3

PAIR OF LINEAR EQUATIONS IN TWO VARIABLES

INTRODUCTION

An equation of the form $Ax + By + C = 0$ is called a linear equation.
Where A is called coefficient of x, B is called coefficient of y and C is the constant term (free from x and y)
$A, B, C \in R$, [$\in \rightarrow$ belongs to, $R \rightarrow$ Real no.] But A and B cannot be simultaneously zero.
If $A \neq 0$, $B = 0$ equation will be of the form $Ax + C = 0$
If $A = 0$, $B \neq 0$ equation will be of the form $By + C = 0$
If $A \neq 0$, $B \neq 0$, $C = 0$ equation will be of the form $Ax + By = 0$ (line passing through origin)
If $A \neq 0$, $B \neq 0$, $C \neq 0$ equation will be of the form $Ax + By + C = 0$.
It is called a linear equation because the two unknowns (x and y) occurs only in the first power and the product of two unknown quantities does not occur.
Since it involves two variables therefore a single equation will have infinite set of solution i.e., interminate solution. So we require a pair of equation i.e., simultaneous equations.
Standard from of linear equation : (Standard form refers to all positive coefficients)

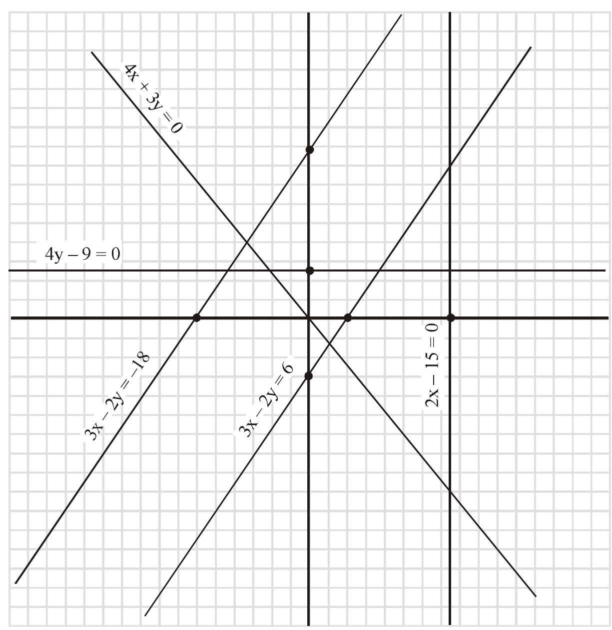
$$a_1x + b_1y + c_1 = 0 \quad \text{.............. (i)} \qquad a_2x + b_2y + c_2 = 0 \quad \text{.............. (ii)}$$

INTERPRETING EQUATIONS

The general form for an equation in the first degree in one variable is $ax + b = 0$. The general form for first-degree equations in two variables is $ax + by + c = 0$
It is interesting and often useful to note what happens graphically when equations differ, in certain ways, from the general form. With this information, we know in advance certain facts concerning the equation in question.

LINE PARALLEL TO THE AXES

If in a linear equation the y term is missing, as in $2x - 15 = 0$, the equation represents a line parallel to the Y axis and 7 1/2 units from it. Similarly, an equation such as $4y - 9 = 0$, which has no x term, represents a line parallel to the X axis and 2¼ units from it. The fact that one of the two variables does not appear in an equation means that there are no limitations on the values the missing variable can assume. When a variable does not appear, it can assume any value from zero to plus or minus infinity. This can happen only if the line represented by the equation lies parallel to the axis of the missing variable.



Lines Passing Through the Origin

A linear equation, such as $4x + 3y = 0$
that has no constant term, represents a line passing through the origin. This fact is obvious since $x = 0$, $y = 0$ satisfies any equation not having a constant term.

Lines Parallel to Each Other

An equation such as $3x - 2y = 6$

has all possible terms present. It represents a line that is not parallel to an axis and does not pass through the origin. Equations that are exactly alike, except for the constant terms, represent parallel lines. As shown in figure, the lines represented by the equations $3x - 2y = -18$ and $3x - 2y = 6$, are parallel.

Parallel lines have the same slope. Changing the constant term moves a line away from or toward the origin while its various positions remain parallel to one another. In figure that the line $3x - 2y = 6$ lies closer to the origin than $3x - 2y = -18$. This is revealed at sight for any pair of lines by comparing their constant terms. That one which has the constant term of greater absolute value will lie farther from the origin. In this case $3x - 2y = -18$ will be farther from the origin since $|-18| > |16|$.

SOLVING EQUATIONS IN TWO VARIABLES

A linear equation in two unknowns is an equation which after simplification contains two unknowns, each one of them in a separate term and having the exponent one.

A solution of a linear equation in two variables consists of a pair of numbers that satisfy the equation. For example, $x = 2$ and $y = 1$ constitute a solution of $3x - 5y = 1$

When 2 is substituted for x and 1 is substituted for y, we have, $3(2) - 5(1) = 1$

The numbers $x = -3$ and $y = -2$ also form a solution.

This is true because substituting -3 for x and -2 for y reduces the equation to an identity:

$$3(-3) - 5(-2) = 1$$
$$-9 + 10 = 1$$
$$1 = 1$$

Each pair of numbers (x, y) such as (2, 1) or (–3, –2) locates a point on the line $3x - 5y = 1$. Many more solutions could be found. Any two numbers that constitute a solution of the equation are the coordinates of a point on the line represented by the equation.

Suppose we were asked to solve a problem such as: Find two numbers such that their sum is 35 and their difference is 5. We could indicate the problem algebraically by letting x represent one number and y the other. Thus, the problem may be indicated by the two equations :

$$x + y = 33$$
$$x - y = 5$$

Considered separately, each of these equations represents a straight line on a graph. There are many pairs of values for x and y which satisfy the first equation, and many other pairs which satisfy the second equation. Our problem is to find one pair of values that will satisfy both equations. Such a pair of values is said to satisfy both equations at the same time, or simultaneously. Hence, two equations for which we seek a common solution are called simultaneous equations. The two equations, taken together, comprise a system of equations.

In general, a solution of a system in two variables is an ordered pair that makes both equations true.

In other words, it is where the two graphs intersect, what they have in common. So if an ordered pair is a solution to one equation, but not the other, then it is not a solution to the system.

A consistent system is a system that has at least one solution. An inconsistent system is a system that has no solution.

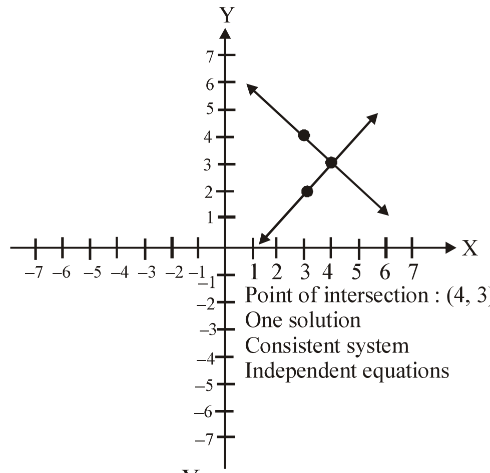
The equations of a system are dependent if all the solutions of one equation are also solutions of the other equation. In other words, they end up being the same line.

The equations of a system are independent if they do not share all solutions. They can have one point in common, just not all of them.

One Solution :

If the system in two variables has one solution, it is an ordered pair that is a solution to both equations. In other words, when you plug in the values of the ordered pair it makes both equations true.

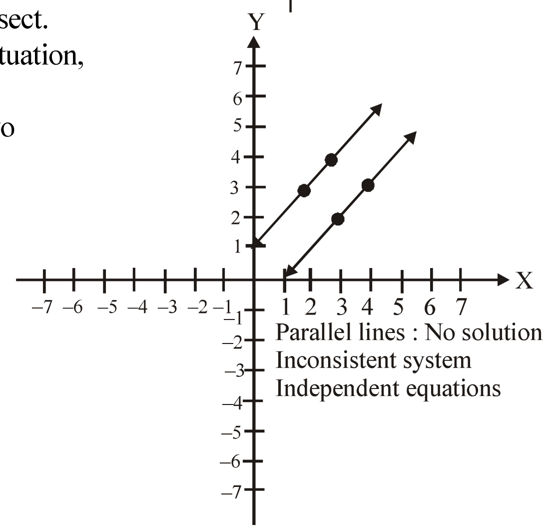
The graph below illustrates a system of two equations and two unknowns that has one solution:



No Solution :

If the two lines are parallel to each other, they will never intersect. This means they do not have any points in common. In this situation, you would have no solution.

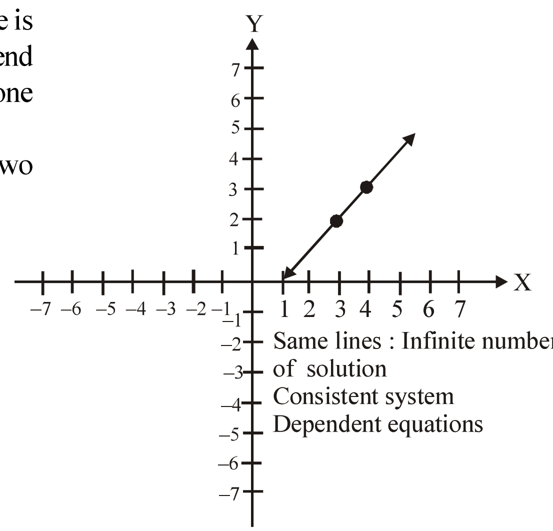
The graph below illustrates a system of two equations and two unknowns that has no solution:



Infinite Solutions :

If the two lines end up lying on top of each other, then there is an infinite number of solutions. In this situation, they would end up being the same line, so any solution that would work in one equation is going to work in the other.

The graph below illustrates a system of two equations and two unknowns that has an infinite number of solutions:



If the lines represented by the equation :

$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$

are (i) intersecting, then $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

(ii) coincident, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(iii) parallel then $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

In fact, the converse is also true for any pair of lines.

SOLVING SYSTEMS OF LINEAR EQUATIONS

There will be six ways that we can use to solve a system of linear equations

Graphically : Graph both equations and find the intersection point.

Inaccurate by hand.

Useful when using technology.

More appropriate for non-linear systems.

Must solve for the equation for y first.

Substitution : Solve one equation for one variable and then substitute that into the other equation.

Best algebraic technique for non-linear systems.

Works well when a variable can be solved for easily, has a coefficient of one.

Works better when fractions and roots aren't involved.

Addition / Elimination : Multiply one or more equations by a constant and then add the two equations together to eliminate one variable. Works well for a linear system when there is no variable with a coefficient of one. Works well for 2 equations with 2 variables systems of equations, but becomes tedious and labor intensive for larger systems.

