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MATHEMATICAL INDUCTION

THEOREM-I

If P (n) is a statement depending upon n, then to prove it by induction, we proceed as follows :

(i) Verify the valdity of P (n) for n = 1

(ii) Assume that P (n) is true for some positive integer m and then using it establish the validity of P (n) for n= m + 1.

Then P (n) is true for each $n \in N$.

THEOREM-II

If P (n) is a statement depending upon n, but beginning with some positive integer k, then to prove P (n), we proceed as follows :

(i) Verify the valdity of P (n) for n = k

(ii) Assume that the statement is true for $n = m \geq k$.

Then using it establish the validity of P (n) for n = m + 1.

Then P (n) is true for each $n \geq k$.

SUMS USEFUL RESULT BASED ON PRINCIPLE OF MATHEMEMATICAL INDUCTION

For any natural number n

(1) $1+2+3+4+.......+n = \Sigma n = \frac{n(n+1)}{2}$

(2) $1^2+2^2+3^2+4^2+.........+n^2 = \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$

(3) $1^3+2^3+3^3+4^3+.........+n^3 = \Sigma n^3 = (\Sigma n)^2 = \left\{\frac{n(n+1)}{2}\right\}^2$

(4) $1^4+2^4+3^4+4^4+.....+n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

(5) $1^5+2^5+3^5+4^5+.........+n^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$

(6) $2+4+6+.......+2n = \Sigma 2n = n(n+1)$

(7) $1+3+5+.....+(2n-1) = \Sigma(2n-1) = n^2$

(8) $x^n - y^n = (x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+.....+xy^{n-2}+y^{n-1})$

(9) $x^n + y^n = (x+y)(x^{n-1}-x^{n-2}y+x^{n-3}y^2+.....-xy^{n-2}+y^{n-1})$ when n is odd positive integer.

NOTE :

(i) Product of r consecutive integers is divisible by r!

(ii) For $x \neq y$, $x^n - y^n$ is divisible by

(a) x + y if n is even (b) x – y if n is even or odd

(iii) For solving objective question related to natural numbers we find out the correct alternative by negative examination of this principle. If the given statement is P(n), then by putting n = 1, 2, 3, ... in P (n) we decide the correct answer. We also use the above formulae established by this principle to find the sum of n terms of a given series. For this we first express T_n as a polynomial in n and then for finding S_n, we put Σ before each term of this polynomial and then use above resulsts of Σn, Σn^2, Σn^3 etc.

ADDITIONAL EXAMPLES

Example 1 :

Prove that $n < 2^n$ for all positive integers n.

Sol. (1) For n = 1, the formula is true, because $1 < 2^1$.

(2) Assuming that $k < 2^k$

you need to show that $k + 1 < 2^{k+1}$.

For n = k, you have $2^{k+1} = 2(2^k) > 2(k) = 2k$.

[By assumption]

Because $2k = k + k > k + 1$ for all $k > 1$, it follows that

$2^{k+1} > 2k > k+1$ or $k+1 < 2^{k+1}$.

So, $n < 2^n$ for all integers $n \geq 1$.

Example 2 :

Prove by mathematical induction that the sum of the products of every pair of squares of the first n natural numbers is $\frac{1}{360}n(n^2-1)(5n+6)$.

Sol. Let $p(n) = 1^2.2^2 + 1^2.3^2 + ... + 1^2.n^2 + 2^2.3^2 + 2^2 + 4^2$

$+.........+ 2^2.n^2 + ... + (n-1)^2.n^2 = \frac{1}{360}n(n^2-1)(5n+6)$

$P(2) = 1^2.2^2 = \frac{2x3x15x16}{360}$ is true

Assume $P(m+1) = \frac{1}{360}m(m^2-1)(4m^2-1)(5m+6)$

$+ \frac{(m+1)^2 m(m+1)(2m+1)}{6}$

$= \frac{m(m+1)(2m+1)}{360}\{(m-1)(2m-1)(5m+6) + 60(m+1)^2\}$

$= \frac{m(m+1)(2m+1)}{360}(10m^3+57m^2+107m+66)$

$= \frac{m(m+1)(2m+1)}{360}(m+2)(2m+3)(5m+11)$

$= (m+1)(m^2+2m)(4(m+1)^2 - 1(5m+11)$

$= \{(m+1)(m+1)^2 - 1\}\{4(m+1)^2 - 1\}\{m+1\} + 6$

This being of the same form the RHS of P (m), P(m + 1) is true. Hence, P(n) is true by mathematical induction.

