5

COMPLEX NUMBERS

IMAGINARY NUMBER

Square root of a negative real number is an imaginary number, while solving equation $x^2 + 1 = 0$ we get $x = \pm\sqrt{-1}$ which is imaginary. So the quantity $\sqrt{-1}$ is denoted by 'i' called 'iota' thus $i = \sqrt{-1}$

Further $\sqrt{-5}$, $\sqrt{-3}$, $\sqrt{-9}$ ....may be expressed as $\pm i\sqrt{5}, \pm i\sqrt{3}, \pm 3i$..............

Integral powers of iota (i)

We have $i = \sqrt{-1}$ and $i^2 = -1$.
So $i^3 = i^2 . i = (-1)\, i = -i$ and $i^4 = (i^2)^2 = (-1)^2 = 1$.
Note that i^0 is defined as 1.
To find the values of i^n, $n > 4$, we first divide n by 4. Let m be the quotient and r be the remainder. Then $n = 4m + r$, where $0 \le r \le 3$.
$\therefore\ i^n = i^{4m+r} = (i^4)^m\, i^r = (1)^m\, i^r = i^r$ $\quad [\because\ i^4 = 1]$
Thus if $n > 4$, then $i^n = i^r$, where r is the remainder when n is divided by 4. The values of the negative integral powers of i are found as given below :

$$i^{-1} = \frac{1}{i} = \frac{i^3}{i^4} = i^3 = -i, \qquad i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1,$$

$$i^{-3} = \frac{1}{i^3} = \frac{i}{i^4} = \frac{i}{1} = i, \qquad i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

Note :

(i) $i^2 = i \times i = \sqrt{-1} \times \sqrt{-1} \ne \sqrt{1}$

(ii) $\sqrt{-a} \times \sqrt{-b} \ne \sqrt{ab}$ so for two real numbers a and b
$\sqrt{ab} = \sqrt{a}\cdot\sqrt{b}$ possible if both a, b are non-negative.

(iii) 'i' is neither positive, zero nor negative. Due to this reason order relations are not defined for imaginary numbers.

Example 1 :

Find the value of $\left[i^{19} + \left(\frac{1}{i}\right)^{25}\right]^2$

Sol. $\left[i^{19} + \left(\frac{1}{i}\right)^{25}\right]^2 = \left[i^{19} + \left(\frac{1}{i^{25}}\right)\right]^2$

$$= \left[i^3 + \left(\frac{1}{i}\right)\right]^2 = \left[-i + \left(\frac{i^3}{i^4}\right)\right]^2$$

$= [-i + i^3]^2 = (-i - i)^2 = 4\, i^2 = -4$

Example 2 :

Find the value of $\dfrac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}} - 1$

Sol. Given expression

$$= \frac{i^{10}\left(i^{582} + i^{580} + i^{578} + i^{576} + i^{574}\right)}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}} - 1$$

$= i^{10} - 1 = (i^2)^5 - 1 = (-1)^5 - 1$
$= -1 - 1 = -2$

COMPLEX NUMBER

A number of the form $z = x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$ is called a complex number where x is called as real part and y is called imaginary part of complex number and they are expressed as $Re\,(z) = x$, $Im\,(z) = y$
Here if $x = 0$ the complex number is purely imaginary and if $y = 0$ the complex number is purely Real.
A complex number may also be defined as an ordered pair of real numbers and may be denoted by the symbol (a, b). If we write $z = (a, b)$ then a is called the real part and b the imaginary part of the complex number z.

ALGEBRAIC OPERATIONS WITH COMPLEX NUMBER

Addition : $(a + ib) + (c + id) = (a + c) + i(b + d)$
Subtraction : $(a + ib) - (c + id) = (a - c) + i\,(b - d)$
Multiplication : $(a + ib)\,(c + id) = ac + iad + ibc + i^2 bd$
$= (ac - bd) + i\,(ad + bc)$

Division : $\dfrac{a + ib}{c + id} = \dfrac{(a + ib)(c - id)}{(c + id)(c - id)}$

(When at least one of c and d is non zero)

$$= \frac{(ac + bd)}{c^2 + d^2} + i\,\frac{(bc - ad)}{c^2 + d^2}$$

Properties of Algebraic Operations with Complex Number :

Let z, z_1, z_2 and z_3 are any complex number then their algebraic operation satisfy following properties

Commutativity : $z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1$

Associativity: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

Identity element : If $0 = (0, 0)$ and $1 = (1, 0)$ then $z + 0 = 0 + z = z$ and $z.\ 1 = 1.\ z = z$.
Thus 0 and 1 are the identity elements for addition and multiplication respectively.

Inverse element : Additive inverse of z is $-z$ and multiplicative inverse of z is $1/z$.

Cancellation law : $\left.\begin{array}{l} z_1 + z_2 = z_1 + z_3 \\ z_2 + z_1 = z_3 + z_1 \end{array}\right\} \Rightarrow z_2 = z_3$ and $z_1 \neq 0$

$$\left.\begin{array}{l} z_1 z_2 = z_1 z_3 \\ z_2 z_1 = z_3 z_1 \end{array}\right\} \Rightarrow z_2 = z_3$$

Distributivity : $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$
and $(z_2 + z_3)z_1 = z_2 z_1 + z_3 z_1$

Multiplicative inverse of a non- zero complex number (Reciprocal of a complex number) : Multiplicative inverse of a nonzero complex number $z = x + iy$ is

$$z^{-1} = \frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2}$$

$$= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} \quad \text{i.e. } z^{-1} = \frac{\text{Re}(z)}{|z|^2} + i\frac{-\text{Im}(z)}{|z|^2}$$

Example 3 :

Find the multiplicative inverse of $z = 3 - 2i$.

Sol. $z^{-1} = \frac{3}{3^2 + (-2)^2} + \frac{i(-(-2))}{3^2 + (-2)^2} = \frac{3}{13} + \frac{2}{13}i = \frac{1}{13}(3 + 2i)$

Equality of complex numbers :

Two complex numbers are said to be equal if and only if their real parts and imaginary parts are separately equal if $a + ib = c + id$, then $a = c$ & $b = d$

Note :

(i) If $z = 0 \Rightarrow x + iy = 0 \Rightarrow x = 0$ and $y = 0$

(ii) $x, y \in R$ and $x, y \neq 0$ then if $x + y = 0 \Rightarrow x = -y$ is correct but $x + iy = 0 \Rightarrow x = -iy$ is incorrect.

(iii) Inequality relation does not hold good in case of complex numbers having nonzero imaginary parts. For example the statement $8 + 5i > 4 + 2i$ makes no sense.

(iv) Complex number '0' is purely real and purely imaginary both.

Example 4 :

If $(x + iy)(2 - 3i) = 4 + i$, then find the value of x and y.

Sol. $x + iy = \frac{4+i}{2-3i} = \frac{(4+i)(2+3i)}{13} = \frac{5+14i}{13}$

$\therefore x = 5/13, y = 14/13.$

Example 5 :

Find the values of x and y satisfying the equation

$$\frac{(1+i)x - 2i}{3+i} + \frac{(2-3i)y + i}{3-i} = i$$

Sol. $\frac{(1+i)x - 2i}{3+i} + \frac{(2-3i)y + i}{3-i} = i$

$\Rightarrow (4 + 2i)x + (9 - 7i)y - 3i - 3 = 10i$

Equating real and imaginary parts, we get

$2x - 7y = 13$ and $4x + 9y = 3$. Hence $x = 3$ and $y = -1$

SQUARE ROOT OF A COMPLEX NUMBER

If $z = x + iy$

Suppose $\sqrt{z} = \sqrt{x+iy} = a + ib$

$\Rightarrow x + iy = a^2 - b^2 + 2iab$

On comparing the real and imaginary parts

$x = a^2 - b^2$, $y = 2ab$

Now, $a^2 + b^2 = \sqrt{x^2 + y^2} = |z|$ .....(i)

$a^2 - b^2 = x$ .....(ii)

From equation (i) and (ii)

$$a = \pm\sqrt{\frac{|z| + x}{2}}, \; b = \pm\sqrt{\frac{|z| - x}{2}}$$

Solving these two equations we shall get the required square roots as follows :

$$\pm\left[\sqrt{\frac{|z|+x}{2}} + i\sqrt{\frac{|z|-x}{2}}\right] \text{ if } y > 0$$

and $\pm\left[\sqrt{\frac{|z|+x}{2}} - i\sqrt{\frac{|z|-x}{2}}\right]$ if $y < 0$

Note : (i) The square root of i is $\pm\left(\frac{1+i}{\sqrt{2}}\right)$ (Here b = 1)

(ii) The square root of $-i$ is $\pm\left(\frac{1-i}{\sqrt{2}}\right)$ (Here b = –1)

Example 6 :

Find the square roots of $7 + 24i$.

Sol. Here $|z| = 25$, $x = 7$

Hence square root

$$= \pm\left[\left(\frac{25+7}{2}\right)^{1/2} + i\left(\frac{25-7}{2}\right)^{1/2}\right] = \pm(4 + 3i)$$

TRY IT YOURSELF-1

Q.1 Evaluate : i^{135}.

Q.2 If $(a + b) - i(3a + 2b) = 5 + 2i$, then find a and b.

Q.3 If $z = x + iy$, $z^{1/3} = a - ib$ and $\frac{x}{a} - \frac{y}{b} = k(a^2 - b^2)$, then find the value of k.

Q.4 If one root of the equation $z^2 - az + a - 1 = 0$ is $(1 + i)$, where a is a complex number, then find the other root.

Q.5 Express $\frac{(1+i)^2}{3-i}$ in the standard form a + i b.

Q.6 Find square root of $9 + 40i$.

Q.7 Express $\left(\frac{1}{3} + 3i\right)^3$ in the standard form a + i b.

Q.8 Find the multiplicative inverse of $\sqrt{5} + 3i$

ANSWERS

(1) –1 **(2)** a = –12, b = 17 **(3)** 4 **(4)** z = 1

(5) $-\frac{1}{5}+\frac{3}{5}i$ **(6)** (5 + 4i) or – (5 + 4i)

(7) $\frac{-242}{27}-26i$ **(8)** $\frac{\sqrt{5}}{14}-\frac{3}{14}i$

REPRESENTATION OF A COMPLEX NUMBER

Cartesian Representation : The complex number z = x + iy = (x, y) is represented by a point P whose coordinates are referred to rectangular axis xox' and yoy', which are called real and imaginary axes respectively. Thus a complex number z is represented by a point in a plane, and corresponding to every point in this plane there exists a complex number such a plane is called Argand plane or Argand diagram or complex plane or gaussian plane.

Note : (i) Distance of any complex number from the origin is called the modulus of complex number and is denoted by

$|z|$ Thus, $|z| = \sqrt{x^2+y^2}$.

(ii) Angle of any complex number with positive direction of x–axis is called amplitude or argument of z.

Thus, amp (z) = arg(z) = $\theta = \tan^{-1}\frac{y}{x}$.

Polar Representation: If z = x + iy is a complex number then z = r (cosθ + i sinθ) is a polar form of complex number z where x = r cosθ, y = r sinθ and r = $\sqrt{x^2+y^2}$ = | z |.

Exponential Form: If z = x + iy is a complex number then its exponential form is z = $re^{i\theta}$ where r is modulus and θ is amplitude of complex number.