Gaussian Elimination / Gauss Jordan Elimination : Uses elementary operations to produce equivalent equations. Works for non-square systems of linear equations.

Built upon the concepts of addition elimination, but instead of obtaining new equations, the old equation is replaced with an equivalent equation.

probably the fastest way to solve a large system of linear equations by hand.

Cramer's Rule

. **Matrix Algebra / Matrix Inverses :** We will deal with only with graphical and algebraic methods to solve pair of linear equations ,we will use other methods in higher classes.

Solving Equations Graphically

Graphical Solution : If there is a pair of numbers that can be substituted for x and y in two different equations, the pair form the coordinates of a point which lies on the graph of each equation. The only way in which a point can lie on two lines simultaneously is for the point to be at the intersection of the lines. Therefore, the graphical solution of two simultaneous equations involves drawing their graphs and locating the point at which the graph lines intersect. Hence steps involve in solving problems graphically.

Step 1: Graph the first equation ; To draw graph use x and y intercept.

x-intercept : The x-intercept is the point where the graph of an equation crosses the x-axis. The x-intercept can be found by substituting $y = 0$ into the equation and solving for x. The x-intercept is also called a solution, root, or zero of the equation.

y-intercept : The y-intercept is the point where the graph of an equation crosses the y-axis. The y-intercept can be found by substitution $x = 0$ into the equation and solving for y.

There is no requirement that an equation have either an x-intercept or y-intercept. It is also possible that there may be more than one of each intercept.

The graphical method is a quick and simple means of finding an approximate solution of two simultaneous equations. Each equation is graphed, and the point of intersection of the two lines is read as accurately as possible. A high degree of accuracy can be obtained but this, of course, is dependent on the precision with which the lines are graphed and the amount of accuracy possible in reading the graph.

Example, the line $5x + 3y = 15$ crosses the Y axis at (0,5). This may be verified by letting $x = 0$ in the equation. The X intercept is (3,0), since x is 8 when y is 0.

Picture shows the Line $5x + 3y = 15$ graphed by means of the X and Y intercepts

Step 2: Graph the second equation on the same coordinate system as the first.

Step 3: Find the solution.

If the two lines intersect at one place, then the point of intersection is the solution to the system. If the two lines are parallel, then they never intersect, so there is no solution.

If the two lines lie on top of each other, then they are the same line and you have an infinite number of solutions. In this case you can write down either equation as the solution to indicate they are the same line.

Step 4: Check the proposed ordered pair solution in both equations.

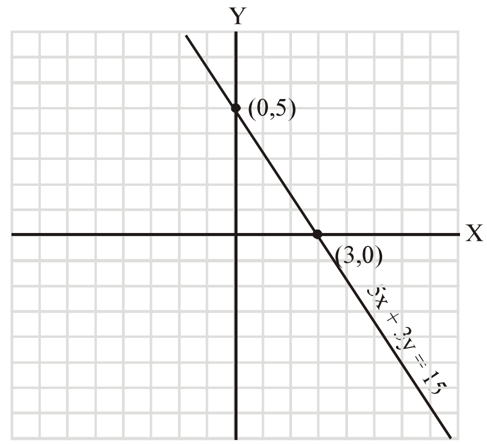


Figure : Graph of 5x + 3y = 15.

You can plug in the proposed solution into both equations. If it makes both equations true then you have your solution to the system. If it makes at least one of them false, you need to go back and redo the problem.

If the graphs of the pair of equations $ax + by + c = 0$ and $a'x + b'y + c' = 0$:

(i) Intersect each other, then the system has one and only one solution, i.e. the system has a unique solution and is said to be a consistent or compatible system.

(ii) Are parallel straight lines, then the system has no solution, and hence is an inconsistent system.

(iii) Are one and the same, i.e., are coincident straight lines, the system has infinite number of solutions and thus is a dependent system. This is also called a consistent system of equations.

Any system of two linear equations in two unknowns has to be one of the types of systems described above. Solving the system graphically is not necessary if you want to determine whether the system of equations is consistent or not. There are other algebraic methods to do so.

Method I. Comparison of y-forms

Let us compare the y-forms of the equation and put the result in the form of a table.

Equations	y-forms	Solutions	Type of System
1. $4x + 3y = 24$	$y = \frac{24 - 4x}{3}$	Unique solution	Consistent
$3y - 2x = 6$	$y = \frac{2x + 6}{3}$		
2. $y - 2x = 3$	$y = 2x + 3$	No solution	Inconsistent
$2y - 4x = 10$	$y = 2x + 5$		
3. $y - 2x = 3$	$y = 2x + 3$	Infinite number of solutions	Dependent
$2y - 4x = 6$	$y = 2x + 3$		

From the above table you can easily conclude that if in the y-forms of the equations :

1. The coefficients of x are different (even if the constant terms are the same), the system of equations has a unique solution.
2. The coefficients of x are the same but the constant terms are different, the system of equations has no solution.
3. The coefficients of 'x' and the constant terms are the same, i.e., the two y-forms are identical, the system has infinite number solutions.

Method 2. The Ratio Method

This is another method to determine whether a system of equations is consistent or not.

A system of linear equations ax + by + c = 0 and a'x + b'y + c' = 0 is :

(i) Consistent, if $\frac{a}{a'} \neq \frac{b}{b'}$ (ii) Inconsistent, if $\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$ (iii) dependent, if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

Example : 1

Solve equations x + y = 33 and x – y = 5

Sol. Draw two lines on graph using x and y intercept. For x + y = 33 (0, 33) and (33, 0) and for x – y = 5 (0, –5) and (5, 0). We see that they intersect in a single point. There is one pair of values comprising coordinates of that point (19, 14), and that pair of values satisfies both equations, as follows:

x + y = 33	x – y = 5
19 + 14 = 33	19 – 14 = 5

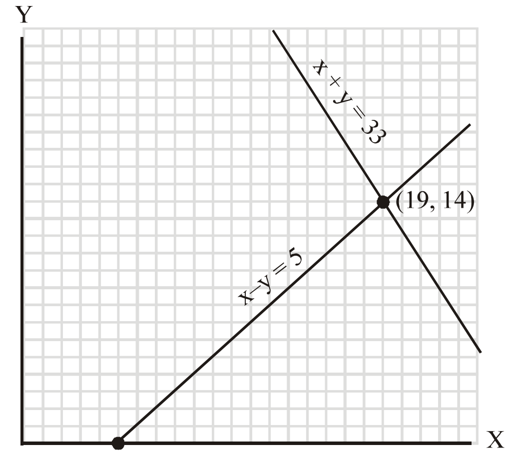


Figure : Graph of x + y = 33 and x - y = 5.

This pair of numbers satisfies each equation. It is the only pair of numbers that satisfies the two equations simultaneously.

Example : 2

Check whether the pair of equations

x + 3y = 6 .... (1) and 2x – 3y = 12 .... (2)

is consistent. If so, solve them graphically.

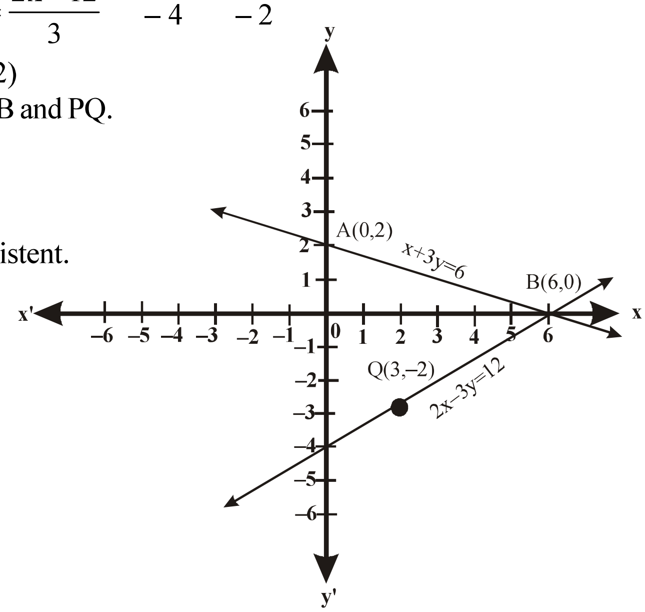
Sol. Let us draw the graphs of the Equations (1) and (2). For this, we find two solutions of each of the equations, which are given in Table.

x	0	6	x	0	3
$y = \frac{6-x}{3}$	2	0	$y = \frac{2x-12}{3}$	– 4	– 2

Plot the points A(0, 2), B(6, 0), P(0, – 4) and Q(3, – 2) on graph paper, and join the points to form the lines AB and PQ.

Point B (6, 0) common to both the lines AB and PQ.

So, the solution of the pair of linear equations is x = 6 and y = 0, i.e., the given pair of equations is consistent.



Example : 3

In each of the following, find whether the system is consistent, inconsistent or dependent :

(i) $5x + 2y = 16$ $7x - 4y = 2$

(ii) $5x + 2y = 16$ $3x + \frac{6}{5}y = 2$

(iii) $5x + 2y = 16$ $\frac{15}{2}x + 3y = 24$

Sol. (i) (a) Method 1 : Using y-forms

The y-forms of the two equations are : $y = -\frac{5}{2} + 8$ and $y = \frac{7}{4} \times -\frac{1}{2}$

Since the coefficient of 'x' are different in the two y-forms, the system of equations is consistent.

(b) Method 2 : The Ratio Method

Here a = 5, b = 2 and c = –16 and a' = 7, b' = –4 and c' = –2

$\therefore \frac{a}{a'} = \frac{5}{7}, \frac{b}{b'} = -\frac{1}{2}$ and $\frac{c}{c'} = 8$. Since, $\frac{a}{a'} \neq \frac{b}{b'}$, the system of equations is consistent.

(ii) Using y-forms of th two equations, we have $y = -\frac{5}{2}x + 8$ and $y = -\frac{5}{2}x + \frac{5}{3}$

Since the coefficient of x are the same and the constant terms are different, the system is inconsistent.