Example 3 :

Prove by mathematical induction or that the sum or otherwise that 3^{2n+2}, 5^{2n} - 3^{3n+2}, 2^{2n} is divisible by 1053 for $n \le 1$.

Sol. Let $f(n) = 3^{2n+2}(5^{2n} - 3n.\, 2^{2n}) = 9^{n+1}(25^n - 12^n)$

$= 81.9^{n-1}(25^n - 12^n)$

Now, $a^n - b^n$ is divisible by $a - b$

$\therefore$ $25^n - 12^n$ is divisible by $25 - 12 = 13$

$\therefore$ $f(n) = 81.9^{n-1}.\, 13 \times k = 1053\, .9^{n-1}.k$

Hence f (n) is divisible by 1053.

Example 4 :

Find P_{k+1} for the following.

(a) $P_k : S_k = \dfrac{k^2(k+1)^2}{4}$

(b) $P_k : S_k = 1 + 5 + 9 + \ldots + [4(k-1) - 3] + (4k - 3)$

(c) $P_k : S_k \ge 2k + 1$

Sol. (a) $P_{k+1} : S_{k+1} = \dfrac{(k+1)^2(k+1+1)^2}{4}$ [Replace k by k + 1]

$= \dfrac{(k+1)^2(k+2)^2}{4}$

(b) $P_k : S_k = 1 + 5 + 9 + \ldots + \{[4(k+1) - 1] - 3\} + [4(k+3) - 3]$

$= 1 + 5 + 9 + \ldots + (4k - 3) + (4k + 1)$

(c) $P_{k+1} : 3^{k+1} \ge 2(k+1) + 1$

$3^{k+1} \ge 2k + 3.$

Example 5 :

Find $\sum_{n=1}^{7} n^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3$

Sol. Using the formula for the sum of the cubes of the first n positive integers, you obtain the following.

$$\sum_{n=1}^{7} n^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 =$$

$$\frac{7^2(7+1)^2}{4} = \frac{49\,(64)}{4} = 784$$

Check this sum by adding the numbers 1, 8, 27, 64, 125, 216 and 343.

Example 6 :

Use mathematical induction to prove the following formula.

$S_n = I + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$.

Sol. Mathematical induction consists of two distinct parts. First, you must show that the formula is true when n = 1.

(1) When n = 1, the formula is valid, because $S_1 = 1 = 1^2$.

The second part of mathematical induction has two steps. The first step is to assume that the formula is valid for some integer k. The second step is to use this assumption to prove that the formula is valid for the next integer, k + 1.

(2) Assuming that the formula

$S_k = 1 + 3 + 5 + 7 + \ldots + (2k - 1) = k^2$

is true, you must show that the formula $S_{k+1} = (k + 1)^2$ is true. $S_{k+1} = 1 + 3 + 5 + 7 + \ldots + (2k - 1) + [2(k + 1) - 1]$

$= [1 + 3 + 5 + 7 + \ldots + (2k - 1)] + (2k + 2 - 1)$

$= S_k + (2k + 1)$ [Group terms to form S_k]

$= k^2 + 2k + 1$ [Replace S_k by k^2]

$= (k + 1)^2$

Combining the results of parts (1) and (2), you can conclude by mathematical induction that the formula is valid for all positive integer values of n.

It occasionally happens that a statement involving natural numbers is not true for the first k – 1 positive integers but is true for all values of $n \ge k$. In these instances, you use a slight variation of the Principle of Mathematical Induction in which you verify P_k rather than P_1. This variation is called the extended principle of mathematical induction. To see the validity of this variation, note from Figure that all but the first k – 1 dominoes can be knocked down by knocking over the kth domino. This suggests that you can prove a statement P_n to be true for $n \ge k$ by showing that P_k is true and that P_k implies P_{k+1}.