Vector Representation: If z = x + iy is a complex number such that it represent point P(x, y) then its vector representation is z = $\overrightarrow{OP}$.

Example 7 :

Find the polar form of – 1 + i.

Sol. ∵ $|-1+i| = \sqrt{2}$, amp (–1 + i) = π – π/4 = 3π/4

∴ $-1 + i = \sqrt{2}$ (cos 3π/4 + i sin 3π/4)

Example 8 :

If z = $re^{i\theta}$, then find the value of $|e^{iz}|$

Sol. If $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

⇒ iz = ir (cosθ + i sinθ) = – r sinθ + ir cosθ

or $e^{iz} = e^{(-r\sin\theta + ir\cos\theta)}) = e^{-r\sin\theta}\, e^{i\, r\cos\theta}$

or $|e^{iz}| = |e^{-r\sin\theta}|\,|e^{r\, i\cos\theta}|$

$= e^{-r\sin\theta}[\cos^2(r\cos\theta)+\sin^2(r\cos\theta)]^{1/2} = e^{-r\sin\theta}$

CONJUGATE OF A COMPLEX NUMBER

In a complex number if we replace i by – i, we get conjugate of complex number. If a + ib is complex number it's conjugate is a – ib. Here both numbers will be conjugate to each other. It is represented by $\bar{z}$ and $\bar{z}$ is mirror image of z in real axis on Argand plane.

Properties of Conjugate Complex Number

Let z = a + ib and $\bar{z}$ = a – ib then

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2a = 2\,\text{Re}(z)$ = purely real

(iii) $z - \bar{z} = 2ib = 2i\,\text{Im}(z)$ = purely imaginary

(iv) $z\bar{z} = a^2 + b^2 = |z|^2$

(v) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$ (vi) $\overline{z_1-z_2} = \bar{z}_1 - \bar{z}_2$

(vii) $\overline{re^{i\theta}} = re^{-i\theta}$ (viii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

(ix) $\overline{z^n} = (\bar{z})^n$ (x) $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$

(xi) $|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)} = (z_1+z_2)(\bar{z}_1+\bar{z}_2)$

$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2$

(xii) $z + \bar{z} = 0$ or $z = -\bar{z}$ ⇒ z = 0 or z is purely imaginary

(xiii) $z = \bar{z}$ ⇒ z is purely real

Example 9 :

Find the conjugate of $\frac{1}{3+4i}$

Sol. $\frac{1}{3+4i} = \frac{3-4i}{(3+4i)(3-4i)} = \frac{1}{25} = (3-4i)$

⇒ conjugate of $\left(\frac{1}{3+4i}\right) = \frac{1}{25}(3+4i)$

Example 10 :

If z is a complex number such that $z^2 = (\bar{z})^2$, then

(1) z is purely real

(2) z is purely imaginary

(3) Either z is purely real or purely imaginary

(4) None of these

Sol. **(3).** Let z = x + iy, then its conjugate $\bar{z}$ = x – iy

Given that $z^2 = (\bar{z})^2 \Rightarrow x^2 - y^2 + 2ixy = x^2 - y^2 - 2ixy$

⇒ 4 ixy = 0 If x ≠ 0 then y = 0 and if y ≠ 0 then x = 0

MODULUS OF A COMPLEX NUMBER

If z = x + iy then modulus of z is equal to $\sqrt{x^2+y^2}$ and it is denoted by | z | . Thus $z = x+iy \Rightarrow |z| = \sqrt{x^2+y^2}$.

Note: Modulus of every complex number is a non negative real number.

Properties of Modulus of a Complex Number :

(i) $|z| \ge 0$ and $|z| = 0$ if and only if z = 0, i.e., x = 0, y = 0

(ii) $-|z| \le \text{Re}(z) \le |z|$ (iii) $-|z| \le \text{Im}(z) \le |z|$

(iv) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$ (v) $z\bar{z} = |z|^2$

(vi) $|z_1 z_2| = |z_1|\,|z_2|$ (vii) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ $(z_2 \neq 0)$

(viii) $|z^2| = |z|^2$ or $|z^n| = |z|^n$, $n \in N$
also $|z_1 z_2 \ldots\ldots z_n| = |z_1|\,|z_2| \ldots\ldots |z_n|$

(ix) $|z| = 1 \Leftrightarrow \bar{z} = \frac{1}{z}$ (x) $z^{-1} = \frac{\bar{z}}{|z|^2}$

(xi) $|z_1 \pm z_2| \leq |z_1| + |z_2|$ (xii) $|z_1 - z_2| \geq ||z_1| - |z_2||$
(xiii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,(|z_1|^2 + |z_2|^2)$
(xiv) $|re^{i\theta}| = r$
(xv) $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\mathrm{Re}\left(z_1\bar{z}_2\right)$

Example 11 :

Find the modulus of $(1+i)\frac{2+i}{3+i}$

Sol. $\left|(1+i)\frac{2+i}{3+i}\right| = |1+i|\frac{|2+i|}{|3+i|} = \sqrt{2}\cdot\frac{\sqrt{5}}{\sqrt{10}} = 1$

Example 12 :

If $z = x + iy$ and $\left|\frac{z-5i}{z+5i}\right| = 1$ then z lies on

(1) x-axis (2) y-axis
(3) line y = 5 (4) None of these

Sol. (1). $\left|\frac{z-5i}{z+5i}\right| = 1$

$\Rightarrow |z-5i|^2 = |z+5i|^2$
$\Rightarrow x^2 + (y-5)^2 = x^2 + (y+5)^2 \Rightarrow y = 0$

Example 13 :

If $z_1 = 3 + i$ and $z_2 = i - 1$, then
(1) $|z_1 + z_2| > |z_1| + |z_2|$ (2) $|z_1 + z_2| < |z_1| - |z_2|$
(3) $|z_1 + z_2| \leq |z_1| + |z_2|$ (4) $|z_1 + z_2| < |z_1| + |z_2|$

Sol. (4). $z_1 + z_2 = 2 + 2i$

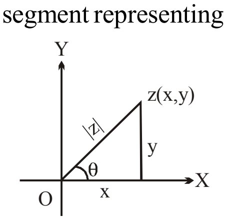
$\Rightarrow |z_1 + z_2| = \sqrt{4+4} = \sqrt{8}$. Now $|z_1| = \sqrt{10}$, $|z_2| = \sqrt{2}$.
It is clear that, $|z_1 + z_2| < |z_1| + |z_2|$

AMPLITUDE OR ARGUMENT OF A COMPLEX NUMBER

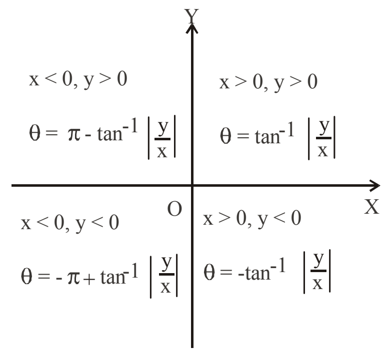
The amplitude or argument of a complex number z is the inclination of the directed line segment representing z, with real axis.

If $z = x + iy$ then

$\mathrm{amp}(z) = \tan^{-1}\left(\frac{y}{x}\right)$.



For finding the argument of any complex number first check that the complex number is in which quadrant and then find the angle θ and amplitude using the adjacent figure.



Note :

(i) Principle value of any complex number lies between $-\pi < \theta \leq \pi$

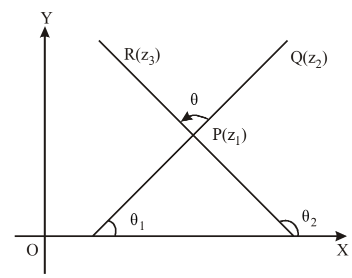
(ii) Amplitude of a complex number is a many valued function. If θ is the argument of a complex number then $(2n\pi + \theta)$ is also argument of complex number.

(iii) Argument of zero is not defined.

(iv) If a complex number is multiplied by iota (i) its amplitude will be increased by π/2 and will be decreased by π/2 If is multiplied by – i.

(v) Amplitude of complex number in I and II quadrant is always positive and in III and IV is always negative.

(vi) Let z_1, z_2, z_3 be the affixes of P, Q, R respectively in the Argand Plane.



Then from the figure the angle between PQ and PR is

$$\theta = \theta_2 - \theta_1 = \arg\,\overline{PR} - \arg\,\overline{PQ} = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

(a) If z_1, z_2, z_3 are collinear, thus θ = 0 therefore $\frac{z_3 - z_1}{z_2 - z_1}$ is purely real.

(b) If z_1, z_2, z_3 are such that PR ⊥ PQ, θ = π/2 So $\frac{z_3 - z_1}{z_2 - z_1}$ is purely imaginary .

Properties of Argument of a complex Number :

(i) amp (any real positive number) = 0
(ii) amp (any real negative number) = π
(iii) amp $(z - \bar{z}) = \pm\pi/2$
(iv) amp $(z_1 . z_2)$ = amp (z_1) + amp (z_2)
(v) amp $\left(\frac{z_1}{z_2}\right)$ = amp (z_1) – amp (z_2)
(vi) amp $(\bar{z})$ = – amp (z) = amp (1/z)

(vii) $\text{amp}(-z) = \text{amp}(z) \pm \pi$
(viii) $\text{amp}(z^n) = n\,\text{amp}(z)$
(ix) $\text{amp}(iy) = \pi/2$ if $y > 0$
$= -\pi/2$ if $y < 0$
(x) $\text{amp}(z) + \text{amp}(\bar{z}) = 0$

Example 14 :

Find the amplitude of $\sin\frac{\pi}{5} + i\left(1 - \cos\frac{\pi}{5}\right)$.

Sol. $\sin\frac{\pi}{5} + i\left(1-\cos\frac{\pi}{5}\right) = 2\sin\frac{\pi}{10}\cos\frac{\pi}{10} + i2\sin^2\frac{\pi}{10}$

$= 2\sin\frac{\pi}{10}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$

For amplitude, $\tan\theta = \dfrac{\sin\frac{\pi}{10}}{\cos\frac{\pi}{10}} = \tan\frac{\pi}{10} \Rightarrow \theta = \frac{\pi}{10}$

Example 15 :

Let z be a complex number such that $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$, then find the value of z.

Sol. Let $z = r(\cos\theta + i\sin\theta)$. Then $r = 4$, $\theta = \frac{5\pi}{6}$

$\therefore z = 4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 4\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = -2\sqrt{3} + 2i$

Example 16 :

Let z_1 and z_2 be two complex numbers with α and β as their principal arguments such that $\alpha + \beta > \pi$, then principal $\arg(z_1 z_2)$ is given by

(1) $\alpha + \beta + \pi$ (2) $\alpha + \beta - \pi$
(3) $\alpha + \beta - 2\pi$ (4) $\alpha + \beta$

Sol. We know that Principal argument of a complex number lie between $-\pi$ and π, but $\alpha + \beta > \pi$, therefore principal $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = \alpha + \beta$, is given by $\alpha + \beta - 2\pi$

Example 17 :

If $z = (1/i)$ then find $\arg(\bar{z})$.