(iii) The ratio method: Here a = 5, b = 2 and c = –16, a' = 15/2, b'= 3 and c' = 24

$\therefore \frac{a}{a'} = \frac{5}{15/2} = \frac{2}{3}, \frac{b}{b'} = \frac{2}{3}$ and $\frac{-16}{-24} = \frac{2}{3}$ $\therefore \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \frac{2}{3}$

SELF CHECK

Q.1 On comparing the ratios $\frac{a_1}{a_2}, \frac{b_1}{b_2}$ and $\frac{c_1}{c_2}$, find out whether the lines representing the following pairs of linear equations intersect at a point, are parallel or coincident :

(i) $5x - 4y + 8 = 0$, $7x + 6y - 9 = 0$

(ii) $9x + 3y + 12 = 0$, $18x + 6y + 24 = 0$

(iii) $6x - 3y + 10 = 0$, $2x - y + 9 = 0$

Q.2 Which of the following pairs of linear equations are consistent/inconsistent ? If consistent, obtain the solution graphically:

(i) $x + y = 5, 2x + 2y = 10$ (ii) $x - y = 8, 3x - 3y = 16$

(iii) $2x + y - 6 = 0, 4x - 2y - 4 = 0$ (iv) $2x - 2y - 2 = 0, 4x - 4y - 5 = 0$

Q.3 Given the linear equation $2x + 3y - 8 = 0$, write another linear equation in two variables such that the geometrical representation of the pair so formed is: (i) intersecting lines (ii) parallel lines (iii) coincident lines

Q.4 Solve the following simultaneous systems graphically :

(i) $x + y = 8$, $x - y = 2$

(ii) $3x + 3y = 12$, $4x + 5y = 2$

ANSWERS

(1) (i) Intersect at a point   (ii) Coincident   (iii) Parallel

(2) (i) Consistent   (ii) Inconsistent   (iii) Consistent   (iv) Inconsistent

The solution of (i) above, is given by y = 5 – x, where x can take any value, i.e., there are infinitely many solutions. The solution (iii) above is x = 2, y = 2, i.e., unique solution.

(3) One possible answer for the three parts : (i) $3x + 2y - 7 = 0$ (ii) $2x + 3y - 12 = 0$ (iii) $4x + 6y - 16 = 0$

(4) x = 5, y = 3   (ii) x = 18, y = – 14

ALGEBRAIC METHODS

Substitution method : In some cases it is more convenient to use the substitution method of solving problems. In this method we solve one equation for one of the variables and substitute the value obtained into the other equation. This eliminates one of the variables, leaving an equation in one unknown

The method of substitution will work with non-linear as well as linear equations.

1. Solve one of the equations for one of the variables.
2. Substitute that expression in for the variable in the other equation.
3. Solve the equation for the remaining variable
4. Back-substitute the value for the variable to find the other variable.
5. Check

The process of back-substitution involves taking the value of the variable found in step 3 and substituting it back into the expression obtained in step 1 (or the original problem) to find the remaining variable.

It is important that both variables be given when solving a system of equations. A common mistake students make is to find one variable and stop there. You need to include a value for all the variables.

It is a good idea to check your answer into the both equations, but is probably sufficient to check in the equation you didn't isolate a variable in the first step. That is, if you solved for y in the first equation in step 1, use the second equation to check the answer.

Example: find the solution of the following system :

$$4x + y = 11 \quad ; \quad x + 2y = 8$$

It is easy to solve for either y in the first equation or x in the second equation. Let us solve for y in the first equation. The result is $y = 11 - 4x$

Since equals may be substituted for equals, we may substitute this value of y wherever y appears in the second equation. Thus, $x + 2(11 - 4x) = 8$

We now have one equation that is linear in x; that is, the equation contains only the variable x.

Removing the parentheses and solving for x, we find that

$x + 22 - 8x = 8$ or $-7x = 8 - 22$

$-7x = -14$ or $x = 2$

To get the corresponding value of y, we substitute $x = 2$ in $y = 11 - 4x$. The result is

$y = 11 - 4(2) = 11 - 8 = 3$

Thus, the solution for the two original equations is $x = 2$ and $y = 9$..

Example 4 :

Solve the following pair of equations by substitution method:

$7x - 15y = 2$ .... (1)

$x + 2y = 3$ .... (2)

Sol. Step 1 : We pick either of the equations and write one variable in terms of the other.

Let us consider the Equation (2) :

$x + 2y = 3$ and write it as $x = 3 - 2y$ .... (3)

Step 2 : Substitute the value of x in Equation (1). We get

$7(3 - 2y) - 15y = 2$

i.e., $21 - 14y - 15y = 2$ i.e., $-29y = -19$ Therefore, $y = 19/29$

Step 3 : Substituting this value of y in Equation (3), we get $x = 3 - 2\left(\frac{19}{29}\right) = \frac{49}{29}$

Therefore, the solution is $x = \frac{49}{29}$, $y = \frac{19}{29}$; Verification : Substituting $x = \frac{49}{29}$, $y = \frac{19}{29}$,

you can verify that both the Equations (1) and (2) are satisfied.

Example 5 :

Solve the following simultaneous equations by using the substitution method:

$x + y = 15 \quad ; \quad y = x + 3$

Sol. Label the equations as follows:

$x + y = 15$ .... (1)

$y = x + 3$ .... (2)

substituting $y = x + 3$ in (1) gives

$x + x + 3 = 15 ; 2x + 3 = 15$

$2x + 3 - 3 = 15 - 3 \quad \text{or} \quad 2x = 12 ; \frac{2x}{2} = \frac{12}{2} \Rightarrow x = 6$

When $x = 6$, $y = 6 + 3 = 9$ [From eq. (2)]. So, the solution is (6, 9).

Example 6 :

Solve the following simultaneous equations by using the substitution method: $x + 4y = 14 ; 7x - 3y = 5$

Sol. $x + 4y = 14$ .... (1) $7x - 3y = 5$ .... (2)

From eq. (1), $x = 14 - 4y$ .... (3)

Substitute the value of x in equation (2), $7(14 - 4y) - 3y = 5$

$\Rightarrow 98 - 28y - 3y = 5 \Rightarrow 98 - 31y = 5 \Rightarrow 93 - 31 \Rightarrow y = \frac{93}{11} \Rightarrow y = 3$

Now substitute value of y in eq. (3), $7x - 3(3) = 5 \Rightarrow 7x = 14 \Rightarrow x = 14/7 = 2$.

Solution is $x = 2, y = 3$

SELF CHECK

Q.1 Solve the following pair of linear equations by the substitution method.

(i) $x - t = 3$, $\frac{s}{3} + \frac{t}{2} = 6$ (ii) $0.2x + 0.3y = 1.3$, $0.4x + 0.5y = 2.3$ (iii) $\frac{3x}{2} - \frac{5y}{3} = -2$, $\frac{x}{3} + \frac{y}{2} = \frac{13}{6}$

Q.2 Solve $2x + 3y = 11$ and $2x - 4y = -24$ and hence find the value of 'm' for which $y = mx + 3$.

Q.3 Form the pair of linear equations for the following problems and find their solution by substitution method.

(i) The difference between two numbers is 26 and one number is three times the other. Find them.

(ii) The larger of two supplementary angles exceeds the smaller by 18 degrees. Find them.

Q.4 Solve the following systems by the substitution method:

(i) $2x - 9y = 1$, $x - 4y = 1$ (ii) $2x + y = 0$, $2x - y = 1$ (iii) $5r + 2s = 23$, $4r + s = 19$ (iv) $t - 4v = 1$, $2t - 9v = 3$

ANSWERS

(1) (i) $s = 9, t = 6$ (ii) $x = 2, y = 3$ (iii) $x = 2, y = 3$ **(2)** $x = -2, y = 5, m = -1$

(3) (i) $x - y = 26$, $x = 3y$, where x and y are two numbers ($x > y$), $x = 39, y = 13$.

(ii) $x - y = 18$, $x + y = 180$, where x and y are the measures of the two angles in degrees, $x = 99, y = 81$.

(4) (i) $x = 5, y = 1$ (ii) $x = 1/4, y = -1/2$ (iii) $r = 5, s = -1$ (iv) $t = -3, v = -1$

ADDITION / ELIMINATION

The idea behind the addition / elimination method is to multiple one or more equations by a constant so when they are added together, one of the variables eliminates. Then you have one equation with one variable and you can solve for that variable.

The addition method of solving systems of equations is illustrated in the following example:

$x - y = 2$

$x + y = 8$

$2x + 0 = 10 \Rightarrow x = 5$

The result in the foregoing example is obtained by adding the left member of the first equation to the left member of the second, and adding the right member of the first equation to the right member of the second.

Having found the value of x, we substitute this value in either of the original equations to find the value of y, as follows: $x - y = 2$

$\Rightarrow (5) - y = 2 \Rightarrow -y = 2 - 5 \Rightarrow -y = -3 \Rightarrow y = 3$

Notice that the primary goal in the addition method is the elimination (temporarily) of ,one of the variables. If the coefficient of y is the same in both equations, except for its sign, adding the equations eliminates y as in the foregoing example. On the other hand, suppose that the coefficient of the variable which we desire to eliminate is exactly the same in both equations.

In the following example, the coefficient of x is the same in both equations, including its sign:

$x + 2y = 4$; $x - 3y = -1$

Adding the equations would not eliminate either x or y. However, if we multiply both members of the second equation by –1, then addition will eliminate x, as follows:

$x + 2y = 4$

$-x + 3y = 1$

$5y = 5 \Rightarrow y = 1$

The value of x is found by substituting 1 for y in either of the original equations, as follows:

$x + 2(1) = 4 \Rightarrow x = 2$

As a second example of the addition method, find the solution of the simultaneous equations

$3x + 2y = 12$; $4x + 5y = 2$

Here both x and y have unlike coefficients. The coefficients of one of the variables must be made the same, except for their signs.

The coefficients of x will be the same except signs, if both members of the first equation are multiplied by 4 and both members of the second equation by –3. Then addition will eliminate x.

Following this procedure to get the value of y, we multiply the first equation by 4 and the second equation by -3, as follows:

$12x + 8y = 48$

$-12x - 15y = -6$

$-7y = 42 \Rightarrow y = -6$

Substituting for y in the first equation to get the value of x, we have

$3x + 2(-6) = 12$

$x + 2(-2) = 4$

$x - 4 = 4 \Rightarrow x = 8$

This solution is checked algebraically by substituting 8 for x and -6 for y in each of the original equations, as follows: (i) $3x + 2y = 12$; $3(8) + 2(-6) = 12$; $24 - 12 = 12$

(ii) $4x + 5y = 2$; $4(8) + 5(-6) = 2$; $32 - 30 = 2$

From above discussion you have following steps to solve equations.

1. Choose a variable to eliminate. Usually the variable that can be eliminated by multiplying by smaller numbers is the better choice.
2. Multiply one or both equations by a constant so that the least common multiple of the coefficients on the variable to be eliminated is obtained. Care should be taken so that one coefficient becomes negative and the other is positive.