Example 7 :

Find a formula for the following finite sum :

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \ldots\ldots + \frac{1}{n\,(n+1)}$$

Sol. Begin by writing out the first few sums.

$$S_1 = \frac{1}{1.2} = \frac{1}{2} = \frac{1}{1+1};\; S_2 = \frac{1}{1.2} + \frac{1}{2.3} = \frac{4}{6} = \frac{2}{3} = \frac{2}{2+1}$$

$$S_3 = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} = \frac{9}{12} = \frac{3}{4} = \frac{3}{3+1}$$

$$S_4 = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} = \frac{48}{60} = \frac{4}{5} = \frac{3}{4+1}$$

From this sequence, it appears that the formula for the kth sum is

$$S_k = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \ldots\ldots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

To prove the validity of this hypothesis, use mathematical induction, as follows. Note that you have already verified the formula for n = 1, so you can begin by assuming that the formula is valid for n = k and trying to show that it is valid for n = k + 1.

$$S_{k+1} = \left[\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \ldots\ldots + \frac{1}{k(k+1)}\right] + \frac{1}{(k+1)\,(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)\,(k+2)} = \frac{k\,(k+2)+1}{(k+1)\,(k+2)}$$

$$= \frac{k^2 + 2k + 1}{(k+1)\,(k+2)} = \frac{(k+1)^2}{(k+1)\,(k+2)} = \frac{k+1}{k+2}.$$

So, the hypothesis is valid.

QUESTION BANK

EXERCISE

Q.1 $2^n > n^2$ when $n \in N$ such that
(A) $n > 2$ (B) $n > 3$
(C) $n < 5$ (D) $n \geq 5$

Q.2 If $n \in N$ and n is odd, then $n(n^2 - 1)$ is divisible by –
(A) 24 (B) 16
(C) 32 (D) 8

Q.3 If $49^n + 16n + \lambda$ is divisible by 64 for all $n \in N$, then the least negative internal value of λ is –
(A) –2 (B) –1
(C) –3 (D) – 4

Q.4 For $n \in N$, $x^{n+1} + (x+1)^{2n-1}$ is divisible by
(A) x (B) x + 1
(C) $x^2 + x + 1$ (D) $x^2 - x + 1$

Q.5 The sum of the terms in the n^{th} bracket of the series $(1) + (2 + 3 + 4) + (5 + 6 + 7 + 8 + 9) + ...$, is –
(A) $(n-1)^3 + n^3$ (B) $(n+1)^3 + 8n^2$
(C) $\frac{(n+1)(n+2)}{6n}$ (D) $(n+1)^3 + n^3$

Q.6 If $n \in N$ and $n > 1$, then –
(A) $n! > \left(\frac{n+1}{2}\right)^n$ (B) $n! \geq \left(\frac{n+1}{2}\right)^n$
(C) $n! < \left(\frac{n+1}{2}\right)^n$ (D) None of these

Q.7 If ω is an imaginary cube root of unity then value of expression $1.(2-\omega).(2-\omega^2) + 2.(3-\omega).(3-\omega^2) + ... + (n-1)(n-\omega).(n-\omega^2)$ is –
(A) $\frac{1}{4}n^2(n+1)^2$ (B) $\frac{1}{4}n^2(n+1)^2 - n$
(C) $\frac{1}{6}n^2(n+1)^2 - 1$ (D) None of these

Q.8 $\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + ...$ upto n terms, is equal to–
(A) $\frac{6n}{n+1}$ (B) $\frac{9n}{n+1}$ (C) $\frac{12n}{n+1}$ (D) $\frac{5n}{n+1}$

Q.9 If $A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, then for some $n \in N$, A^n is equal to –
(A) $\begin{pmatrix} n & k \\ 0 & n \end{pmatrix}$ (B) $\begin{pmatrix} 1 & k^n \\ 0 & 1 \end{pmatrix}$ (C) $\begin{pmatrix} 1 & nk \\ 0 & 1 \end{pmatrix}$ (D) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Q.10 $\frac{3}{4} + \frac{15}{16} + \frac{63}{64} + ...$to n terms =
(A) $n - \frac{4^n}{3} - \frac{1}{3}$ (B) $n + \frac{4^{-n}}{3} - \frac{1}{3}$
(C) $n + \frac{4^n}{3} - \frac{1}{3}$ (D) $n - \frac{4^n}{3} + \frac{1}{3}$

Q.11 Let $S(K) = 1 + 3 + 5 + ... + (2K-1) = 3 + K^2$. Then which of the following is true?
(A) S (1) is correct (B) $S(K) \Rightarrow S(K+1)$
(C) $S(K) \not\Rightarrow S(K+1)$
(D) Principle of mathematical induction can be used to prove the formula