Sol. $z = \frac{1}{i} = \frac{1}{i} \times \frac{-i}{-i} = \frac{-i}{+i} - i$

$\therefore \bar{z} = i$, which is the positive Imaginary quantity

$\therefore \arg(\bar{z}) = \pi/2$

Example 18 :

If $z = \frac{3-i}{2+i} + \frac{3+i}{2-i}$ then find $\arg(zi)$

Sol. $z = \frac{3-i}{2+i} + \frac{3+i}{2-i} = \frac{(3-i)(2-i)+(3+i)(2+i)}{(2+i)(2-i)}$

$z = 2 \Rightarrow (iz) = 2i$, which is the positive Imaginary quantity

$\therefore \arg(iz) = \pi/2$

TRY IT YOURSELF-2

Q.1 Find the real part of $(1-i)^{-i}$.

Q.2 Solve the equation $|z| = z + 1 + 2i$.

Q.3 Find real values of x and y for which complex numbers $-3 + i x^2 y$ and $x^2 + y + 4i$ are conjugate of each other.

Q.4 If $|z_1| = 1, |z_2| = 2, |z_3| = 3$ and $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$, then find the value of $|z_1 + z_2 + z_3|$.

Q.5 Find the greatest and the least value of $|z_1 + z_2|$ if $z_1 = 24 + 7i$ and $|z_2| = 6$.

Q.6 Find the amplitude of $\frac{1+\sqrt{3}i}{\sqrt{3}+i}$

Q.7 If $\arg(z_1) = 170°$ and $\arg(z_2) = 70°$, then find the principal argument of z_1z_2.

Q.8 Find the modulus and the arguments of $z = -1 - i\sqrt{3}$

Q.9 Solve : $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$

Q.10 If $\arg(z) < 0$, then $\arg(-z) - \arg(z) =$

(A) π (B) $-\pi$
(C) $-\pi/2$ (D) $\pi/2$

ANSWERS

(1) $e^{-\pi/4}\cos\left(\frac{1}{2}\log 2\right)$ **(2)** $x + iy = \frac{3}{2} - 2i$

(3) $(x^2+4)(x^2-1) = 0$ **(4)** 2

(5) 19, 25 **(6)** $\pi/6$ **(7)** $-120°$

(8) $2, -\frac{2\pi}{3}$ **(9)** $\frac{\sqrt{2} \pm i\sqrt{34}}{2\sqrt{3}}$ **(10)** (A)

DEMOIVRE'S THEOREM

(i) If n is any integer then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(ii) If $p, q \in I$ and $q \neq 0$ then

$(\cos\theta + i\sin\theta)^{p/q} = \cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)$

where $k = 0, 1, 2, 3, \ldots\ldots\ldots, q-1$

Note: (i) This theorem is not valid when n is not a rational number or the complex number is not in the form of $\cos\theta + i\sin\theta$

Ex. $(\cos\theta + i\sin\theta)^{\sqrt{5}} \neq (\cos\sqrt{5}\theta + i\sin\sqrt{5}\theta)$

$(\sin\theta + i\cos\theta)^n \neq \sin n\theta + i\cos n\theta$

$(\cos\theta + i\sin\theta)^8 = \cos 8\theta + i\sin 8\theta$

(ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots\ldots\ldots(\cos\theta_n + i\sin\theta_n)$
$= \cos(\theta_1 + \theta_2 + \theta_3 \text{-----} \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 \ldots\ldots\theta_n)$

(iii) The term $(\cos\theta + i\sin\theta)$ is also denoted by $\text{cis}\,\theta$

Example 19:

Find the value of $\left(\sin\frac{\pi}{5}+i\cos\frac{\pi}{5}\right)^5$

Sol. $\left(\sin\frac{\pi}{5}+i\cos\frac{\pi}{5}\right)^5$

$$=\left\{\cos\left(\frac{\pi}{2}-\frac{\pi}{5}\right)+i\sin\left(\frac{\pi}{2}-\frac{\pi}{5}\right)\right\}^5=\left(\cos\frac{3\pi}{10}+i\sin\frac{3\pi}{10}\right)^5$$

$$=\cos 5.\frac{3\pi}{10}+i\sin 5.\frac{3\pi}{10}=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}$$

$$=0+i(-1)=-i$$

Example 20 :

If $\frac{1}{x}+x=2\cos\theta$, then find the value of $x^n+\frac{1}{x^n}$

Sol. $\frac{1}{x}+x=2\cos\theta \Rightarrow x^2-2x\cos\theta+1=0$

$\Rightarrow x=\cos\theta \pm i\sin\theta \Rightarrow x^n=\cos n\theta \pm i\sin n\theta$

$\Rightarrow \frac{1}{x}=\frac{1}{\cos\theta\pm i\sin\theta} \Rightarrow \frac{1}{x}=\cos\theta \mp i\sin\theta$

$\Rightarrow \frac{1}{x^n}=\cos n\theta \mp i\sin n\theta$. Thus, $x^n+\frac{1}{x^n}=2\cos n\theta$

POWERS OF COMPLEX NUMBERS

To find the value of any power of a complex number $z=x+iy$ first we express z into the polar form.

i.e. $z=x+iy=r(\cos\theta+i\sin\theta)$, where $-\pi<\theta\le\pi$

then we use De-moivre's theorem to find z^n

i.e. $z^n=r^n(\cos\theta+i\sin\theta)^n$

$=r^n(\cos n\theta+i\sin n\theta)$

Thus, we have

No.	x+iy form	Polar form	General
1	$1+i0$	$\cos 0+i\sin 0$	$\cos 2n\pi+i\sin 2n\pi$
–1	$-1+i0$	$\cos\pi+i\sin\pi$	$\cos(2n+1)\pi+i\sin(2n+1)\pi$
i	$0+i(1)$	$\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$	$\cos(4n+1)\frac{\pi}{2}+i\sin(4n+1)\frac{\pi}{2}$
–i	$0+i(-1)$	$\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}$	$\cos(4n+1)\frac{\pi}{2}-i\sin(4n+1)\frac{\pi}{2}$

Example 21 :

Find the value of $\frac{(1+i)^8}{\left(1-i\sqrt{3}\right)^3}$

Sol. Exp. $=\frac{(\sqrt{2})^8(\cos\pi/4+i\sin\pi/4)^8}{2^3(\cos\pi/3-i\sin\pi/3)^3}=2\frac{\cos 2\pi+i\sin 2\pi}{\cos\pi-i\sin\pi}$

$=2(\cos 3\pi+i\sin 3\pi)=-2$

Example 22 :

If α and β are roots of the equation $x^2-2x+4=0$ then the find the value of $\alpha^{12}+\beta^{12}$

Sol. Solving the equation $x^2-2x+4=0$

we get $\alpha=1+i\sqrt{3}$; $\beta=1-i\sqrt{3}$

Here $\alpha^{12}+\beta^{12}=(1+i\sqrt{3})^{12}+(1-i\sqrt{3})^{12}$

Now $1+i\sqrt{3}=2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$

and $1-i\sqrt{3}=2(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3})$

$\therefore\ \alpha^{12}+\beta^{12}=[2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})]^{12}$

$+[2\cos(\frac{\pi}{3})-i\sin(\frac{\pi}{3})]^{12}$

$=2^{12}[\cos 4\pi+i\sin 4\pi]+2^{12}[\cos 4\pi-i\sin 4\pi]$

$=2^{12}(1+0)+2^{12}(1-0)=2^{12}+2^{12}=2^{12}(1+1)$

$\therefore\ \alpha^{12}+\beta^{12}=2.2^{12}=2^{13}$

EULER'S FORMULA

$e^{i\theta}=\cos\theta+i\sin\theta$ ....(1)

$e^{-i\theta}=\cos\theta-i\sin\theta$ ....(2)

From (1) and (2)

$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}\ \&\ \sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$$

Thus, $(e^{i\theta})^n=e^{i(n\theta)}=\cos n\theta+i\sin n\theta$

and $(e^{i\theta})^{-n}=e^{i(-n\theta)}=\cos n\theta-i\sin n\theta$

$$i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=e^{i\pi/2}$$

$$\log i=\log e^{\frac{i\pi}{2}}=\frac{i\pi}{2},\ \log(\log i)=\log\left(\frac{i\pi}{2}\right)$$

$$=\log i+\log\left(\frac{\pi}{2}\right)=\frac{i\pi}{2}+\log(\pi/2)$$

Example 23 :

If $x+\frac{1}{x}=2\cos\theta$, then find the value of $x^{12}+\frac{1}{x^{12}}$

Sol. Let $x=\cos\theta+i\sin\theta=e^{i\theta}$

then $x^{12}+\frac{1}{x^{12}}=e^{i12\theta}+\frac{1}{e^{i12\theta}}$

$=e^{i12\theta}+e^{-i12\theta}=\cos 12\theta+i\sin 12\theta+\cos 12\theta-i\sin 12\theta$

$=2\cos 12\theta$

Example 24 :

Find the value of i^i.

Sol. We know $(i)^i = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^i = (e^{i\,\pi/2})^{\,i} = e^{-\pi/2}$

APPLICATION OF DE-MOIVRE'S THEOREM

n^{th} Roots of Complex Number ($z^{1/n}$) :

To find the roots of a complex number, first we express it in polar form, then write the general value of amplitude and use the De-Moivre's theorem so,

$z^{1/n} = (x + iy)^{1/n} = r^{1/n}[\cos\theta + i\sin\theta]^{1/n}$

$= r^{1/n}[\cos(2m\pi + \theta) + i\sin(2m\pi + \theta)]^{1/n}$

$$= r^{1/n}\left[\cos\left(\frac{2m\pi+\theta}{n}\right) + i\sin\theta\left(\frac{2m\pi+\theta}{n}\right)\right]$$

where m = 0, 1, 2,............ (n – 1)

Thus there will be n distinct roots and these can be obtained by corresponding to m = 0, 1, 2, 3,..., (n –1) when m = 0, corresponding value is called the principal value of $z^{1/n}$.

Properties of the roots of $z^{1/n}$:

(i) Modulus of all roots of $z^{1/n}$ are equal & each equal to $r^{1/n}$ or $|z|^{1/n}$

(ii) All roots of $z^{1/n}$ lies on the circumference of a circle whose centre is origin and radius equal to $|z|^{1/n}$, Also these roots divides the circle into n equal parts and forms a polygon of n sides.

(iii) Amplitude of all the roots of $z^{1/n}$ are in A.P. with common difference $\frac{2\pi}{n}$

(iv) All roots of $z^{1/n}$ are in G.P. With common ratio $e^{2\pi i/n}$

(v) Sum of all roots of $z^{1/n}$ is always equal to zero.

(vi) Product of all roots of $z^{1/n} = (-1)^{n-1} z$

Roots of unity :

Consider the equation $x^n - 1 = 0$

$\therefore\ x = (1)^{1/n} = (1 + i\,0)^{1/n}$

$\Rightarrow\ x = [\cos 2m\pi + i\sin 2m\pi]^{1/n}$

$\Rightarrow\ x = [\cos\frac{2m\pi}{n} + i\sin\frac{2m\pi}{n}]$

$= e^{i(2m\pi/n)}$ where m = 0, 1, 2, ... (n – 1)

$= 1, e^{i(2\pi/n)}, e^{i(4\pi/n)}.........e^{\frac{i\,2(n-1)\pi}{n}}$

$= 1, \alpha, \alpha^2, ...\alpha^{n-1}$ where $\alpha = e^{i(2\pi/n)}$

Note :

(i) n^{th} root of unity are always in a G. P. with common ratio $e^{i(2\pi/n)}$

(ii) The sum of roots of unity is always zero.