3. Add the two equations together so the variable is eliminated.
4. Solve the resulting equation for the remaining variable.
5. Back-substitute that value into the one of the two original equations to find the remaining variable.
6. Check your answer into the other equation.
As an alternative to step 5, and this is extremely helpful when the answer is a fraction or decimal value and not pleasant to work with, you can go through the elimination process again with the other variable, and then you don't have to work with the fractions until the check process.

Example 7 :

The ratio of incomes of two persons is 9 : 7 and the ratio of their expenditures is 4 : 3. If each of them manages to save Rs 2000 per month, find their monthly incomes.

Sol. Let us denote the incomes of the two person by Rs 9x and Rs 7x and their expenditures by Rs 4y and Rs 3y respectively. Then the equations formed in the situation is given by :

$9x - 4y = 2000$ .... (1)

and $7x - 3y = 2000$ .... (2)

Step 1 : Multiply Equation (1) by 3 and Equation (2) by 4 to make the coefficients of y equal.
Then we get the equations:

$27x - 12y = 6000$ .... (3)

$28x - 12y = 8000$ .... (4)

Step 2 : Subtract Equation (3) from Equation (4) to eliminate y, because the coefficients of y are the same. So, we get $(28x - 27x) - (12y - 12y) = 8000 - 6000$ i.e., $x = 2000$

Step 3 : Substituting this value of x in (1), we get

$9(2000) - 4y = 2000$ i.e., $y = 4000$

So, the solution of the equations is $x = 2000$, $y = 4000$.
Therefore, the monthly incomes of the persons are Rs 18,000 and Rs 14,000, respectively.
Verification : 18000 : 14000 = 9 : 7.
Also, the ratio of their expenditures = 18000 – 2000 : 14000 – 2000 = 16000 : 12000 = 4 : 3

Example 8 :

Solve the following simultaneous equations by using the elimination method:

$2x + 3y = 15$; $4x - 3y = 3$

Sol. Label the equations as follows:

$2x + 3y = 15$ .... (1)

$4x - 3y = 3$ .... (2)

Notice that 3y appears on the left-hand side of both equations. Adding the left-hand side of (1) and (2), and then the right-hand sides, gives:

$$2x + 3y + 4x - 3y = 15 + 3 \; ; \; 6x = 18 \; ; \; \frac{6x}{6} = \frac{18}{6} \Rightarrow x = 3$$

We have added equals to equals, and addition eliminates y.
Substituting x = 3 in (1) gives : $2 \times 3 + 3y = 15$; $6 + 3y = 15$

$6x + 3y - 6 = 15 - 6$; $3y = 9$

$$\frac{3y}{3} = \frac{9}{3} \Rightarrow y = 3$$

So, the solution is (3, 3)

Example 9 :

Solve $9x - 4y = 8$; $13x + 7y = 101$

Sol. $9x - 4y = 8$ .... (1)

$13x + 7y = 101$ .... (2)

Multiply eq. (1) by 7 and eq. (2) by 4, we get

$63x - 28y = 56$

Add $52x + 28y = 404$

$115x = 460$

$\Rightarrow x = \frac{460}{115} \Rightarrow x = 4$

Substitute x = 4 in eq. (1), $9(4) - 4y = 8 \Rightarrow 36 - 8 = 4y \Rightarrow 28 - 4y \Rightarrow y = \frac{28}{4} = 7$ $\therefore x = 4, y = 7$

Example 10 :

The sum of a two-digit number and the number obtained by reversing the digits is 66. If the digits of the number differ by 2, find the number. How many such numbers are there ?

Sol. Let the ten's and the unit's digits in the first number be x and y, respectively.

So, the first number may be written as $10x + y$ in the expanded form (for example, 56 = 10(5) + 6).

When the digits are reversed, x becomes the unit's digit and y becomes the ten's digit. This number, in the expanded notation is $10y + x$ (for example, when 56 is reversed, we get 65 = 10(6) + 5).

According to the given condition.

$(10x + y) + (10y + x) = 66$

i.e., $11(x + y) = 66$ i.e., $x + y = 6$ .... (1)

We are also given that the digits differ by 2, therefore,

either $x - y = 2$ .... (2) or $y - x = 2$ .... (3)

If $x - y = 2$, then solving (1) and (2) by elimination, we get x = 4 and y = 2.

In this case, we get the number 42.

If $y - x = 2$, then solving (1) and (3) by elimination, we get x = 2 and y = 4.

In this case, we get the number 24. Thus, there are two such numbers 42 and 24.

Verification : Here 42 + 24 = 66 and 4 – 2 = 2. Also 24 + 42 = 66 and 4 – 2 = 2.

SELF CHECK

Q.1 Solve the following problems:

(i) $x + y = 24$, $x - y = 12$ (ii) $5t + 2v = 9$, $3t - 2v = -5$ (iii) $x - 2y = -1$, $2x + 3y = 12$ (iv) $2x + 7y = 3$, $3x - 5y = 51$

Q.2 Solve the following pair of linear equations by the elimination method and the substitution method :

(i) $x + y = 5$ and $2x - 3y = 4$ (ii) $3x - 5y - 4 = 0$ and $9x = 2y + 7$

Q.3 Form the pair of linear equations in the following problems, and find their solutions (if they exist) by the elimination method :

(i) If we add 1 to the numerator and subtract 1 from the denominator, a fraction reduces to 1. It becomes 1/2 if we only add 1 to the denominator. What is the fraction?

(ii) The sum of the digits of a two-digit number is 9. Also, nine times this number is twice the number obtained by reversing the order of the digits. Find the number.

(iii) A lending library has a fixed charge for the first three days and an additional charge for each day thereafter. Saritha paid Rs 27 for a book kept for seven days, while Susy paid Rs 21 for the book she kept for five days. Find the fixed charge and the charge for each extra day.

ANSWERS

(1) (i) x = 18, y = 6   (ii) t = 1/2, v = 13/4   (iii) x =3, y = 2   (iv) x = 12, y = –3

(2) (i) $x = \frac{19}{5}, y = \frac{6}{5}$ (ii) $x = \frac{9}{13}, y = -\frac{5}{13}$

(3) (i) x – y + 2 = 0, 2x – y – 1 = 0, where x and y are the numeragtor and denominator of the fraction, 3/5

(ii) x + y = 9 and 8x – y = 0, where x and y are respectively the tens and units digits of the number, 18

(iii) x + 4y = 27, x + 2y = 21, where x is the fixed charge (in Rs.) and y is the additional charge (in Rs.) per day, x = 15, y = 3.

CROSS- MULTIPLICATION METHOD

For any pair of linear equations in two variables of the form

$a_1x + b_1y + c_1 = 0$ .... (1)

and $a_2x + b_2y + c_2 = 0$ .... (2)

To obtain the values of x and y as shown above, we follow the following steps:

Step 1 : Multiply Equation (1) by b_2 and Equation (2) by b_1, to get

$b_2a_1x + b_2b_1y + b_2c_1 = 0$ .... (3)

$b_1a_2x + b_1b_2\,y + b_1c_2 = 0$ .... (4)

Step 2 : Subtracting Equation (4) from (3), we get:

$(b_2a_1 - b_1a_2)\,x + (b_2b_1 - b_1b_2)\,y + (b_2c_1 - b_1c_2) = 0$

i.e., $(b_2a_1 - b_1a_2)\,x = b_1c_2 - b_2c_1)$

So, $x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$, provided $a_1b_2 - a_2b_1 \neq 0$ .... (5)

Step 3 : Substituting this value of x in (1) or (2), we get

$y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$ .... (6)

Now, two cases arise :

Case 1 : $a_1b_2 - a_2b_1 \neq 0$. In this case $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$. Then the pair of linear equations has a unique solution.

Case 2 : $a_1b_2 - a_2b_1 = 0$. If we write $\frac{a_1}{a_2} = \frac{b_1}{b_2} = k$, then $a_1 = k\,a_2$, $b_1 = k\,b_2$.

Substituting the values of a_1 and b_1 in the Equation (1), we get

$k\,(a_2x + b_2y) + c_1 = 0$ .... (7)

It can be observed that the Equations (7) and (2) can both be satisfied only if

$c_1 = k\,c_2$, i.e., $\frac{c_1}{c_2} = k$

If $c_1 = k\,c_2$, any solution of Equation (2) will satisfy the Equation (1), and vice versa. So, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = k$, then there are infinitely many solutions to the pair of linear equations given by (1) and (2).

If $c_1 \neq k\,c_2$, then any solution of Equation (1) will not satisfy Equation (2) and vice versa. Therefore the pair has no solution. Thus,

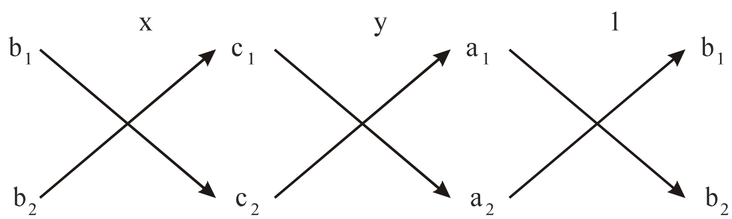
(i) When $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, we get a unique solution. (ii) When $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, there are infinitely many solutions.

(iii) When $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, there is no solution.

Note that you can write the solution given by Equations (5) and (6) in the following form :

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1} \quad .... (8)$$

In remembering the above result, the following diagram may be helpful to you :



The arrows between the two numbers indicate that they are to be multiplied and the second product is to be subtracted from the first.

For solving a pair of linear equations by this method, we will follow the following

Steps :

Step 1 : Write the given equations in the form (1) and (2).

Step 2 : Taking the help of the diagram above, write Equations as given in (8).

Step 3 : Find x and y, provided $a_1b_2 - a_2b_1 \neq 0$

Step 2 above gives you an indication of why this method is called the cross-multiplication method.