Q.12 **Statement-1:** For every natural number $n \geq 2$
$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}} > \sqrt{n}$$
Statement -2: For every natural number $n \geq 2$
$$\sqrt{n(n+1)} < n+1$$
(A) Statement-1 is true, Statement -2 is true; Statement-2 is a correct explanation for Statement-1
(B) Statement-1 is true, Statement -2 is true; Statement-2 is not a correct explanation for Statement-1
(C) Statement-1 is true, Statement -2 is false
(D) Statement-1 is false, Statement-2 is true

Q.13 $1^3 + 2^3 + 3^3 + ... + n^3 =$
(A) $\left(\frac{n(n+1)}{2}\right)^2$ (B) $\left(\frac{n(n+1)}{2}\right)^3$
(C) $\left(\frac{n(n-1)}{2}\right)^3$ (D) $\left(\frac{n(n-1)}{3}\right)^2$

Q.14 $10^{2n-1} + 1$ is divisible by –
(A) 12 (B) 13
(C) 11 (D) 14

Q.15 $1^2 + 3^2 + 5^2 + ... + (2n-1)^2 =$
(A) $\frac{n(n-1)(2n+1)}{3}$ (B) $\frac{n(2n-1)(2n+1)}{3}$
(C) $\frac{n(n-1)(n+1)}{3}$ (D) $\frac{n(2n-1)(n+1)}{3}$

Q.16 $3^{2n+2} - 8^n - 9$ is divisible by –
(A) 2 (B) 3
(C) 4 (D) 8

Q.17 $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2n+1)(2n+3)}=$

(A) $\frac{n}{(2n+3)}$ (B) $\frac{n}{(2n-3)}$

(C) $\frac{n}{3(2n+3)}$ (D) $\frac{2n}{(2n-3)}$

Q.18 n (n + 1) (n + 2) is divisible by –

(A) 4 (B) 2

(C) 6 (D) 8

Q.19 $1.3+2.3^2+3.3^3+........n.3^n=$

(A) $\frac{(2n-1)3^{n+1}+3}{4}$ (B) $\frac{(n-1)3^{n+1}+3}{4}$

(C) $\frac{(2n-1)3^{n+1}+2}{4}$ (D) $\frac{(2n-1)3^{n+1}}{2}$

Q.20 n (n + 1) (n + 5) is a multiple of –

(A) 3 (B) 4

(C) 5 (D) 6

Q.21 $1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+.....+\frac{1}{(1+2+3+.....)}=$

(A) $\frac{n}{(n+1)}$ (B) $\frac{n}{(n-1)}$

(C) $\frac{2n}{(n-1)}$ (D) $\frac{2n}{(n+1)}$

Q.22 $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)...\left(1+\frac{(2n+1)}{n^2}\right)=$

(A) $2(n+1)^2$ (B) $(n+1)^2$

(C) $(n-1)^2$ (D) $(n+1)^3$

Q.32 $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...\left(1+\frac{1}{n}\right)=$

(A) $(n+1)$ (B) $(n+1)^2$

(C) $(n-1)^2$ (D) $(n+1)^3$

Q.24 $x^{2n}-y^{2n}$ is divisible by –

(A) x – y (B) x + y

(C) x – 2y (D) x + 2y

Q.25 $1+3+3^2+.....+3^{n-1}=$

(A) $\frac{(3^n-1)}{2}$ (B) $\frac{(3^n+1)}{2}$

(C) $\frac{(3^n+1)}{3}$ (D) $\frac{(3^n-1)}{3}$

Q.26 $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+...+\frac{1}{(3n-2)(3n+1)}=$

(A) $\frac{2n}{(3n+1)}$ (B) $\frac{2n}{(3n-1)}$

(C) $\frac{2n}{(2n-1)}$ (D) $\frac{n}{(3n+1)}$

Q.27 41^n-14^n is a multiple of –

(A) 27 (B) 30

(C) 9 (D) 81

ANSWER KEY

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
A	D	A	B	C	A	C	B	A	C	B	C	B	A	A	B	D	C	C	A	A
Q	21	22	23	24	25	26	27													
A	D	B	A	B	A	D	A													

CHAPTER- 3 : MATHEMATICAL INDUCTION QUESTION BANK SOLUTIONS

SOLUTIONS EXERCISE

(1) **(D).** Let the given statement by P (n), then
$P(1) \Rightarrow 2^1 > 1^2$ which is true
$P(2) \Rightarrow 2^2 > 2^2$ which is false
$P(3) \Rightarrow 2^3 > 3^2$ which is false
$P(4) \Rightarrow 2^4 > 4^2$ which is false
$P(5) \Rightarrow 2^5 > 5^2$ which is true
$P(6) \Rightarrow 2^6 > 6^2$ which is true
$\therefore$ P (n) is true when $n \geq 5$