Cube roots of unity :

In above case if n = 3, then for cube root of unity

$(1)^{1/3} = \cos\frac{2m\pi}{3} + i\sin\frac{2m\pi}{3}$, m = 0, 1, 2

$= 1,\ \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

$= 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i = 1, -\frac{1}{2}(1 \pm i\sqrt{3})$

Now if $\omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ then its square

$\omega^2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ and vice versa

Here $(1)^{1/3} = 1, \omega, \omega^2$ and $1 + \omega + \omega^2 = 0$, $\omega^3 = 1$

Note : (i) Cube root of unity are the vertices of an equilateral triangle.

(ii) If n = 4 the fourth roots of unity are $(1)^{1/4} = \pm 1, \pm i$

(iii) Fourth root of unity are vertices of a square which lies on coordinate axes.

Some Identities :

(a) $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$

(b) $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$

(c) $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$

(d) $x^2 - xy + y^2 = (x + y\omega)(x + y\omega^2)$

(e) $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$

Continued product of the roots :

If $z = r(\cos\theta + i\sin\theta)$ i. e. $|z| = r$ and amp. $z = \theta$

then continued product of roots of $z^{1/n}$ is $r(\cos\phi + i\sin\phi)$

$$\text{where}\quad \phi = \sum_{m=0}^{n-1}\frac{2m\pi+\theta}{n} = (n-1)\pi + \theta$$

Thus continued product of roots of

$z^{1/n} = r[\cos\{(n-1)\pi + \theta\} + i\sin\{(n-1)\pi + \theta\}]$

$$= \begin{cases} z, \text{if } n \text{ is odd} \\ -z, \text{if } n \text{ is even} \end{cases}$$

Similarly, the continued product of values of $z^{m/n}$ is

$$¾ \begin{cases} z^m, \text{if } n \text{ is odd} \\ (-z)^m, \text{if } n \text{ is even} \end{cases}$$

Sum of p^{th} Powers of n^{th} Roots of Unity :

The sum of p^{th} powers of n^{th} roots of unity

$$= \begin{cases} n, \text{when } p \text{ is a multiple of } n \\ 0, \text{when } p \text{ is not a multiple of } n \end{cases}$$

Example 25 :

If $x = a + b$, $y = a\omega + b\omega^2$ and $z = a\omega^2 + b\omega$, then find the value of $x^3 + y^3$.

Sol. $\because\ x + y + z = a(1 + \omega + \omega^2) + b(1 + \omega + \omega^2) = 0$

$(\because\ 1 + \omega + \omega^2 = 0)$

$\Rightarrow x^3 + y^3 + z^3 = 3xyz$

$= 3(a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega)$

$= 3(a+b)[a^2\omega^3 + b^2\omega^3 + ab(\omega^2 + \omega^4)]$
$= 3(a+b)[a^2 + b^2 + ab(\omega^2 + \omega)]$
$= 3(a+b)(a^2 + b^2 - ab) = 3(a^3 + b^3)$

Example 26 :
If $x = a$, $y = b\omega$, $z = c\omega^2$, where ω is a complex cube root of unity, then find the value of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c}$.

Sol. Given that $x = a$, $y = b\omega$, $z = c\omega^2$

Then $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \frac{a}{a} + \frac{b\omega}{b} + \frac{c\omega^2}{c} = 1 + \omega + \omega^2 = 0$

Example 27 :
Find the roots of $(2 - 2i)^{1/3}$.

Sol. Using De- Moivre's theorem
$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$
and putting n = 0, 1, 2 then we get roots as

$$\sqrt{2}\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right) ; \sqrt{2}\left(-\sin\frac{\pi}{12} + i\cos\frac{\pi}{12}\right), -1 - i$$

Example 28 :
Find the sum of 14th power of 10th roots of unity.

Sol. Here p = 14 and n = 10
$\because$ 14 is not a multiple of 10 hence
the sum of 14th power of 10th root of unity $= 0$

MISCELLANEOUS RESULTS

(i) If $z = \cos\theta + i\sin\theta$, then $1/z = \cos\theta - i\sin\theta$

Hence $z + \frac{1}{z} = 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$z - \frac{1}{z} = 2i\sin\theta \Rightarrow \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

(ii) If $z = \cos\theta + i\sin\theta$, using De-Moivre's theorem

$$z^n + \frac{1}{z^n} = 2\cos n\theta ; z^n - \frac{1}{z^n} = 2i\sin n\theta$$

(iii) If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$ and given $x + y + z = 0$, then

(i) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$ (ii) $yz + zx + xy = 0$

(iii) $x^2 + y^2 + z^2 = 0$ (iv) $x^3 + y^3 + z^3 = 3xyz$

then, putting, values if x, y, z in these results $x + y + z = 0$

$\Rightarrow \cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$
$yz + zx + xy = 0$

$$\Rightarrow \begin{cases}\cos(\beta+\gamma) + \cos(\gamma+\alpha) + \cos(\alpha+\beta) = 0\\ \sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta) = 0\end{cases}$$

$$x^2 + y^2 + z^2 = 0 \Rightarrow \begin{cases}\Sigma\cos 2\alpha = 0\\ \Sigma\sin 2\alpha = 0\end{cases},$$

the summation consists 3 terms

$x^3 + y^3 + z^3$ xyz, gives similarly
$\Sigma\cos 3\alpha = 3\cos(\alpha + \beta + \gamma)$
$\Sigma\sin 3\alpha = 3\sin(\alpha + \beta + \gamma)$
If the condition given be $x + y + z = xyz$, then
$\Sigma\cos\alpha = \cos(\alpha + \beta + \gamma)$ etc.

TRY IT YOURSELF-3

Q.1 Evaluate $\sqrt{-2 + 2\sqrt{-2 + 2\sqrt{-2 + \ldots\infty}}}$

Q.2 If $z + z^{-1} = 1$, then find the value of $z^{100} + z^{-100}$.

Q.3 If ω is a cube root of unity, then find the value of the $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8)$

Q.4 If $z = (i)^{(i)^{(i)}}$ where $i = \sqrt{-1}$, then $|z|$ is equal to –
(A) 1 (B) $e^{-\pi/2}$
(C) $e^{-\pi}$ (D) None of these

Q.5 Sum of common roots of the equation $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$ is –
(A) –1 (B) 1
(C) 0 (D) 1

Q.6 Let z_1 and z_2 be nth roots of unity which subtend a right angle at the origin. Then n must be of the form
(A) 4k + 1 (B) 4k + 2
(C) 4k + 3 (D) 4k

Q.7 If ω ($\neq$ 1) be a cube root of unity and $(1 + \omega^2)^n = (1 + \omega^4)^n$, then the least positive value of n
(A) 2 (B) 3
(C) 5 (D) 6

Q.8 A man walks a distance of 3 units from the origin towards the north-east (N 45°E) direction. From there, he walks a distance of 4 units towards the north-west (N 45° W) direction to reach a point P. Then the position P in the Argand plane is
(A) $3e^{i\pi/4} + 4i$ (B) $(3 - 4i)e^{i\pi/4}$
(C) $(4 + 3i)e^{i\pi/4}$ (D) $(3 + 4i)e^{i\pi/4}$

Q.9 Let $\omega = e^{\frac{i\pi}{3}}$, and a, b, c, x, y, z be non-zero complex numbers such that
$a + b + c = x$; $a + b\omega + c\omega^2 = y$; $a + b\omega^2 + c\omega = z$. Then

the value of $\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is

ANSWERS

(1) $x = \pm\sqrt{2}\omega$	**(2)** –1	**(3)** 9
(4) (A)	**(5)** (A)	**(6)** (D)
(7) (B)	**(8)** (D)	**(9)** 3

GEOMETRY OF COMPLEX NUMBERS

(i) **Distance Formula :** Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers represented by points P and Q respectively in Argand Plane then –

$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = |(x_2 - x_1) + i\,(y_2 - y_1)|$
$= |z_2 - z_1|$

(ii) Section Formula : If the line segment joining A (z_1) and B (z_2) is divided by the point P (z) internally in the ratio

$m_1 : m_2$ then $z = \dfrac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$

But if P divides AB externally in the ratio $m_1 : m_2$, then

$$z = \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}$$

If P is mid point of AB, then $z = \dfrac{z_1 + z_2}{2}$

(iii) Area of a triangle : Area of triangle ABC with vertices A (z_1), B (z_2) and C (z_3) is given by

$$\Delta = \frac{1}{4}\left\| \begin{matrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{matrix} \right\|$$

(iv) Condition for collinearity : Three points z_1, z_2 and z_3 will be collinear if there exists a relation $az_1 + bz_2 + cz_3 = 0$ (a, b & c are real), such that a + b + c = 0. In other words.

Three points z_1, z_2 and z_3 are collinear if $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$

(v) Equation of Straight Line : Equation of straight line through z_1 and z_2 is given by

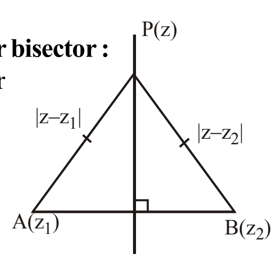
$$\frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \Rightarrow \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

The general equation of straight line is

$\bar{a}z + a\bar{z} + b = 0$, where b is a real number

(vi) If z_1, z_2, z_3, z_4 are vertices of parallelogram then $z_1 + z_3 = z_2 + z_4$

(vii) Equation of the perpendicular bisector : The equation of perpendicular bisector of the line segment joining points A (z_1) and B (z_2) is $|z - z_1| = |z - z_2|$



(viii) Equation of a circle : The equation of a circle with centre z_0 and radius r is $|z - z_0| = r$

The general equation of a circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$, where b is real number.

The centre of this circle is '–a' and its radius is $\sqrt{a\bar{a} - b}$.

(a) $\left|\dfrac{z - z_1}{z - z_2}\right| = k$ is a circle if $k \neq 1$ and is a line if $k = 1$

(b If arg $\left[\dfrac{(z_2 - z_3)(z_1 - z_4)}{(z_1 - z_3)(z_2 - z_4)}\right] = \pm\pi, 0$, then the points z_1, z_2, z_3, z_4 are concyclic.

(c) $|z - z_0| < r$ represents interior of the circle $|z - z_0| = r$ and $|z - z_0| > r$ represents exterior of the circle $|z - z_0| = r$.

(ix) Equation of ellipse :
If $|z - z_1| + |z - z_2| = 2a$, where $2a > |z_1 - z_2|$, then the point z describes an ellipse having foci at z_1 and z_2, $a \in R^+$.

(x) Equation of hyperbola :
If $|z - z_1| - |z - z_2| = 2a$, where $2a < |z_1 - z_2|$, then the point z describes a hyperbola having foci at z_1 and z_2, $a \in R^+$.

(xi) Some properties of triangle

(a) If z_1, z_2, z_3 are the vertices of triangle then centroid z_0 may by given as $z_0 = \dfrac{z_1 + z_2 + z_3}{3}$

(b) If z_1, z_2, z_3 are the vertices of an equilateral triangle then the circumcentre z_0 may be given as

$z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.

(c) If z_1, z_2, z_3 be the vertices of an equilateral triangle when

$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$

or $\dfrac{1}{z_1 - z_2} + \dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} = 0$

(d) If z_1, z_2, z_3 be the vertices of an isosceles triangle, right angled at z_2 then $z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$

(e) If z_1, z_2, z_3 are the vertices of isosceles triangle right angled at z_3 then $(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$.