Example 11 :

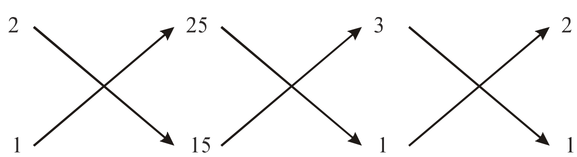
Solve 3x + 2y + 25 = 0, x + y + 15 = 0

Sol. $3x + 2y + 25 = 0$ .... (1) $x + y + 15 = 0$ .... (2)

Here, $a_1 = 3, b_1 = 2, c_1 = 25$

$a_2 = 1, b_2 = 1, c_2 = 15$

$\therefore$



$$\frac{x}{2\times 15 - 25\times 1} = \frac{y}{25\times 1 - 15\times 3} = \frac{1}{3\times 1 - 2\times 1} \; ; \; \frac{x}{30-25} = \frac{y}{25-45} = \frac{1}{3-2} \; ; \; \frac{x}{5} = \frac{y}{-20} = \frac{1}{1}$$

$$\frac{x}{5} = 1, \quad \frac{y}{-20} = 1 \quad \Rightarrow \quad x = 5, y = -20$$

Example 12 :

For what values of k will the following pair of linear equations have infinitely many solutions?

kx + 3y – (k – 3) = 0 ; 12x + ky – k = 0

Sol. Here, $\frac{a_1}{a_2} = \frac{k}{12}, \; \frac{b_1}{b_2} = \frac{3}{k}, \; \frac{c_1}{c_2} = \frac{k-3}{k}$

For a pair of linear equations to have finitely many solutions :

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$. So, we need $\frac{k}{12} = \frac{3}{k} = \frac{k-3}{k}$ or $\frac{k}{12} = \frac{3}{k}$ which gives $k^2 = 36$ i.e., $k = \pm 6$

Also, $\frac{3}{k} = \frac{k-3}{k}$ gives $3k = k^2 - 3k$, i.e., $6k = k^2$, which means $k = 0$ or $k = 6$.

Therefore, the value of k, that satisfies both the conditions, is k = 6. For this value, the pair of linear equations has infinitely many solutions.

Example 13 :

Find the value of k for which the system of linear equation : kx + 4y = k – 4, 16x + ky = k, has many solutions.

Sol. kx + 4y = k – 4 .... (1) 16x + ky = k .... (2)

$a_1 = k$, $b_1 = 4$, $c_1 = -(k-4)$; $a_2 = 16$, $b_2 = k$, $c_2 = -k$

Here condition is $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$; $\frac{k}{16} = \frac{4}{k} = \frac{k-4}{k}$; $\frac{k}{16} = \frac{4}{k}$ $\Rightarrow k^2 = 64 = \pm 8$

Also, $\frac{4}{k} = \frac{k-4}{k}$ $\Rightarrow$ $4k = k^2 - 4k$ $\Rightarrow$ $k^2 - 8k = 0 \Rightarrow k(k-8) = 0$

k = 0 or k = 8 but k = 0 is not possible otherwise equation will be one variable.

$\therefore$ k = 8 is correct value for many solution.

SELF CHECK

Q.1 Which of the following pairs of linear equations has unique solution, no solution, or infinitely many solutions. In case there is a unique solution, find it by using cross multiplication method.

(i) x – 3y – 3 = 0
3x – 9y – 2 = 0

(ii) x – 3y – 7 = 0
3x – 3y – 15 = 0

Q.2 For which values of a and b does the following pair of linear equations have an infinite number of solutions

2x + 3y = 7
(a – b) x + (a + b) y = 3a + b – 2

Q.3 Solve the following pair of linear equations by the substitution and cross-multiplication methods :

8x + 5y = 9
3x + 2y = 4

Q.4 Form the pair of linear equations in the following problems and find their solutions (if they exist) by any algebraic method :

(i) A fraction becomes $\frac{1}{3}$ when 1 is subtracted from the numerator and it becomes $\frac{1}{4}$ when 8 is added to its denominator. Find the fraction.

(ii) Places A and B are 100 km apart on a highway. One car starts from A and another from B at the same time. If the cars travel in the same direction at different speeds, they meet in 5 hours. If they travel towards each other, they meet in 1 hour. What are the speeds of the two cars ?

ANSWERS

(1) (i) No solution (ii) Unique solution, x = 4, y = – 1 **(2)** a = 5, b = 1

(3) x = – 2, y = 5

(4) (i) 3x – y – 3 = 0, 4x – y – 8 = 0, where x and y are the numerator and denominator of the fraction, $\frac{5}{12}$

(ii) u – v = 20, u + v = 1000, where u and v are the speeds (in km/h) of the atwo cars, u = 60, v = 40.

RECIPROCAL EQUATION

An equation which contains variables in reciprocal (or fractional) form is called reciprocal equation. To solve the simultaneous equation in reciprocal form we multiply the equations by such numbers which equate the coefficients of one of the variables. Now we add or subtract the resulting equation, thus obtaining a single equation in one variable only. We then find the value of this variable. Lastly we substitute the value of this variable thus obtained in either of the given equations and then we obtain the value of the other variable.

Example 14 :

Solve $\frac{15}{x}+\frac{2}{y}=17, \frac{1}{x}+\frac{1}{y}=\frac{36}{5}$

Sol. We have the equations : $\frac{15}{x}+\frac{2}{y}=17$ .... (1) and $\frac{1}{x}+\frac{1}{y}=\frac{36}{5}$ .... (2)

Multiplying eq. (2) by 2, we get, $\frac{2}{x}+\frac{2}{y}=\frac{72}{5}$ .... (3)

Subtracting (3) from (1), we get

$$\frac{15}{x}-\frac{2}{y}=17-\frac{72}{5} \Rightarrow \frac{13}{x}=\frac{85-72}{5} \Rightarrow \frac{13}{x}=\frac{13}{5} \Rightarrow x \times 13 = 13 \times 5 \Rightarrow x=\frac{13\times 5}{13}=5$$

Putting the value of x in (2)

$$\frac{1}{5}+\frac{1}{y}=\frac{36}{5} \Rightarrow \frac{1}{y}=\frac{36}{5}-\frac{1}{5} \Rightarrow \frac{1}{y}=\frac{35}{5} \Rightarrow 35 \times y = 5 \Rightarrow y=\frac{5}{35} \therefore y=\frac{1}{7} \text{ Hence, } x=5,\ y=\frac{1}{7}$$

SELF CHEK

Q.1 Solve the following pairs of equations by reducing them to a pair of linear equations:

(i) $\frac{1}{2x}+\frac{1}{3y}=2$; $\frac{1}{3x}+\frac{1}{2y}=\frac{13}{6}$ (ii) $\frac{4}{x}+3y=14$; $\frac{3}{x}-4y=23$

(iii) $\frac{5}{x-1}+\frac{1}{y-2}=2$; $\frac{6}{x-1}-\frac{3}{y-2}=1$ (iv) $6x+3y=6xy$; $2x+4y=5xy$

(v) $\frac{1}{3x+y}+\frac{1}{3x-y}=\frac{3}{4}$; $\frac{1}{2(3x+y)}-\frac{1}{2(3x-y)}=\frac{-1}{8}$

ANSWERS

(1) (i) $x=\frac{1}{2}, y=\frac{1}{3}$ (ii) $x=\frac{1}{5}, y=-2$ (iii) x = 4, y = 5 (iv) x = 1, y = 2 (v) x =1, y = 1

WORD PROBLEMS

Many problems can be solved quickly and easily using one equation with one variable. Other problems that might be rather difficult to solve in terms of one variable can easily be solved using two equations and two variables. The difference in the two methods is shown in the following example, solved first by using one variable and then using two.

Example : Find the two numbers such that half the first equals a third of the second and twice their sum exceeds three times the second by 4.

Solution using one variable :

1. Let x = the first number.

2. Then $\frac{\pi}{2} = \frac{1}{3}$ of the second number, **3.** Thus $\frac{3x}{2} =$ the second number.

From the statement of the problem, we then have

$$2\left(x + \frac{3x}{2}\right) = 3\left(\frac{3x}{2}\right) + 4 \quad ; \quad 2x + 3x = \frac{9x}{2} + 4$$

$$10x = 9x + 8 \;;\; x = 8 \text{ (first number)}, \quad \frac{3x}{2} = 12 \text{ (second number)}$$

SOLUTION USING TWO VARIABLES

If we let x and y be the first and second numbers, respectively, we can write two equations almost directly from the statement of the problem. Thus, $\frac{x}{2} = \frac{y}{3}$ or 2 (x + y) = 3y + 4

Solving for x in the first equation and substituting this value in the second, we have

$$x = \frac{2y}{3} \text{ or } 2\left(\frac{2y}{3} + y\right) = 3y + 4 \;;\; \frac{4y}{3} + 2y = 3y + 4 \Rightarrow 4y + 6y = 9y + 12 \Rightarrow y = 12 \text{ (Second number)}$$

$$\frac{x}{2} = \frac{12}{3} \Rightarrow x = 8 \text{ (first number)}$$

Thus, we see that the solution using two variables is more direct and simple. Often it would require a great deal of skill to manipulate a problem so that it might be solved using one variable; whereas the solution using two variables might be very simple. The use of two variables, of course, involves the fact that the student must be able to form two equations from the information given in the problem.

Steps you can use to solve problems;

(i) Represent the unknown quantities by variable x and y, which are to be determined.

(ii) Find the conditions given in the problem and translate the verbal conditions into a pair of simultaneous linear equation.

(iii) Solve these equations and obtain the required quantities with appropriate units.

Type of problems : Determining two no.'s when the relation between them is given, Problems regarding fractions, digits of a number, ages of person, current of a river, regarding time and distance, mensuration and geometry, time and work, regarding mixtures, cost of articles, profit & loss, discount.

Example 15 :

The numerator of a fraction is 4 less than the denominator if the numerator is decreased by 2 and the denominator is increased by 1, then the denominator is eight times the numerator find the fraction.

Sol. y – x = 4 .... (1)

y + 1 = 8 (x – 2) .... (2) Answer : 3/7

Example 16 :

The sum of two-digit number and the number obtained by reversing the order of its digits is 165. If the digits differ by 3, find the number.

Sol. Let unit digit be x ten's digit be y no. will be 10 y + x.

Acc. to problem, (10y + x) + (10x + y) = 165

$x + y = 15$ .... (1)

and $x - y = 3$ .... (2)

or $-(x - y) = 3$ .... (3)

$x + y = 15$	$x + y = 15$
$x - y = 3$	$-x + y = 3$
$2x = 18$	$2y = 18$
$\therefore\ x = 9$	$\therefore\ y = 9$
$\therefore\ y = 6$	$\therefore\ x = 6$
$\therefore$ No. will be 6, 9	$\therefore$ No. will be 9, 6

Example 17 :

A boat goes 12 km. upstream and 40 km downstream in 8 hours. It can go 16 km, upstream and 32 km downstream in the same time. Find the speed of the boat in still water and the speed of the stream.

Sol. Let the speed of the boat in still water be x km/hour and the speed of the stream be y km/hr. then speed of boat in downstream is $(x + y)$ km/hr. and the speed of boat upstream is $(x - y)$ km/hr.

In Ist case : Distance covered in upstream = 12 km $\therefore$ time = $\frac{12}{x - y}$ hr.

distance covered in downstream = 40 km $\therefore$ time = $\frac{40}{x + y}$ hr.