(2) **(A).** Let $P(n) = n(n^2 - 1)$ then
$P(1) = 1(0) = 0$ which is divisible by every $n \in N$
$P(3) = 3(8) = 24$ which is divisible by 24 and 8
$P(5) = 5(24) = 120$ which is divisible by 24 and 8
But we know that a number which is divisible by 24 is also divisible by 8 but its converse may not be true.
Hence P (n) is divisible by 24.

(3) **(B).** For n = 1, we have
$49^n + 16n + \lambda = 49 + 16 + \lambda = 65 + \lambda$
$= 64 + (\lambda + 1)$, which is divisible by 64 if $\lambda = -1$
For n = 2, we have
$49^n + 16n + \lambda = 49^2 + 16 \times 2 + \lambda = 2433 + \lambda$
$= 64 \times 38 + (\lambda + 1)$, which is divisible by 64 if $\lambda = -1$
Hence, $\lambda = -1$

(4) **(C).** For n = 1, we have
$x^{n+1} + (x+1)^{2n-1} = x^2 + (x+1) = x^2 + x + 1$,
which is divisible by $x^2 + x + 1$
For n = 2, we have
$x^{n+1} + (x+1)^{2n-1} = x^3 + (x+1)^3 = (2x+1)(x^2+x+1)$,
which is divisible by $x^2 + x + 1$.
Hence, option (C) is true.

(5) **(A).** For n = 1, we have
Sum of the terms in first bracket = 1
And, $(n-1)^3 + n^3 = (1-1)^3 + 1^3 = 1$
For n = 2, we have
Sum of the terms in the second bracket = 2 + 3 + 4 = 9
And, $(n-1)^3 + n^3 = (2-1)^3 + 2^3 = 1 + 8 = 9$

(6) **(C).** When n = 2 then

$$n = 2, \left(\frac{n+1}{2}\right)^n = \frac{9}{4} \Rightarrow n! < \left(\frac{n+1}{2}\right)^n$$

$$\text{When } n = 3, \text{ then } n! = 6, \left(\frac{n+1}{2}\right)^n = 8 \Rightarrow n! < \left(\frac{n+1}{2}\right)^n$$

$$\text{When } n = 4, \text{ then } n! = 24, \left(\frac{n+1}{2}\right)^n = \frac{625}{16} \Rightarrow n! < \left(\frac{n+1}{2}\right)^n$$

$$\therefore \text{ It is seen that} \Rightarrow n! < \left(\frac{n+1}{2}\right)^n$$

(7) **(B).** $\text{Sum} = \sum_{n=2}^{n} (n-1)(n-\omega)(n-\omega^2)$

$$= \sum_{n=1}^{n} (n-1)[(n^2 - n(\omega + \omega^2) + \omega^3]$$

[$\because$ when n = 1, sum = 0]
$= \Sigma (n-1)(n^2 + n + 1)$

$$= \Sigma (n^3 - 1) = \Sigma n^3 - \Sigma 1 = \frac{1}{4} n^2 (n+1)^2 - n$$

(8) **(A).**

$$T_n = \frac{2n+1}{1^2 + 2^2 + ... + n^2} = \frac{(2n+1)\,6}{n(n+1)(2n+1)}$$

$$= \frac{6}{n(n+1)} = 6\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$$\Rightarrow S_n = 6\left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + ... + \left(\frac{1}{n} - \frac{1}{n+1}\right)\right]$$

$$= 6\left[1 - \frac{1}{n+1}\right] = \frac{6n}{n+1}$$

(9) **(C).** We find that

$$A^2 = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3k \\ 0 & 1 \end{pmatrix}$$

$$\text{Similarly, } A^4 = \begin{pmatrix} 1 & 4k \\ 0 & 1 \end{pmatrix}, A^5 = \begin{pmatrix} 1 & 5k \\ 0 & 1 \end{pmatrix} \text{ etc.}$$

$$\text{So, } A^n = \begin{pmatrix} 1 & nk \\ 0 & 1 \end{pmatrix}$$

(10) **(B).** For n = 1, we have

$$n - \frac{4^n}{3} - \frac{1}{3} = 1 - \frac{4}{3} - \frac{1}{3} = -\frac{2}{3}$$

$$n + \frac{4^{-n}}{3} - \frac{1}{3} = 1 + \frac{4^{-1}}{3} - \frac{1}{3} = \frac{3}{4}$$

$$n + \frac{4^n}{3} - \frac{1}{3} = 1 + \frac{4}{3} - \frac{1}{3} = 2$$

$$n - \frac{4^{-n}}{3} + \frac{1}{3} = 1 - \frac{4^{-1}}{3} + \frac{1}{3} = \frac{5}{4}$$