(f) If three points z_1, z_2, z_3 are collinear then,

$\dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{\bar{z}_3 - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$

Example 29 :

The points represented by the complex numbers

$1 + i, -2 + 3i, \dfrac{5}{3}i$ on the Argand diagram are

(1) Vertices of an equilateral triangle
(2) Vertices of an isosceles triangle
(3) Collinear
(4) None of these

Sol. **(3).** Let $z_1 = 1 + i$, $z_2 = -2 + 3i$ and $z_3 = 0 + \dfrac{5}{3}i$

Then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 3 & 1 \\ 0 & 5/3 & 1 \end{vmatrix}$

$= 1\left(3 - \dfrac{5}{3}\right) + 1(2) + 1\left(\dfrac{-10}{3}\right) = \dfrac{4}{3} + 2 - \dfrac{10}{3} = \dfrac{4 + 6 - 10}{3} = 0$

Example 30 :

If the complex numbers, z_1, z_2, z_3 represented the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, then find the value of $z_1 + z_2 + z_3$.

Sol. Let the complex number z_1, z_2, z_3 denote the vertices A, B, C of an equilateral triangle ABC. Then, if O be the origin we have $OA = z_1$, $OB = z_2$, $OC = z_3$,
Therefore $|z_1| = |z_2| = |z_3| \Rightarrow OA = OB = OC$
i.e. O is the circumcentre of ΔABC
Hence $z_1 + z_2 + z_3 = 0$.

TRY IT YOURSELF-4

Q.1 Identify the locus of z if $\bar{z} = \dfrac{\bar{a}r^2}{(z-a)}$, $r > 0$

Q.2 If z be any complex number such that $|3z-2| + |3z+2| = 4$, then identify the locus of z.

Q.3 If $\left|\dfrac{z-2}{z-3}\right| = 2$ represents a circle, then find its radius.

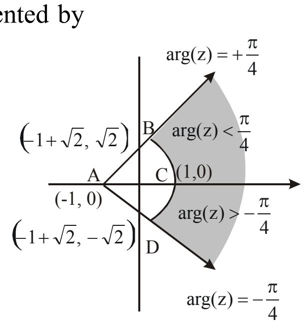
Q.4 Locus of z if $\arg[z-(1+i)] = \begin{cases} 3\pi/4, & \text{when } |z| \le |z-2| \\ -\pi/4, & \text{when } |z| > |z-2| \end{cases}$ is
(A) Straight lines passing through (2, 0).
(B) Straight lines passing through (2, 0), (1, 1).
(C) a line segment
(D) a set of two rays.

Q.5 If z is complex number then the locus of z satisfying the condition $|2z-1| = |z-1|$ is –
(A) Perpendicular bisector of line segment joining 1/2 & 1.
(B) circle
(C) parabola
(D) none of the above curves.

Q.6 If $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$, then area of the triangle whose vertices are $z_1\ z_2\ z_3$ is –
(A) $3\sqrt{3}/4$ (B) $\sqrt{3}/4$
(C) 1 (D) 2

Q.7 If $z = \dfrac{3}{2+\cos\theta + i\sin\theta}$, then locus of z is –
(A) a straight line
(B) a circle having centre on y-axis.
(C) a parabola
(D) a circle having centre on x-axis.

Q.8 The locus of z which lies in shaded region (excluding the boundaries) is best represented by
(A) $z : |z+1| > 2$ and $|\arg(z+1)| < \pi/4$
(B) $z : |z-1| > 2$ and $|\arg(z-1)| < \pi/4$
(C) $z : |z+1| > 2$ and $|\arg(z+1)| < \pi/2$
(D) $z : |z-1| > 2$ and $|\arg(z+1)| < \pi/2$



ANSWERS

(1) circle	**(2)** Line	**(3)** 2/3
(4) (D)	**(5)** (B)	**(6)** (A)
(7) (D)	**(8)** (A)	

ADDITIONAL EXAMPLES

Example 1 :

Find the value of $\dfrac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} + \dfrac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}$

Sol. Multiplying the numerator and denominator by ω and ω^2 respectively I and II expansion

$$= \frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} + \frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}$$

$$= \frac{\omega(a+b\omega+c\omega^2)}{(b\omega+c\omega^2+a)} + \frac{\omega^2(a+b\omega+c\omega^2)}{(c\omega^2+a+b\omega)} = \omega+\omega^2 = -1.$$

Example 2 :

Find the continued product of four roots of

$$\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{3/4}$$

Sol. $\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)^{3/4} = (e^{\pi i/3})^{3/4} = (e^{\pi i})^{1/4} = (-1)^{1/4}$

Hence continued product of four roots of $(-1)^{1/4} = (-1)^{4-1}(-1) = 1$

Example 3 :

If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$, then find the value of $\sin 3\alpha + \sin 3\beta + \sin 3\gamma$.

Sol. If $a = \cos\alpha + i\sin\alpha$; $b = \cos\beta + i\sin\beta$; $c = \cos\gamma + i\sin\gamma$,
then $a + b + c = (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$
$= 0 + i\,0 = 0$
$\Rightarrow a^3 + b^3 + c^3 = 3abc$
$\Rightarrow \Sigma(\cos\alpha + i\sin\alpha)^3$
$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$
$\Rightarrow \Sigma\cos 3\alpha + i\Sigma\sin 3\alpha = 3\cos(\alpha+\beta+\gamma) + 3i\sin(\alpha+\beta+\gamma)$
$\Rightarrow \sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

Example 4 :

Let $z = \left(\dfrac{\sqrt{3}}{2}\right) - \left(\dfrac{i}{2}\right)$. Then the smallest positive integer n such that $(z^{95} + i^{67})^{94} = z^n$ is –
(A) 12 (B) 10 (C) 9 (D) 8

Sol. **(B).** From the hypothesis we have

$$z = \frac{\sqrt{3}}{2} - \frac{i}{2} = i\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) = i\omega \text{ where } \omega = \left(-\frac{1}{2}\right) - \left(\frac{i\sqrt{3}}{2}\right)$$

which is a cube root unity.

Now, $z^{95} = (i\omega)^{95} = -i\omega^2$ (since $\omega^3 = 1$) & $i^{67} = i^3 = -i$

Therefore, $z^{95} + i^{67} = -i(1+\omega^2) = (-i)(-\omega) = i\omega$

$(z^{95} + i^{67})^{94} = (i\omega)^{94} = i^2\omega = -\omega$

Now, $-\omega = z^n = (i\omega)^n \Rightarrow i^n \cdot \omega^{n-1} = -1$

$\Rightarrow$ n = 2, 6, 10, 14, ... and n – 1 = 3, 6, 9, ...

Therefore, n = 10 is the required least positive integer.

Example 5 :

If $\text{Re}\left(\frac{iz+1}{iz-1}\right) = 2$, then z lies on the curve

(A) $4x^2 + 4y^2 + x - 6y + 2 = 0$
(B) $x^2 + y^2 + 4y + 3 = 0$
(C) $3(x^2 + y^2) - 2x - 4y = 0$
(D) $x^2 + y^2 - x + 2y - 1 = 0$

Sol. **(B).** $\text{Re}\left(\frac{iz+1}{iz-1}\right) = 2 \Rightarrow \text{Re}\left(\frac{z-i}{z+1}\right) = 2$

Let z = x + i y then

$\Rightarrow \text{Re}\left(\frac{x+(y-1)i}{x+(y+1)i}\right) = 2 \Rightarrow \text{Re}\left(\frac{x^2+y^2-1+i2x}{x^2+(y+1)^2}\right) = 2$

$\Rightarrow x^2 + y^2 - 1 = 2x^2 + 2(y+1)^2 \Rightarrow x^2 + y^2 + 4y + 3 = 0$

Example 6 :

Let $z_k = \cos\left(\frac{2k\pi}{10}\right) + i\sin\left(\frac{2k\pi}{10}\right)$; k = 1, 2, ... 9

then $\frac{|1-z_1||1-z_2|...|1-z_9|}{10} =$

(A) 1 (B) 2
(C) 3 (D) 4

Sol. **(A).** $z^{10} - 1 = 0$

$\Rightarrow (z-z_1)(z-z_2).....(z-z_9) = 1 + z + z^2 + ... + z^9$

So, $|1-z_1||1-z_2|.......|1-z_9| = 10$

Example 7 :

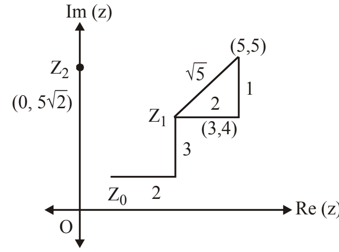
A particle starts from a point $z_0 = 1 + i$, where $i = \sqrt{-1}$. It moves horizontally away from origin by 2 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 particle moves $\sqrt{5}$ units in the direction of $2\hat{i} + \hat{j}$ and then it moves through an angle of $\text{cosec}^{-1}\sqrt{2}$ in anticlockwise direction of a circle with centre at origin to reach a point z_2. The arg z_2 is given by

(A) $\sec^{-1} 2$ (B) $\cot^{-1} 0$

(C) $\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$ (D) $\cos^{-1}\left(\frac{-1}{2}\right)$

Sol. **(B).** Clearly $z_1 = 3 + 4i$

After moving by $\sqrt{5}$ distance in direction of $2\hat{i} + \hat{j}$, particle will react at point $(5\hat{i} + 5\hat{j})$. If particle moves by an angle $\pi/4$ then it will reach at y-axis.



At $z_2 = 0 + 5\sqrt{2}i$ hence, amp $(z_2) = \frac{\pi}{2} = \cot^{-1} 0$

Example 8 :

The continued product of all the four values of the complex number $(1+i)^{3/4}$ is –

(A) $2^3(1+i)$ (B) $2(1-i)$
(C) $2(1+i)$ (D) $2^3(1-i)$

Sol. **(B).** Let $z = 1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$. Therefore,

$z^{3/4} = 2^{3/8} = \left[\cos\left(2k\pi + \frac{\pi}{4}\right)\frac{3}{4} + i\sin\left(2k\pi + \frac{\pi}{4}\right)\frac{3}{4}\right]$

For k = 0, 1, 2, 3, the product of the values of this is equal to

$2^{3/2}\left[\text{cis}\left(\frac{\pi}{4} + \frac{9\pi}{4} + \frac{17\pi}{4} + \frac{25\pi}{4}\right)\frac{3}{4}\right]$

$= 2^{3/2}\text{cis}\left(\frac{52\pi}{4}\cdot\frac{3}{4}\right) = 2^{3/2}\text{cis}\frac{39\pi}{4}$

$= 2^{3/2}\text{cis}\left(9\pi + \frac{3\pi}{4}\right) = 2^{3/2}\text{cis}\left(10\pi - \frac{3\pi}{4}\right)$

$= 2^{3/2}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right] = 2(1-i)$

Example 9 :

If $f(x) = g(x^3) + xh(x^3)$ is divisible by $x^2 + x + 1$, then –

(A) g (x) is divisible by (x – 1) but not by h (x).
(B) h (x) is divisible by (x – 1) but not by g (x).
(C) both g (x) and h (x) are divisible by (x – 1).
(D) None of these

Sol. **(C).** $f(x) = g(x^3) + xh(x^3)$

Let $f_1(x) = 1 + x + x^2$

Clearly, the roots of $f_1(x) = 0$ and ω and ω^2 (where ω is a ron-real cube root of unity). As $f_1(x)$ divides f (x).