Total time is 8 hrs. $\therefore\ \frac{12}{x - y} + \frac{40}{x + y} = 8$

In IInd case : Distance covered in upstream = 16 km $\therefore$ time = $\frac{16}{x - y}$ hr.,

downstream = 32 km $\therefore$ time = $\frac{32}{x + y}$ hr.

Total time taken = 8 hrs. $\therefore\ \frac{16}{x - y} + \frac{32}{x + y} = 8$

Solve them to get , x = Speed of boat = 6 km/hr, y = speed of stream = 2 km/hr.

Example 18 :

Points A and B are 90 km apart from each other on a highway. A car starts from A and another from B at the same time. If they go in the same direction, they meet in 9 hrs. and if they go in opposite directions, they meet in 9/7 hours. Find their speeds.

Sol. Let the speeds of the cars starting than A and B x km/hr and y km/hr. respectively

Acc. to problem, $9x - 90 = 9y$ .... (1)

and $\frac{9}{7}x + \frac{9}{7}y = 90$ .... (2)

Solving we get x = 40 km/hr, y = 30 km/hr.

Speed of car A = 40 km/hr speed of car B = 30 km/hr.

Example 19 :

A vessel contain's mixture of 24ℓ milk and 6ℓ water and a second vessel contains a mixture of 15ℓ milk and 10ℓ water. How much mixture of milk and water should be taken from the first and the second vessel separately and kept in a third vessel so that the third vessel may contain a mixture of 25ℓ milk and 10ℓ water ?

Sol. Let xℓ of mixture be taken from Ist vessel and yℓ of the mixture be taken from 2nd vessel and kept in 3rd vessel so that $(x + y)\ell$ of the mixture in third vessel may contain 25ℓ of milk and 10ℓ of water.

A mixture of $x\,\ell$ from 1st vessel contains $\frac{24}{30}x = \frac{4}{5}x\ell$ of mile and $\frac{x}{5}\ell$ of water. And a mixture of yℓ from 2nd vessel contains $\frac{3y}{5}\ell$ of milk and $\frac{2y}{5}\ell$ of water. $\therefore\ \frac{4}{5}x + \frac{3}{5}y = 25$ .... (1) ; $\frac{x}{5} + \frac{2}{5}y = 10$ .... (2)

Solve it to get x and y, i.e., $x = 20\ell,\ y = 15\ell$

SELF CHECK

Q.1 Solve the following pairs of equations by reducing them to a pair of linear equationFormulate the following problems as a pair of equations, and hence find their solutions:

(i) Ritu can row downstream 20 km in 2 hours, and upstream 4 km in 2 hours. Find her speed of rowing in still water and the speed of the current.

(ii) 2 women and 5 men can together finish an embroidery work in 4 days, while 3 women and 6 men can finish it in 3 days. Find the time taken by 1 woman alone to finish the work, and also that taken by 1 man alone.

(iii) Roohi travels 300 km to her home partly by train and partly by bus. She takes 4 hours if she travels 60 km by train and the remaining by bus. If she travels 100 km by train and the remaining by bus, she takes 10 minutes longer. Find the speed of the train and the bus separately.

Q.2 2 men and 3 boys together can do a piece of work in 8 days. The same swork is done in 6 days by 3 men and 2 boys together. How long would 1 boy alone or 1 man along take to complete the work.

Q.3 The sum of two no's is 18, the sum of their reciprocal is 1/4. Find the numbers.

Q.4 A man sold a chair and a table together for Rs. 1520 there by making a profit of 25% on the chair and 10% on table. By selling them together for Rs. 1535, he would have made a profit of 10% on the chair and 25% on the table. Find cost price of each.

Q.5 A man went to the Reserve Bank of India with a note of Rs. 500. He asked the cashier to give him Rs. 5 and Rs. 10 notes in return. The cashier gave him 70 notes in all. Find how many notes of Rs. 5 and Rs. 10 did the man receive.

Q.6 The sum of the digits of a two-digit number is 12. The number obtained by the inter changing the two digits exceeds the given number by 18. Find the number.

Q.7 A farmer wishest to purchase a number of sheep found that if they cost him Rs. 42 a head, he would not have money enough by Rs. 28, bu if they cost him Rs. 40 a head, he would then have Rs. 40 more than he required, find the number of sheeps and the money which he had.

ANSWERS

(1) (i) $u + v = 10,\ u - v = 2$, where u and v are respectively speeds (in km/h) of rowing and current, $u = 6, v = 4$. (ii) $\frac{2}{n} + \frac{5}{n} = \frac{1}{4},\ \frac{3}{n} + \frac{6}{m} = \frac{1}{3}$, where n and m are the number of days taken by 1 woman and 1 man to finish the embroideey work, n = 18, m = 36 (iii) $\frac{60}{u} + \frac{240}{v} = 4,\ \frac{100}{u} + \frac{200}{v} = \frac{25}{6}$, where u and v are respectively the speeds (in km/h) of the train and bus, $u = 60, v = 80$.

(2) One boy can do in 120 days and one man can do in 20 days. (3) No's are 12 and 6

(4) Chair = Rs. 600, table = Rs. 700 (5) 40, Rupee 5 notes, 30 Rupee 10 notes

(6) 57 (7) 34 sheep, Rs. 1400

ADDITIONAL EXAMPLES

Example 1 :

Solve the following simultaneous equations by using the substitution method:

$x = 2y + 10\ ,\ \ 2x + y = 5$

Sol. Label the equations as follows:

$x = 2y + 10$ ..... (1) ; $2x + y = 5$ ..... (2)

Sustituting $x = 2y + 10$ in (2) gives : $2(2y + 10) + y = 5$

$$4y + 20 + y = 5 \Rightarrow 5y + 20 = 5 \Rightarrow 5y = -15\ ;\ \frac{5y}{5} = \frac{-15}{2} \Rightarrow y = -3$$

Substituting $y = -3$ in (1) gives : $x = 2(-3) + 10 = 4$, So, the solution is $(4, -3)$

Example 2 :

Solve the following equation : $2(2x + 3) - 10 \le 6(x - 2)$

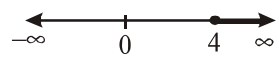
Sol. We have, $2(2x + 3) - 10 \le 6(x - 2)$

$\Rightarrow 4x + 6 - 10 \le 6x - 12 \quad \Rightarrow \quad 4x - 4 \le 6x - 12$

$\Rightarrow 4x - 6x \le -12 + 4$ [Transposing – 4 to RHS and 6x to LHS]

$$\Rightarrow -2x \le -8 \quad \Rightarrow -\frac{2x}{-2} \ge \frac{-8}{-2} \Rightarrow x \ge 4 \Rightarrow x \in [4, \infty)$$

Hence, the solution set of of the given inequation is $[4, \infty)$ which can be graphed on real line as shown in figure.



Example 3 :

Draw the graph of the equation $2x + 3y = 7$ and verify from the graph if $x = 2$ and $y = 3$ is a solution of the equation.

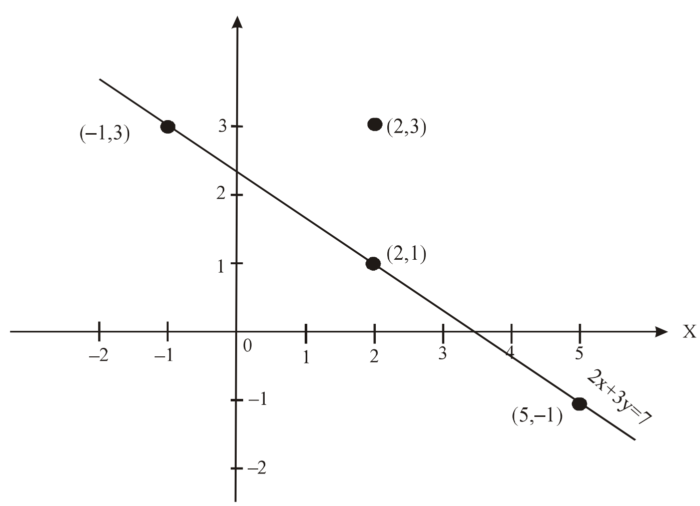
Sol. Equation $2x + 3y = 7 \Rightarrow 3y = 7 - 2x$

$$\Rightarrow y = \frac{7 - 2x}{3}$$

When $x = -1$, $y = \frac{7 - 2(-1)}{3} = 3$

When $x = 2$, $y = \frac{7 - 2(2)}{3} = 1$

When $x = 5$, $y = \frac{7 - 2(5)}{3} = -1$



2x + 3y = 7

x	–1	2	5
y	3	1	–1

Since the point $x = 2, y = 3$ does not lie on the line, it is not a solution of the equation.

Example 4 :

Solve the systems of equations graphically : $\frac{2x+1}{3}+\frac{3y-1}{2}=2$; $\frac{3x-1}{2}+\frac{2y+1}{3}=2$

Sol. $\frac{2x+1}{3}+\frac{3y-1}{2}=2$ .... (1) $\frac{3x-1}{2}+\frac{2y+1}{3}=2$ .... (2)

$\frac{2x+1}{3}+\frac{3y-1}{2}=2$

Multiplying both sides by 6

$2(2x+1)+3(3y-1)=12$

$\Rightarrow 4x+2+9y-3=12$

$\Rightarrow 4x+9y=13$

$\Rightarrow 9y=13-4x$

$\Rightarrow y=\frac{13-4x}{9}$

When x = 1, $y=\frac{13-4(1)}{9}=1$

When x = 10, $y=\frac{13-4(10)}{9}=-3$

When x = 19, $y=\frac{13-4(19)}{9}=-7$

Equation (1)

x	1	10	19
y	1	–3	–2

$\frac{3x-1}{2}+\frac{2y+1}{3}=2$

Multiplying both sides by 6

$3(3x-1)+2(2y+1)=12$

$\Rightarrow 9x-3+4y+2=12$

$\Rightarrow 9x+4y=13$

$\Rightarrow 4y=13-9x$

$\Rightarrow y=\frac{13-9x}{4}$

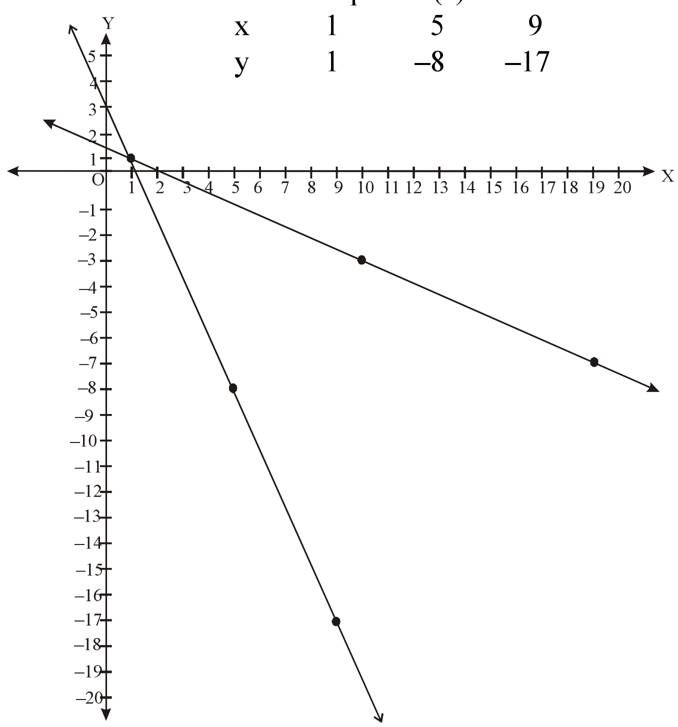
When x = 1, $y=\frac{13-9(1)}{4}=1$

When x = 5, $y=\frac{13-9(5)}{4}=-8$

When x = 9, $y=\frac{13-9(9)}{4}=-17$

Equation (2)

x	1	5	9
y	1	–8	–17



From the graph, the solution is the point of intersection of the lines, i.e., x = 1, y = 1