Also, for n = 2, we have

$$n + \frac{4^{-n}}{3} - \frac{1}{3} = 2 + \frac{1}{48} - \frac{1}{3} = \frac{27}{16} \text{ and } \frac{3}{4} + \frac{15}{16} = \frac{27}{16}$$

Hence, option (B) is correct.

(11) **(C).** $S(K) = 1 + 3 + 5 + 7 + ... + (2K - 1) = 3 + K^2$
For $K = 1$
L.H.S. = 1 and R.H.S. = 4
Option (A) cancel out
$\because$ we know $1 + 3 + 5 + 7 + ... + (2K - 1) = K^2$
but in question $S(K) = 3 + K^2$
$\therefore$ by principle of mathematical induction
$S(K) \not\Rightarrow S(K+1)$
$\{\because S(K)$ is not true for $K = 1, 2, 3, ...\}$

(12) **(B).** Let $p(n) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}}$

$p(2) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}$

Let us assume that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... \frac{1}{\sqrt{k}} > \sqrt{k}$ is true.

$\therefore p(k+1) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$

has to be true.

$\therefore \text{L.H.S.} > \sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k(k+1)} + 1}{\sqrt{k+1}}$

Since, $\sqrt{k(k+1)} > k \ (\forall k \geq 0)$

$\therefore \frac{\sqrt{k(k+1)} + 1}{\sqrt{k+1}} > \frac{k+1}{\sqrt{k+1}} = \sqrt{k+1}$

L.H.S. $> \sqrt{k+1}$
$\therefore$ Statement (1) is true.

Let $p(n) = \sqrt{n(n+1)} < n+1$

$P(2) = \sqrt{2(2+1)} < 2+1$

If $p(k) = \sqrt{k(k+1)} < k+1$ is true

Now, $p(k+1) = \sqrt{(k+1)(k+2)} < k+2$ has to be true.
Since $(k+1) < k+2$

$\therefore \sqrt{(k+1)(k+2)} < k+2$

Hence statement 2 is also true but not correct explanation of statement (1).

(13) **(A).**
(14) **(A).**
(15) **(B).**
(16) **(D).**
(17) **(C).**
(18) **(C).** Let P (n) be the statement "n (n + 1) (n + 2) is divisible by 6", i.e. $P(n) = n(n+1)(n+2)$ is divisible by 6.
Step 1 : For n = 1, we have
$P(1) = 1(1+1)(1+2) = 1 \times 2 \times 3 = 6$, which is divisible by 6. Thus, P (1) is true for n = 1.
Step 2 : For n = k, Let P (k) be true,
i.e., $P(k) = k(k+1)(k+2)$ is divisible by 6.
Let, $k(k+1)(k+2) = 6\lambda$, for some $\lambda \in N$. .... (1)
Step 3 : For n = k + 1, we have to show that P (k + 1) is true,
i.e. P (k + 1) is divisible by 6.
$P(k+1) = (k+1)(k+1+1)(k+1+2)$
$= (k+1)(k+2)(k+3)$
$= k(k+1)(k+2) + 3(k+1)(k+2)$
$= 6\lambda + 3(k+1)(k+2)$ [From (1)
$= 6\lambda + 6t = 6(\lambda + t)$
Thus, P (k + 1) is divisible by 6.
Therefore, P (k + 1) is true.
$\therefore$ P (k) is true $\Rightarrow$ P (k + 1) is true.
Hence, by principle of mathematical induction, P (n) is true for all natural number n.

(19) **(A).** Let P (n) be the statement given by

$P(n) = 1.3 + 2.3^2 + 3.3^3 + ...n.3^n = \frac{(2n-1)3^{n+1} + 3}{4}$

Step 1 : For n = 1, we have

$P(1) : 1 : 3^1 = \frac{(2-1)3^{1+1} + 3}{4}$ or $3 = \frac{9+3}{4}$ or $3 = 3$

Thus, P (1) is true
Step 2 : For n = k, assume that P (k) is true.