$\Rightarrow f(\omega) = 0, f(\omega^2) = 0 \Rightarrow g(\omega^3) + \omega h(\omega^3) = 0$

and $g(\omega^6) + \omega^2 h(\omega^6) = 0$

$\Rightarrow g(1) + \omega h(1) = 0, g(1) + \omega^2 h(1) = 0$

$\Rightarrow 2g(1) + h(1)(\omega + \omega^2) = 0$

$\Rightarrow 2g(1) - h(1) = 0 \Rightarrow h(1) = 2g(1)$

$\Rightarrow g(1) + \omega.2g(1) = 0$

$\Rightarrow g(1)(1 + 2\omega) = 0 \Rightarrow g(1) = 0$

$\Rightarrow$ x = 1 is the root of g (x) = 0 and h (x) = 0.

Thus, g (x) and h (x) both are divisible by x – 1.

QUESTION BANK

CHAPTER 5 : COMPLEX NUMBERS

EXERCISE - 1 [LEVEL-1]

PART 1 : POWER OF IOTA, ALGEBRAIC OPERATIONS AND EQUALITY OF COMPLEX NUMBERS

Q.1 Find the value of $[i]^{198}$

(A) –1 (B) 0
(C) 1 (D) i

Q.2 Find the value of $i^n + i^{n+1} + i^{n+2} + i^{n+3}$

(A) –1 (B) 0
(C) 1 (D) i

Q.3 If $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is purely real, then θ is equal to-

(A) $n\pi \pm \pi/6$ (B) $n\pi$
(C) $2n\pi \pm \pi/3$ (D) $n\pi \pm \pi/3$

Q.4 If complex number $\frac{z-1}{z+1}$ is purely imaginary then locus of z is -

(A) a circle (B) a straight line
(C) a parabola (D) None of these

Q.5 If for any complex number z, $|z-4| < |z-2|$, then

(A) $R(z) > 2$ (B) $R(z) < 0$
(C) $R(z) > 0$ (D) $R(z) > 3$

Q.6 $\sqrt{-2}\sqrt{-3}$ is equal to -

(A) $i\sqrt{6}$ (B) $-\sqrt{6}$
(C) $\sqrt{6}$ (D) None of these

Q.7 If $z = x + iy$, $z^{1/3} = a - ib$ and $\frac{x}{a} - \frac{y}{b} = k(a^2 - b^2)$, then k equals -

(A) – 2 (B) 2
(C) 4 (D) 0

Q.8 The values of z for which $|z+i| = |z-i|$ are

(A) Any real number (B) Any complex number
(C) Any natural number (D) None of these

Q.9 The vector $z = 3 - 4i$ is turned anticlockwise through an angle of 180° and stretched 2.5 times. The complex number corresponding to the newly obtained vector is

(A) $\frac{15}{2} - 10i$ (B) $\frac{-15}{2} + 10i$
(C) $\frac{-15}{2} - 10i$ (D) None of these

Q.10 Let $\frac{1-ix}{1+ix} = a - ib$ and $a^2 + b^2 = 1$, where a and b are real, then x =

(A) $\frac{2a}{(1+a)^2+b^2}$ (B) $\frac{2b}{(1+a)^2+b^2}$
(C) $\frac{2a}{(1+b)^2+a^2}$ (D) $\frac{2b}{(1+b)^2+a^2}$

Q.11 The least positive integer n, for which $\frac{(1+i)^n}{(1-i)^{n-2}}$ is positive, is –

(A) 3 (B) 4
(C) 1 (D) 2

Q.12 The value of $\left|\frac{1+i\sqrt{3}}{\left(1+\frac{1}{i+1}\right)^2}\right|$ is –

(A) 4/5 (B) 5/4
(C) 9 (D) 20

PART 2 : SQUARE ROOT, CONJUGATE, MODULULS AND ARGUMENT OF COMPLEX NUMBER

Q.13 The amplitude of $\frac{a+ib}{a-ib}$ is equal to-

(A) $\tan^{-1}\left(\frac{a^2-b^2}{a^2+b^2}\right)$ (B) $\tan^{-1}\left(\frac{2ab}{a^2-b^2}\right)$
(C) $\tan^{-1}\left(\frac{2ab}{a^2+b^2}\right)$ (D) $\tan^{-1}\left(\frac{a^2-b^2}{2ab}\right)$

Q.14 If $|z_1| = |z_2| = ... = |z_n| = 1$, then $\left|\frac{1}{z_1} + \frac{1}{z_2} + ... + \frac{1}{z_n}\right|$

(A) $= |z_1 + z_2 + ... + z_n|$ (B) $< |z_1 + z_2 + ... z_n|$
(C) $> |z_1 + z_2 + ... + z_n|$ (D) $= 1$

Q.15 If $z = (1/2, 1)$, then the value of z^{-1} is-

(A) $\left(-\frac{2}{5}, \frac{4}{5}\right)$ (B) $\left(\frac{1}{5}, -\frac{2}{5}\right)$
(C) $\left(\frac{1}{5}, \frac{2}{5}\right)$ (D) $\left(\frac{2}{5}, -\frac{4}{5}\right)$

Q.16 If $\frac{\tan\theta - i\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)}{1 + 2i\sin\frac{\theta}{2}}$ is purely imaginary then general value of θ is -

(A) $n\pi + \frac{\pi}{4}$ (B) $2n\pi + \frac{\pi}{4}$
(C) $n\pi + \frac{\pi}{2}$ (D) $2n\pi + \frac{\pi}{2}$

Q.17 For any two non real complex numbers z_1, z_2 if $z_1 + z_2$ and z_1z_2 are real numbers, then
(A) $z_1 = 1/z_2$ (B) $z_1 = \bar{z}_2$
(C) $z_1 = -z_2$ (D) $z_1 = z_2$

Q.18 If z_1, z_2 be two complex numbers $(z_1 \neq z_2)$ satisfying $|z_1^2 - z_2^2| = |\bar{z}_1^2 + \bar{z}_2^2 - 2\bar{z}_1\bar{z}_2|$, then –
(A) $\frac{z_1}{z_2}$ is purely imaginary (B) $\frac{z_1}{z_2}$ is purely real
(C) $|\arg z_1 - \arg z_2| = \pi$ (D) $|\arg z_1 - \arg z_2| = \pi/3$

Q.19 If z_1, z_2 are any two complex numbers and a, b are any two real numbers, then $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$ is equal to-
(A) $(a^2+b^2)(|z_1|^2+|z_2|^2)$ (B) $a^2b^2(|z_1|^2 + |z_2|^2)$
(C) $(a+b)^2(|z_1|^2+|z_2|^2)$ (D) None of these

Q.20 The amplitude of $1 - \cos\theta - i\sin\theta$ is -
(A) $\frac{1}{2}(\pi - \theta)$ (B) $\frac{\theta}{2}$
(C) $-\frac{\pi}{2} + \frac{\theta}{2}$ (D) $\frac{\pi}{2} + \frac{\theta}{2}$

Q.21 The polar form of complex number
$$z = \frac{\{\cos(\pi/3) - i\sin(\pi/3)\}\,(\sqrt{3}+i)}{i-1} \text{ is-}$$
(A) $\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$ (B) $\sqrt{2}\left(\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right)$
(C) $\sqrt{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$ (D) None of these

Q.22 If $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$ then (z_1/z_2) is
(A) zero or purely imaginary (B) purely imaginary
(C) purely real (D) None of these

Q.23 Square root of $-8-6i$ is –
(A) $\pm(3+i)$ (B) $\pm(1+i\sqrt{3})$
(C) $\pm(1-3i)$ (D) $\pm(1+3i)$

Q.24 The complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other when –
(A) $x = 0$ (B) $x = \left(n+\frac{1}{2}\right)\pi$
(C) $x = n\pi$ (D) no value of x

Q.25 If $|z + 2i| \le 1$, then greatest and least value of $|z - \sqrt{3} + i|$ are-
(A) 3, 1 (B) ∞, 0
(C) 1, 3 (D) None of these

Q.26 If complex number $z = x + iy$ is taken such that the amplitude of fraction $\frac{z-1}{z+1}$ is always $\frac{\pi}{4}$, then
(A) $x^2 + y^2 + 2y = 1$ (B) $x^2 + y^2 - 2y = 0$
(C) $x^2 + y^2 + 2y = -1$ (D) $x^2 + y^2 - 2y = 1$

Q.27 The values of x and y for which the numbers $3 + ix^2y$ and $x^2 + y + 4i$ are conjugate complex can be
(A) (–2, –1) or (2, –1) (B) (–1, 2) or (–2, 1)
(C) (1, 2) or (–1, –2) (D) None of these

Q.28 If the conjugate of $(x + iy)(1 - 2i)$ be $1 + i$, then –
(A) $x = 1/5$ (B) $y = 3/5$
(C) $x + iy = \frac{1-i}{1-2i}$ (D) $x - iy = \frac{1-i}{1+2i}$

Q.29 The maximum value of $|z|$ where z satisfies the condition $\left|z + \frac{2}{z}\right| = 2$ is
(A) $\sqrt{3} - 1$ (B) $\sqrt{3} + 1$
(C) $\sqrt{3}$ (D) $\sqrt{2} + \sqrt{3}$

Q.30 If z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part & z_2 has negative imaginary part, then $\frac{(z_1+z_2)}{(z_1-z_2)}$ may be
(A) Purely imaginary (B) Real and positive
(C) Real and negative (D) None of these

Q.31 If $|z| = 1$ and $\omega = \frac{z-1}{z+1}$ (where $z \neq -1$), then $\text{Re}(\omega)$ is
(A) 0 (B) $-\frac{1}{|z+1|^2}$
(C) $\left|\frac{z}{z+1}\right| \cdot \frac{1}{|z+1|^2}$ (D) $\frac{\sqrt{2}}{|z+1|^2}$

Q.32 If $z = \frac{1-i\sqrt{3}}{1+i\sqrt{3}}$, then arg (z) =
(A) 60° (B) 120°
(C) 240° (D) 300°

Q.33 If $z_1.z_2........z_n = z$, then $\arg z_1 + \arg z_2 + ... + \arg z_n$ and arg z differ by a
(A) Multiple of π (B) Multiple of $\pi/2$
(C) Greater than π (D) Less than π

Q.34 The argument of the complex number $\frac{13-5i}{4-9i}$ is
(A) $\pi/3$ (B) $\pi/4$
(C) $\pi/5$ (D) $\pi/6$

Q.35 The modulus and amplitude of $\frac{1+2i}{1-(1-i)^2}$ are –
(A) $\sqrt{2}$ and $\frac{\pi}{6}$ (B) 1 and 0
(C) 1 and $\pi/3$ (D) 1 and $\pi/4$

Q.36 If $z_1 = 1 + 2i$ and $z_2 = 3 + 5i$, and then $\text{Re}\left(\frac{\bar{z}_2 z_1}{z_2}\right) =$
(A) –31/17 (B) 17/22
(C) –17/31 (D) 22/17

