^3 - 6x^2 + 11x - 6$, Putting x = 1 one get P (1) = 0 than 1 is a zero of polynomial P (x).

$\Rightarrow$ 0 may be a zero of a polynomial

$\Rightarrow$ Linear polynomial can have at most one zero

$\Rightarrow$ Quadratic polynomial can have at most two zeroes

$\Rightarrow$ Cubic polynomial can have at most three zeroes

* **Geometrical Meaning of the Zeroes of a Polynomial :**

1. The zeroes of a polynomial p(x) are the **x – coordinates** of the points, where the graph of y = p (x) intersects the x-axis.

2 Number of zeroes of a polynomial is the number of times the graph intersects the x-axis.

* **Relationship between Zeroes and Co-efficients of a Polynomial**

1. It α & β are the zeroes of the quadratic polynomial $ax^2 + bx + c$, then

$$\alpha + \beta = -\frac{b}{a} \; ; \qquad \alpha\beta = \frac{c}{a}$$

2. If α, β, γ are the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d$, then

$$\alpha + \beta + \gamma = -\frac{b}{a}; \quad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

* **Division Algorithm**
The division algorithm states that for any given polynomial p(x) and any non-zero polynomial g(x) there are polynomials q(x) and r(x) such that

P (x) = g (x) × q (x) + r (x)

Dividend = Divisor × Quotient + Remainder

Where r(x)=0 or degree r(x)< degree g(x)

* Number of zeroes of a polynomial will be the number of distinct points where the graph of the polynomial cuts x-axis.

* **To find zeroes of a quadratic polynomial**

(i) Write down the given quadratic polynomial in the form $ax^2 + bx + c$.

(ii) Find the factors of $ax^2 + bx + c$ by writting the middle term as the sum of two expressions. For this ac > 0, then find two numbers p and q having same signs as that of b such that p + q = b and pq = ac.
But if ac < 0, then find two numbers p and q having opposite signs such that p + q = b and pq = ac.

(iii) Equate each factor thus obtained to zero to get the value of x.

(iv) These values of x will be the zeroes of the given quadratic polynomial.