Example 5 :

Solve the equations : $\frac{2x+1}{3}+\frac{3y+2}{5}=2$

Sol. $\frac{2x+1}{3}+\frac{3y+2}{5}=2$ .... (1) and $\frac{2(2x+1)}{3}-\frac{3(3y+2)}{5}=-1$ .... (2)

Let $\frac{2x+1}{3}=u$ and $\frac{3y+2}{5}=v$

The the equations become

$u+v=2$ .... (3) $\quad 2u-3v=-1$ .... (4)

Multiplying (3) by 3,

$3u+3v=6$ .... (5)

Adding (4) and (5), $5u=5 \Rightarrow u=1$

Substituting this value of u in (3), $1+v=2 \Rightarrow v=2-1=1$

Then $\frac{2x+1}{3}=u=1$ and $\frac{3y+2}{5}=v=1$

$\Rightarrow$ $2x+1=3$ and $3y+2=5$

$\Rightarrow$ $2x=3-1=2$ and $3y=5-2=3$

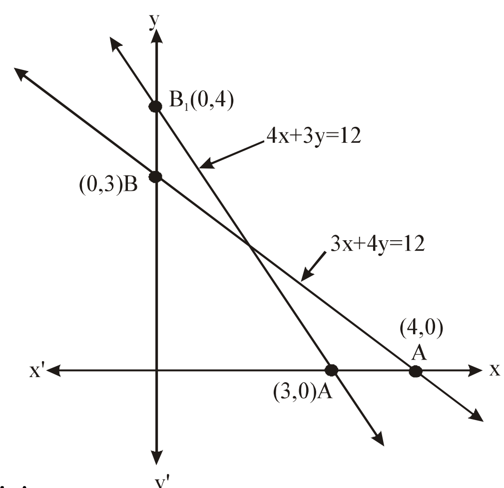
$\Rightarrow$ $x=1$ and $y=1$

Therefore, the solution is $x=1, y=1$

Example 6 :

Exhibit graphically the solution set of the linear inequations :

$3x+4y \le 12,\ 4x+3y \le 12,\quad x \ge 0, y \ge 0$



Sol. Converting the inequations into equations, the inequations reduce to $3x+4y=12$, $4x+3y=12$, $x=0$ and $y=0$. Region represented by $3x+4y \le 12$. The line $3x+4y=12$ meets the coordinate axes at A (4, 0) and B (0, 3). Draw a thick line joining A and B. Point (0, 0) satisfies inequation $3x+4y \le 12$. So, the portion containing the origin represents the solution set of the inequation $3x+4y \le 12$. Region represented by $3x+4y \le 12$. The line $4x+3y=12$ meets the x and y-axis at A_1 (3, 0) and B_1 (0, 4) respectively.
Join these two points by a thick line. Clearly, the region containing the origin is represented by the inequation $3x+4y \le 12$ as shown in figure. Region repersented by $x \ge 0$ and $y \ge 0$. Clearly, $x \ge 0$ and $y \ge 0$ represents the solution set of the given linear inequatons.

Example 7 :

Solve the systems of equations by the cross-multiplication method.

$ax+by=c$; $bx-ay=c$

Sol. $ax+by=c$, $bx-ay=c$. Using the cross-multiplication method,

$$\frac{x}{\begin{vmatrix} c & b \\ c & -a \end{vmatrix}}=\frac{y}{\begin{vmatrix} a & c \\ b & c \end{vmatrix}}=\frac{z}{\begin{vmatrix} a & b \\ b & -a \end{vmatrix}} \quad \Rightarrow \quad \frac{x}{-ac-bc}=\frac{y}{ac-bc}=\frac{1}{-a^2-b^2}$$

$$\Rightarrow x = \frac{-ac - bc}{-a^2 - b^2} = \frac{-c(a+b)}{-(a^2+b^2)} = \frac{c(a+b)}{a^2+b^2} \quad \text{and} \quad y = \frac{ac - bc}{-a^2 - b^2} = \frac{c(a-b)}{-(a^2+b^2)} = -\frac{c(a+b)}{a^2+b^2}$$

Therefore, $x = \frac{c(a+b)}{a^2+b^2}$, $y = -\frac{c(a+b)}{a^2+b^2}$

Example 8 :

Solve the system of equations : $ax + by = 1$; $bx + ay = \frac{2ab}{a^2+b^2}$

Sol. $ax + by = 1$ .... (1) $\quad bx + ay = \frac{2ab}{a^2+b^2}$ .... (2)

$$(a+b)\,x + (a+b)\,y = 1 + \frac{2ab}{a^2+b^2} \;;\; (a+b)\,x + (a+b)\,y = \frac{a^2+b^2+2ab}{a^2+b^2} \Rightarrow (a+b)\,(x+y) = \frac{(a+b)^2}{a^2+b^2}$$

$\Rightarrow x + y = \frac{a+b}{a^2+b^2}$ .... (3)

Subtracting (2) from (1), $(a-b)x + (b-a)y = 1 - \frac{2ab}{a^2+b^2} \Rightarrow (a-b)x - (a-b)y = \frac{a^2+b^2-2ab}{a^2+b^2}$

$\Rightarrow (a-b)(x-y) = \frac{(a-b)^2}{a^2+b^2} \Rightarrow x - y = \frac{a-b}{a^2+b^2}$ .... (4)

Adding (3) and (4), $2x = \frac{a+b}{a^2+b^2} + \frac{a-b}{a^2+b^2} = \frac{2a}{a^2+b^2} \Rightarrow x = \frac{a}{a^2+b^2}$

Subtracting (4) from (3), $2y = \frac{a+b}{a^2+b^2} - \frac{a-b}{a^2+b^2} = \frac{2b}{a^2+b^2} \Rightarrow y = \frac{a}{a^2+b^2}$

Therefore the solution is , $x = \frac{a}{a^2+b^2}$, $y = \frac{b}{a^2+b^2}$

Example 9 :

A part of the monthly expenses of a family is constant and the remaining varies with the price of wheat. When the price of wheat is Rs. 250 per quintal, the total monthly expenses are Rs. 1000 and when it is Rs. 240 per quintal, the total monthly expenses Rs. per 980. Find the total monthly expenses of the family when the cost of wheat is Rs. 350 per quintal.

Sol. Let the constant part of the expenditure = Rs. x
and the variable part = Rs. y × price of wheat.
Given that when the price of wheat is Rs. 250 per quintal, the total expenses are Rs. 1000.
$\therefore x + 250y = 1000$ .... (1)
Given alo that when the price of wheat is Rs. 240 per qunital, the total expenses are Rs. 980
$\therefore x + 240y = 980$ .... (2)
Subtracting (2) from (1), $10y = 20 \Rightarrow y = 2$
Substituting this value of y in (1), $x + 250\,(2) = 1000 \Rightarrow x = 1000 - 500 = 500$
Therefore, when the price of wheat is Rs. 350 per quintal,
total expenses = x + 350 y = 500 + 350 (2) = Rs. 1200 $\quad \therefore$ total expenses = Rs. 1200

Example 10 :

It takes 12 hours to fill a swimming pool using 2 pipes. If the larger pipe is used for 4 hours and the smaller pipe for 9 hours, only half the pool is filled. How long would it take for each pipe alone to fill the pool ?

Sol. Let the time taken to fill the pool by the larger pipe = x hours and that by the smaller pipe = y hours.
Therefore, in 1 hour, volume of pool filled by the larger pipe = 1/x
and by the smaller pipe = 1/y
Given that both pipes can fill the pool in 12 hours.

$$\therefore \frac{12}{x}+\frac{12}{y}=1 \quad \text{.............} (1)$$

Given also that if the larger pipe is used for 4 hours and the smaller pipe for 9 hours, only half the pool is filled.

$$\therefore \frac{4}{x}+\frac{9}{y}=\frac{1}{2} \quad \text{.............} (2)$$

Multiplying (2) by 3, $\frac{12}{x}+\frac{27}{y}=\frac{3}{2}$ .... (3)

Subtracting (3) from (1), $\frac{12}{y}-\frac{27}{y}=1-\frac{3}{2} \Rightarrow -\frac{15}{y}=-\frac{1}{2} \Rightarrow y = 30$

Subtracting the value of y in (1), $\frac{12}{x}+\frac{12}{30}=1 \Rightarrow \frac{12}{x}=1-\frac{12}{30}=1-\frac{2}{5}=\frac{3}{5} \Rightarrow x = 20$

$\therefore$ time taken by the larger pipe = 20 hr. and time taken by the smaller pipe = 30 hr.

Example 11 :

The sum of the digits of a two-digit number is 8. If the digits are reversed, the number is decreased by 54. Find the original number.

Sol. Let the two-digit number be 10x + y.
Then, we have : x + y = 8 .... (i)
and 10y + x = 10x + y 54 or, x – y = 54/9 = 6 .... (ii)
Solving equations (i) and (ii), we get x = (8 + 6)/2 = 7 and y = 1
$\therefore$ The required number = 7 × 10 + 1 = 71
But the same question can be solved using this sample formula. The required number

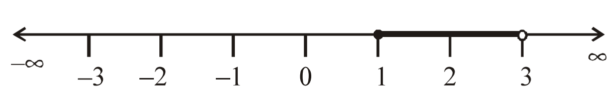
$$= 5\left[\text{Sum of digits}+\frac{\text{Decrease}}{9}\right]+\frac{1}{2}\left[\text{Sum of digits}-\frac{\text{Decrease}}{9}\right]$$

= 5(8 + 6) + 1/2 (8 – 6) = 70 + 1 = 71.