Q.37 If $x + iy = \sqrt{\dfrac{a+ib}{c+id}}$, then $(x^2 + y^2)^2 =$

(A) $\dfrac{a^2+b^2}{c^2+d^2}$ (B) $\dfrac{a+b}{c+d}$

(C) $\dfrac{c^2+d^2}{a^2+b^2}$ (D) $\left(\dfrac{a^2+b^2}{c^2+d^2}\right)^2$

Q.38 If $\sqrt{a+ib} = x + iy$, then possible value of $\sqrt{a-ib}$ is

(A) $x^2 + y^2$ (B) $\sqrt{x^2+y^2}$

(C) $x + iy$ (D) $x - iy$

Q.39 If z_1, z_2 are the roots of the quadratic equation $az^2 + bz + c = 0$ such that Im $(z_1, z_2) \neq 0$ then –

(A) a, b, c are all real

(B) at least one of a, b, c is real

(C) at least one of a, b, c is imaginary

(D) all of a, b, c are imaginary

Q.40 $3 + i\,x^2y$ and $x^2 + y + 4i$ are complex conjugate numbers, then $x^2 + y^2 =$

(A) 4 (B) 2

(C) 3 (D) 5

Q.41 The point of intersection the curves

$\arg(z - i + 2) = \dfrac{\pi}{6}$ and $\arg(z + 4 - 3i) = -\dfrac{\pi}{4}$ is given by

(A) $(-2 + i)$ (B) $2 - i$

(C) $2 + i$ (D) None of these

Q.42 If z is a complex number and the minimum value of $|z| + |z - 1| + |2z - 3|$ is λ and if $y = 2[x] + 3 = 3[x - \lambda]$ then find the value of $[x + y]$ (where [.] denotes the greatest integer function)

(A) 30 (B) 20

(C) 21 (D) 25

Q.43 If $i z^2 - \bar{z} = 0$, the $|z|$ is equal to –

(A) 1 (B) 0

(C) 0 or 1 (D) None of these

Q.44 If $|z + 4| \leq 3$, then the greatest and the least value of $|z + 1|$ are –

(A) 6, – 6 (B) 6, 0

(C) 7, 2 (D) 0, – 1

Q.45 If the conjugate of $(x + i y)(1 - 2i)$ is $1 + i$, then –

(A) $x = -1/5$ (B) $x - iy = \dfrac{1+i}{1-2i}$

(C) $x + iy = \dfrac{1-i}{1-2i}$ (D) $x = \dfrac{1}{5}$

Q.46 The modulus and amplitude of $\dfrac{1+2i}{1-(1-i)^2}$ are –

(A) $\sqrt{2}$ and $\pi/6$ (B) 1 and $\pi/4$

(C) 1 and 0 (D) 1 and $\pi/3$

Q.47 If $Z = \dfrac{(\sqrt{3}+i)^3(3i+4)^2}{(8+6i)^2}$ then $|Z|$ is equal to –

(A) 0 (B) 1

(C) 2 (D) 3

PART 3 : GEOMETRY OF COMPLEX NUMBERS

Q.48 If $A \equiv 1 + 2i$, $B \equiv -3 + i$, $C \equiv -2 - 3i$ and $D \equiv 2 - 2i$ are vertices of a quadrilateral, then it is a

(A) rectangle (B) parallelogram

(C) square (D) rhombus

Q.49 If $\left|\dfrac{z-3i}{z+3i}\right| = 1$ then the locus of z is -

(A) x axis

(B) $x - y = 0$

(C) Circle passing through origin

(D) y axis

Q.50 If z is a complex number satisfying $|z - i\,\text{Re}(z)| = |z - \text{Im}(z)|$ then z lies on –

(A) $y = 2x$ (B) $y = -x$

(C) $y = x + 1$ (D) $y = -x + 1$

Q.51 The complex numbers z_1, z_2 and z_3 satisfying $\dfrac{z_1 - z_3}{z_2 - z_3} = \dfrac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is –

(A) Of area = 0 (B) Right angled isosceles

(C) Equilateral (D) Obtuse angled isosceles

Q.52 A complex number z is such that $\arg\left(\dfrac{z-2}{z+2}\right) = \dfrac{\pi}{3}$. The points representing this complex number will lie on –

(A) An ellipse (B) A parabola

(C) A circle (D) A straight line

Q.53 If complex numbers z_1, z_2 and 0 are vertices of an equilateral triangle, then $z_1^2 + z_2^2 - z_1 z_2$ is equal to-

(A) 0 (B) $z_1 - z_2$

(C) $z_1 + z_2$ (D) 1

Q.54 If $w = \dfrac{z - (1/5)i}{z}$ and $|w| = 1$, then complex number z lies

(A) a parabola (B) a circle

(C) a line (D) None of these

Q.55 If $z = x + iy$, and if $\log_{\sqrt{3}} \dfrac{|z|^2 - |z| + 1}{2 + |z|} < 2$ then z lies in the interior of the circle

(A) $|z| = 4$ (B) $|z| = 3$

(C) $|z| = 2$ (D) $|z| = 5$

Q.56 If z_0 is the circumcenter of an equilateral triangle with vertices z_1, z_2, z_3, then $z_1^2 + z_2^2 + z_3^2$ is equal to

(A) z_0^2 (B) $2z_0^2/3$

(C) $3z_0^2$ (D) $z_0^2/3$

Q.57 In a complex plane z_1, z_2, z_3, z_4 taken in order are vertices of parallelogram if
(A) $z_1 + z_2 = z_3 + z_4$ (B) $z_1 + z_3 = z_2 + z_4$
(C) $z_1 + z_4 = z_2 + z_3$ (D) None of these

Q.58 If A, B and C are respectively the complex numbers $3 + 4i, 5 - 2i, -1 + 16i$, then A, B, C are-
(A) collinear
(B) vertices of right-angle triangle
(C) vertices of isosceles triangle
(D) vertices of equilateral triangle

Q.59 The complex number z having least positive argument which satisfy the condition $|z - 25i| \le 15$ is -
(A) $25i$ (B) $12 + 25i$
(C) $16 + 12i$ (D) $12 + 16i$

Q.60 Let z be a complex number satisfying $|z - 5i| \le 1$ such that amp z is minimum. Then z is equal to
(A) $\frac{2\sqrt{6}}{5} + \frac{24i}{5}$ (B) $\frac{24}{5} + \frac{2\sqrt{6}i}{5}$
(C) $\frac{2\sqrt{6}}{5} - \frac{24i}{5}$ (D) None of these

Q.61 If three complex numbers are in A.P., then they lie on –
(A) A circle in the complex plane
(B) A straight line in the complex plane
(C) A parabola in the complex plane
(D) None of these

Q.62 ABCD is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy BD = 2AC. If the points D and M represents the complex numbers 1 + i and 2 – i respectively, then A represents the complex number
(A) $3 - \frac{1}{2}i$ or $1 - \frac{3}{2}i$ (B) $\frac{3}{2} - i$ or $\frac{1}{2} - 3i$
(C) $\frac{1}{2} - i$ or $1 - \frac{1}{2}i$ (D) None of these

Q.63 For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is
(A) 0 (B) 2
(C) 7 (D) 17

Q.64 For any complex no. Z, the minimum value of $|Z| + |Z - 1|$
(A) 1 (B) 0
(C) 1/2 (D) 3/2

Q.65 The points Z on complex plane satisfying $Z + |Z| = 0$, lie on–
(A) The x-axis, $x \le 0$ (B) The x-axis, $x > 0$
(C) The y-axis (D) None of these

Q.66 If P (x, y) denotes z = x + iy in Argand's plane and $\left|\frac{z-1}{z+2i}\right| = 1$, then the locus of P is a/an –
(A) straight line (B) circle
(C) ellipse (D) hyperbola

PART 4 : DE-MOIVER'S THEOREM AND ROOTS OF UNITY

Q.67 The value of $\left(1 + i\sqrt{3}\right)^6 + (1 - i)^8$ is-
(A) $16(2 - i)$ (B) $32(3 - 2i)$
(C) 80 (D) 48

Q.68 If ω is a cube root of unity, then
$\sin\left\{\left(\omega^{35} + \omega^{25}\right)\pi + \frac{\pi}{2}\right\} + \cos\left\{\left(\omega^{10} + \omega^{23}\right)\pi - \frac{\pi}{4}\right\}$ is
(A) $\frac{2+\sqrt{2}}{2}$ (B) $\frac{2+\sqrt{2}}{\sqrt{2}}$
(C) $-\frac{(2+\sqrt{2})}{2}$ (D) $\frac{2-\sqrt{2}}{2}$

Q.69 If z_1, z_2, z_3, z_4 are the roots of the equation $z^4 + z^3 + z^2 + z + 1 = 0$ then $\left|\sum_{i=1}^{4} z_i^4\right|$ equal to
(A) 2 (B) 3
(C) 1 (D) 4

Q.70 If $x_n = \cos\left(\frac{\pi}{3^n}\right) + i\sin\left(\frac{\pi}{3^n}\right)$, then $x_1.x_2.x_3....x_\infty$ is equal to –
(A) 1 (B) – 1
(C) i (D) – i

Q.71 If $x_n = \cos(\pi/2^n) + i\sin(\pi/2^n)$, then $x_1 x_2 x_3........\infty$ is equal to-
(A) – 1 (B) 1
(C) 0 (D) ∞

Q.72 Number of solution of the equation, $z^3 + \frac{3(\bar{z})^2}{z} = 0$ where z is a complex number is –
(A) 2 (B) 3
(C) 6 (D) 5

Q.73 The value of $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$ is -
(A) – i (B) 0
(C) – 1 (D) i

Q.74 If $z = i\log(2 - \sqrt{3})$, then cos z =
(A) i (B) 2i
(C) 1 (D) 2

Q.75 $(-1 + i\sqrt{3})^{20}$ is equal to
(A) $2^{20}(-1 + i\sqrt{3})^{20}$ (B) $2^{20}(1 - i\sqrt{3})^{20}$
(C) $2^{20}(-1 - i\sqrt{3})^{20}$ (D) None of these

Q.76 The area of the triangle whose vertices are represented by the complex numbers 0, z, $ze^{i\alpha}$, $(0 < \alpha < \pi)$ equals

(A) $\frac{1}{2}|z|^2 \cos\alpha$ (B) $\frac{1}{2}|z|^2 \sin\alpha$

(C) $\frac{1}{2}|z|^2 \sin\alpha\cos\alpha$ (D) $\frac{1}{2}|z|^2$

Q.77 Number of solutions of the equation $z^3 = \bar{z}\, i\, |z|$ are –

(A) 2 (B) 3
(C) 4 (D) 5

Q.78 Integral solution of equation $(1 - i)^x = 2^x$ are –

(A) 0 (B) 4n, $n \in N$
(C) 0, 1 (D) None of these

Q.79 Common roots of the equations $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$ are

(A) ω, ω^2 (B) ω, ω^3
(C) ω^2, ω^3 (D) None of these

Q.80 If the cube roots of unity are 1, ω, ω^2 then the roots of the equation $(x - 2)^3 + 27 = 0$ are –

(A) –1, –1, –1 (B) $-1, -\omega, -\omega^2$
(C) $-1, 2 + 3\omega, 2 + 3\omega^2$ (D) $-1, 2 - 3\omega, 2 - 3\omega^2$