Then, $1.3 + 2.3^2 + 3.3^3 + ...k.3^k = \frac{(2k-1)3^{k+1} + 3}{4}$ ... (1)

Step 3 : For n = k + 1, we have to show that
$1.3 + 2.3^2 + 3.3^3 + ...k.3^k + (k+1)3^{k+1}$

$= \frac{[2(k+1) - 1].3^{(k+1)+1} + 3}{4}$

Now, L.H.S. $= 1.3 + 2.3^2 + 3.3^3 + ...k.3^k + (k+1)3^{k+1}$

$= \frac{(2k-1).3^{k+1} + 3}{4} + (k+1)3^{k+1}$ [Using (1)]

$= \frac{(2k-1).3^{k+1} + 3 + 4(k+1)3^{k+1}}{4}$

$= \frac{(2k - 1 + 4k + 4).3^{k+1} + 3}{4} = \frac{(6k+3)3^{k+1} + 3}{4}$

$= \frac{3(2k+1).3^{k+1} + 3}{4} = \frac{(2k+1)3^{k+2} + 3}{4}$

$= \frac{[2(k+1) - 1].3^{(k+1)-1} + 3}{4}$

Therefore, P (k + 1) is true. Thus, P (k) is true $\Rightarrow$ P (k + 1) is true.
Hence, by principle of mathematical induction P (n) is true for all $n \in N$.

(20) **(A).**
(21) **(D).**

(22) **(B).** Let P (n) be the statement given by :

$$P(n): \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)...\left(1+\frac{(2n+1)}{n^2}\right)=(n+1)^2$$

Step 1 : For n = 1, we have

$$P(1)=\left(1+\frac{2\times1+1}{1^2}\right)=(1+1)^2 \text{ or } \left(1+\frac{3}{1}\right)=2^2 \text{ or } 4=4$$

Thus, P (1) is true.

Step 2 : For n = k, assume that P (k) is true. Then,

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)......\left(1+\frac{2k+1}{k^2}\right)=(k+1)^2 \quad .....(1)$$

Step 3 : For n = k + 1, we have to show that

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)......\left(1+\frac{2k+1}{k^2}\right)\left(1+\frac{2(k+1)+1}{(k+1)^2}\right)$$

$=(k+1+1)^2$ Now,

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)......\left(1+\frac{2k+1}{k^2}\right)\times\left[1+\frac{2(k+1)+1}{(k+1)^2}\right]$$

$$=(k+1)^2\left[1+\frac{2k+3}{(k+1)^2}\right]=(k+1)^2\left[\frac{(k+1)^2+2k+3}{(k+1)^2}\right]$$

$$=(k^2+2k+1+2k+3)=(k^2+4k+4)=(k+2)^2$$

$$=(\overline{k+1}+1)^2$$

Therefore, P (k + 1) is true.

Thus P (k) is true $\Rightarrow$ P (k + 1) is true

Hence, by principle of mathematical induction P (n) is true for all $n \in N$

(23) **(A).**

(24) **(B).**

(25) **(A).** Let P(n) be the statement given by

$$P(n): 1+3+3^2+.....+3^{n-1}=\frac{(3^n-1)}{2}$$

Step 1 : For n = 1, we have

$$P(1): 3^{1-1}=\frac{3^1-1}{2}\Rightarrow 1=1$$

Thus, P (1) is true.

Step 2 : For n = k, assume that P (k) is true.

$$\text{Then } 1+3+3^2+.....+3^{k-1}=\frac{3^k-1}{2}$$

Step 3 : For n = k + 1, we have to show that

$$1+3+3^2+.....+3^{k-1}+3^{k+1-1}=\frac{3^{k+1}-1}{2}$$

Now, L.H.S. $=1+3+3^2+.....+3^{k-1}+3^{k+1-1}$

$$=\frac{3^k-1}{2}+3^{k+1-1}$$

$$=\frac{3^k-1}{2}+3^k=\frac{3^k-1+2.3^k}{2}=\frac{3^k(1+2)-1}{2}$$

$$=\frac{3.3^k-1}{2}=\frac{3^{k+1}-1}{2}$$

There P (k + 1) is true. Thus, p (k) is true $\Rightarrow$ P (k + 1) is true for all $n \in N$.

(26) **(D).**

(27) **(A).**