Example 12 :

Solve and graph the solution set of $3x + 6 \geq 9$ and $-5x > -15$, $x \in R$

Sol. $3x + 6 \geq 9$ and $-5x > -15$
$\Rightarrow 3x \geq 3 \quad \Rightarrow -x \geq -3$
$\Rightarrow x \geq 1 \quad \Rightarrow x < 3$
Combining the solution
So, the solution is $x \in [1, 3)$



Example 13 :

Solve and graph the solution set of $-2 < 2x < -6$ and $-2x + 5 \geq 13$, $x \in R$

Sol. $-2 < 2x < -6$

$\Rightarrow$ $2x - 6 > -2$ and $-2x + 5 \geq 13$

$\Rightarrow$ $2x > 4$ and $-2x \geq 13 - 5$

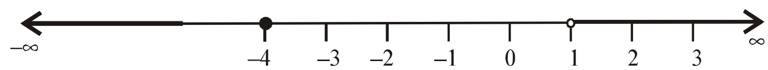
$\Rightarrow$ $x > 2$ and $-2x \geq 8$

$\Rightarrow$ $x > 2$ and $-x \geq 4$

$\Rightarrow$ $x > 2$ and $x \leq -4$

$\therefore$ $x > 2$ or $x \leq -4$

or $x \in (-\infty, -4] \cup (2, \infty)$



Example 14 :

Solve the following inequations : $\dfrac{5x-2}{3} - \dfrac{7x-3}{5} > \dfrac{x}{4}$

Sol. Given : $\dfrac{5x-2}{3} - \dfrac{7x-3}{5} > \dfrac{x}{4} \Rightarrow \dfrac{5(5x-2)-3(7x-3)}{15} > \dfrac{x}{4} \Rightarrow \dfrac{25x-10-21x+9}{15} > \dfrac{x}{4} \Rightarrow \dfrac{4x-1}{15} > \dfrac{x}{4}$

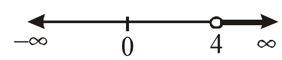
$\Rightarrow$ $4(4x-1) > 15x$ [Multiplying both sides by 60 i.e., 1 cm of 15 and 4]

$\Rightarrow$ $16x - 4 > 15x \Rightarrow$ $16x - 15x > 4$ [Transposing 15x to LHS – 4 to RHS]

$\Rightarrow$ $x > 4$ $\Rightarrow$ $x \in (4, \infty)$

Hence, the solution set of the given inequation is $(4, \infty)$.

This can be graphed on the real number line as shown in figure.



Example 15 :

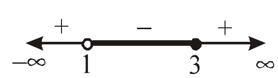
Solve the following inequatiosn : $\dfrac{2x+4}{x-1} \geq 5$

Sol. We have, $\dfrac{2x+4}{x-1} \geq 5 \Rightarrow \dfrac{2x+4}{x-1} - 5 \geq 0 \Rightarrow \dfrac{2x+4-5(x-1)}{x-1} \geq 0$

$\Rightarrow$ $\dfrac{-3x+9}{x-1} \geq 0 \Rightarrow \dfrac{3x-9}{x-1} \leq 0$ [Multiplying both sides by –1]

$\Rightarrow$ $\dfrac{3(x-3)}{x-1} \leq 0 \Rightarrow \dfrac{x-3}{x-1} \leq 0$ [Dividing both sides by 3]

$\Rightarrow$ $1 < x \leq 3$ $\Rightarrow x \in (1, 3]$



Hence the soution set of the given inequations (1, 3].

Example 16 :

Solve : $-5 \leq \dfrac{2-3x}{4} \leq 9$

Sol. $-5 \times 4 \leq \dfrac{2-3x}{4} \times 4 \leq 9 \times 4$ [Multiplying throughout by 4]

$-20 \leq 2 - 3x \leq 36$

$-20 - 2 \leq -3x \leq 36 - 2$ [Subtracting 2 throughout]

$-22 \le -3x \le 34$

$\frac{-22}{-3} \le x \le \frac{34}{-3}$ [Dividing throughout by –3]

$\frac{22}{3} \ge x \ge \frac{-34}{3}$ or $\frac{-34}{3} \le x \le \frac{22}{3}$ or $x \in [-34/3, 22/3]$

Hence, the internal [–34/3, 22/3] is the solution set of the given system of inequations.

Example 17 :

Find x from $1 > |x| < 2$ and represent it on number line.

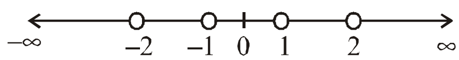
Sol. $1 < |x| \Rightarrow |x| > 1 \Rightarrow x > 1$ or $x < -1$

$\therefore$ $x \in (-\infty, -1) \cup (1, \infty)$ .... (1)

also $|x| < 2 \Rightarrow x < 2$ or $x > -2$ $\therefore$ x lies between – 2 and 2.

$x \in (-2, 2)$ .... (2)

Combining the two results $1 < |x| < 2 \Rightarrow \{-2 < x < -1\} \cup \{1 < x < 2\}$ i.e., $x \in (-2, -1) \cup (1, 2)$



Example 18 :

Find x satisfying $|x - 5| \le 3$

Sol. As $|x - a| \le 3 \Leftrightarrow a - r \le x \le a + r$ i.e., $x \in [a - r, a + r]$

$\therefore$ $|x - 5| \le 3 \Leftrightarrow 5 - 3 \le x \le 5 + 3$ i.e., $2 \le x \le 8$ i.e., $x \in [2, 8]$

Example 19 :

Solve the following inequations graphically : $2x + 3y \le 6$

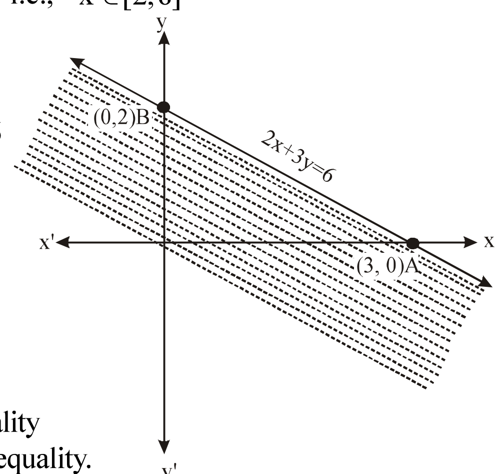
Sol. Converting the given inequation into equation, the equation is 2x + 3y = 6. Putting y = 0 and x = 0 respectively in this equation, x = 3 and y = 2 are the points at y = 0 and x = 0 respecitvely. So, this line meets x-axis at A (3, 0) and y-axis at B (0, 2).

Plot these points and join them by a thick line. This line divides the xy-plane in two parts.

The determine the region represented by the given inequality consider the point O (0, 0). Clearly, (0, 0) satisfies the inequality.

So, the region containing the origin is represented by the given inequation as shown in figure.

This represents the solution set of the given inequations.



Example 20 :

Solve the following inequations graphically : $|x - y| \ge 1$

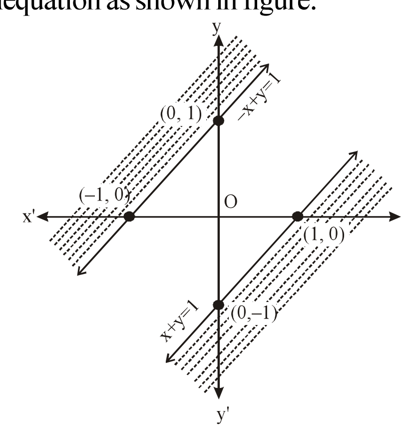
Sol. We have, $|x - y| \ge 1 \Leftrightarrow x - y \ge 1$ or $x - y \le -1$

$\Leftrightarrow x - y - 1 \ge 0$ or $x - y + 1 \le 0$

The required region is the union of regions represented by $x - y - 1 \ge 0$ and $x - y + 1 \le 0$ as shown in figure.

The shaded region represents the solution set of the given in equation.

Hence, these cannot be solved to find the values of x and y.



CONCEPT MAP

PAIR OF LINEAR EQUATION IN TWO VARIABLES

* **Equations :** If the polynomial equates to zero, it is called equation.
* A equation is called **Linear equation** if the highest power of variable is 1
  $ax + b = 0;\ a \neq 0$
* A equation is called **Quadratic equation** if highest power of variable is 2
  $ax^2 + bx + c = 0;\ a \neq 0$
* A equation is called **Cubic equation** if highest power of variable is 3
  $ax^3 + bx^2 + cx + d = 0;\ a \neq 0$

System of linear equations in two variables are of type $a_1x + b_1y + c_1 = 0$,
$a_2x + b_2y + c_2 = 0$,
Linear equation in three variables is of type $ax + by + cz + d = 0,\ b \neq 0,\ c \neq 0$

* **Solution of linear equation is the value of variables which satisfy the equation.**
  Solution of linear equation in one variable $\rightarrow$ There is only one solution
  Solution of linear equation in two variables $\rightarrow$ There are infinite number of solutions
  Solution of **system of linear equations** in two variable $\rightarrow$ It is the value of variables which satisfy both the equations. These equations have either

1. **A unique solution** $\rightarrow$ If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, such a system is called **consistent**. The graph consists of two intersecting lines.
2. **No solution** $\rightarrow$ If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, such a system is called **inconsistent**. The graph consists of two parallel lines.
3. **Infinitely many solutions** :
   If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, such a system is called **dependent**. The graph consists of two coincident lines.

* **Methods of Solution of system of linear equations**
1. Graphical solution
2. Elimination method
   (a) By Substitution
   (b) By equating coefficients
3. Cross multiplication

* **To solve problems based on time, distance and speed :**
  (i) Let the unknown quantities be x and y.
  (ii) Using given conditions in the question form two equations in x and y.
  Using the following results whichever is required. (a) Distance = speed × time
  (b) Speed along the current
  = speed of swimmer + speed of current
  (c) Speed against the current
  = speed of swimmer – speed of current
  (iii) Solve the two equations to get the values of x and y.

* To find the nuumber
  (i) Read the question carefully and let y and x be the digit at units and tens places respectively.
  (ii) Required number will be $10x + y$.
  (iii) Form two equations in x & y using the given conditions.
* For $a_1x + b_1y + c_1 = 0$ .... (1)
  and $a_2x + b_2y + c_2 = 0$ .... (2)

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$