Q.81 If $x + iy = (-1 + i\sqrt{3})^{2010}$, then x =

(A) -2^{2010} (B) 2^{2010}
(C) 1 (D) –1

Q.82 If ω is an imaginary cube root of unity, then the value of $(1 - \omega + \omega)^2 (1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$ .... (2n factors) is

(A) 0 (B) 1
(C) 2 (D) 2^{2n}

Q.83 If α is a complex number such that $\alpha^2 - \alpha + 1 = 0$, then $\alpha^{2011} =$

(A) 1 (B) $-\alpha^2$
(C) α^2 (D) α

Q.84 If $2x = -1 + \sqrt{3}i$, then the value of $(1 + x^2 + x)^6 - (1 - x + x^2)^6 =$

(A) 32 (B) 64
(C) – 64 (D) 0

Q.85 If 1, ω, ω^2 are three cube roots of unity, then $(1 - \omega + \omega^2)(1 + \omega - \omega^2)$ is –

(A) 1 (B) 2
(C) 3 (D) 4

Q.86 The real part of $(1 - \cos\theta + i\sin\theta)^{-1}$ is –

(A) $\frac{1}{1+\cos\theta}$ (B) $\cot\frac{\theta}{2}$

(C) $\frac{1}{2}$ (D) $\tan\frac{\theta}{2}$

PART 5 : MISCELLANEOUS

Q.87 If $z = i^i$, where $i = \sqrt{-1}$, then –

(A) z is purely real (B) z is purely imaginary
(C) $|z| = 1$ (D) $\arg(z) = \pi - \tan^{-1}(1/\sqrt{2})$

Q.88 $Z \in C$ satisfies the condition $|z| \geq 3$. Then the least value of $\left|z + \frac{1}{z}\right|$ is

(A) 3/8 (B) 8/5
(C) 8/3 (D) 5/8

Q.89 If $z = x + iy$ and $\left|\frac{z - 5i}{z + 5i}\right| = 1$ then z lies on

(A) x-axis (B) y-axis
(C) line y = 5 (D) None of these

Q.90 If $|z| = 5$, then the points representing the complex number $-i + \frac{15}{z}$ lies on the circle –

(A) whose centre is (0, 1) and radius = 3
(B) whose centre is (–1, 0) and radius = 15
(C) whose centre is (1, 0) and radius = 15
(D) whose centre is (0, –1) and radius = 3

Q.91 The equation $Z^3 + iZ - 1 = 0$ has

(A) three real roots (B) one real root
(C) no real roots (D) no real or complex roots

Q.92 A point Z moves on the curve $|Z - 4 - 3i| = 2$ in argand plane. The maximum values of |Z| are

(A) 2, 1 (B) 6, 5
(C) 4, 3 (D) 7, 3

Q.93 If $z = x + iy$, $w = \frac{1 - iz}{z - i}$ and $|w| = 2$, then in the Argand's plane z lies on –

(A) real axis (B) imaginary axis
(C) a circle (D) none of these

Q.94 If α, β are the complex numbers, then the maximum value of $\left|\frac{\alpha\bar{\beta} + \bar{\alpha}\beta}{|\alpha\beta|}\right|$ is –

(A) 1 (B) 3
(C) 2 (D) 4

Q.95 For any two non zero complex numbers z_1, z_2, the value of $(|z_1| + |z_2|)\left|\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}\right|$ is

(A) less than $2(|z_1| + |z_2|)$
(B) greater than $2(|z_1| + |z_2|)$
(C) greater than or equal to $2(|z_1| + |z_2|)$
(D) less than or equal to $2(|z_1| + |z_2|)$

Q.96 The number of solutions of the equation in Z,

$Z\bar{Z} - (3-i)Z - (3-i)\bar{Z} - 6 = 0$ is
(A) 0 (B) 1
(C) 2 (D) Infinite

Q.97 The solutions of the equation in Z,
$|Z|^2 - (Z+\bar{Z}) + i(Z-\bar{Z}) + 2 = 0$ are
(A) $2+i, 1-i$ (B) $1+i, 1-i$
(C) $1+2i, -1-i$ (D) $1+i, 1+i$

Q.98 The region represented by the inequality $|2Z-3i| < |3Z-2i|$ is
(A) the unit disc with its centre at $Z = 0$
(B) the exterior of the unit circle with its centre at $Z = 0$
(C) the interior of a square of side 2 units with its centre at $Z = 0$
(D) none of these

EXERCISE - 2 [LEVEL-2]

ONLY ONE OPTION IS CORRECT

Q.1 If a complex number z satisfies $|2z - 10 + 10i| \le 5\sqrt{3} - 5$, then the least principal argument of z is –
(A) $-\pi/3$ (B) $-2\pi/3$
(C) $-5\pi/6$ (D) $-3\pi/4$

Q.2 If $z = re^{i\theta}$, then $|e^{iz}|$ equals
(A) $e^{r\sin\theta}$ (B) $e^{-r\sin\theta}$
(C) $e^{-r\cos\theta}$ (D) $e^{r\cos\theta}$

Q.3 For any two complex numbers Z_1 and Z_2 with $|Z_1| \ne |Z_2|$
$\left|\sqrt{2}Z_1 + i\sqrt{3}\bar{Z}_2\right|^2 + \left|\sqrt{3}\bar{Z}_1 + i\sqrt{2}Z_2\right|^2$ is –
(A) less than $5\left(|Z_1|^2 + |Z_2|^2\right)$
(B) greater than $10|Z_1 Z_2|$
(C) equal to $2|Z_1|^2 + 3|Z_2|^2$
(D) zero

Q.4 A and B represent the complex numbers $1 + ai$ and $3 + bi$ and ΔOAB is an isosceles triangle right-angled at A. Then the values of a and b can be
(A) $a = 2, b = -1$ (B) $a = 1, b = -2$
(C) $a = 2, b = 1$ (D) $a = 2, b = -2$

Q.5 If $\begin{vmatrix} 1 & Z_1 & \bar{Z}_1 \\ 1 & Z_2 & \bar{Z}_2 \\ 1 & Z_3 & \bar{Z}_3 \end{vmatrix} = 0$, the points Z_1, Z_2, Z_3 in an argand plane
(A) Form an isosceles triangle
(B) Form an equilateral triangle
(C) Are collinear
(D) Lie on a circle

Q.6 The roots of $Z^n = (Z + a)^n$, $a > 0$, lie on-
(A) The circle $\left|Z - \frac{a}{2}\right| = \frac{a}{2}$
(B) The circle $\left|Z + \frac{a}{2}\right| = \frac{a}{2}$
(C) The straight line $\text{Re}(Z) + \frac{a}{2} = 0$
(D) The straight line $\text{Re}(Z) - \frac{a}{2} = 0$

Q.7 If there exists an Z satisfying both $|z - m| = m + 5$ and $|z - 4| < 3$. then the set of all permissible values of m belong to the set
(A) $(-3, 3)$ (B) $(-3, 9)$
(C) $(-5, -3)$ (D) $(4, 9)$

Q.8 $\dfrac{3 + 2i\sin\theta}{1 - 2i\sin\theta}$ will be purely imaginary, if θ equals
(A) $2n\pi \pm \frac{\pi}{3}$ (B) $n\pi + \frac{\pi}{3}$
(C) $n\pi \pm \frac{\pi}{3}$ (D) None of these

Q.9 In the argand plane the inequality
$$\left|\left(\sqrt{3}+i\right)Z - \left(\sqrt{2}-i\right)\bar{Z}\right|^2 + \left|\left(\sqrt{2}+i\right)Z + \left(\sqrt{3}-i\right)\bar{Z}\right|^2 < 28$$
represents
(A) The region enclosed by a triangle
(B) The region enclosed by a circle of radius 4
(C) The region enclosed by an ellipse
(D) None of these

Q.10 A triangle with vertices represented by complex numbers z_0, z_1, z_2 has opposite side lengths in ratio $2 : \sqrt{6} : \sqrt{3} - 1$ respectively. Then –
(A) $(z_2 - z_0)^4 = -9\,(7 + 4\sqrt{3})\,(z_1 - z_0)^4$
(B) $(z_2 - z_0)^4 = 9\,(7 + 4\sqrt{3})\,(z_1 - z_0)^4$
(C) $(z_2 - z_0)^4 = (7 + 4\sqrt{3})\,(z_1 - z_0)^4$
(D) None of these

Q.11 Number of ordered pair(s) (a, b) of real numbers such that $(a + ib)^{2008} = a - ib$ holds good, is –
(A) 2008 (B) 2009
(C) 2010 (D) 1

Q.12 If a, b, c are three distinct non-zero complex number such that $|a| = |b| = |c|$ and the equation $az^2 + bz + c = 0$ has a root whose modulus is 1, then –
(A) $b^2 = ac$ (B) $c^2 = ab$
(C) $a^2 = bc$ (D) None of these

Q.13 Let z_r $(1 \le r \le 4)$ be complex numbers such that $|z_r| = \sqrt{r+1}$ and $|30z_1 + 20z_2 + 15z_3 + 12z_4|$ $= k|z_1z_2z_3 + z_2z_3z_4 + z_3z_4z_1 + z_4z_1z_2|$. Then the value of k equals –

(A) $|z_1z_2z_3|$ (B) $|z_2z_3z_4|$
(C) $|z_3z_4z_1|$ (D) $|z_4z_1z_2|$

Q.14 A particle starts to travel from a point P on the curve $C_1 : |z - 3 - 4i| = 5$, where $|z|$ is maximum. From P, the particle moves through an angle $\tan^{-1}\frac{3}{4}$ in anticlockwise direction on $|z - 3 - 4i| = 5$ and reaches at point Q. From Q, it comes down parallel to imaginary axis by 2 units and reaches at point R. Complex number corresponding to point R in the Argand plane is –

(A) $(3 + 5i)$ (B) $(3 + 7i)$
(C) $(3 + 8i)$ (D) $(3 + 9i)$

Q.15 If z_1, z_2, z_3 be three points on $|z| = 1$ and $z_1 + z_2 + z_3 = 0$. If θ_1, θ_2 and θ_3 be the arguments z_1, z_2, z_3 respectively, then $\cos(\theta_1 - \theta_2) + \cos(\theta_2 - \theta_3) + \cos(\theta_3 - \theta_1) =$

(A) 0 (B) –1
(C) 3/2 (D) –3/2

Q.16 If A (z_1) and B (z_2) are two points on circle $|z| = r$ then the tangents to the circle at A and B will intersect at –

(A) $\frac{z_1^2 + z_2^2}{z_1 + z_2}$ (B) $\frac{z_1z_2}{z_1 + z_2}$

(C) $\frac{2z_1z_2}{z_1 + z_2}$ (D) $\frac{z_1^2 + z_2^2}{2(z_1 + z_2)}$

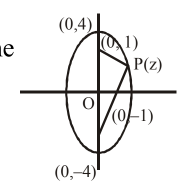
Q.17 If $x^2 - 2x\cos\theta + 1 = 0$, then the value of $x^{2n} - 2x^n \cos n\theta + 1$, $n \in N$ is equal to –

(A) $\cos 2n\theta$ (B) $\sin 2n\theta$
(C) 0 (D) some real number greater than 0

Q.18 If $\omega = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}$, then value of $1 + \omega + \omega^2 + ... + \omega^{n-1}$ is

(A) $1 + i$ (B) $1 + i\tan(\pi/n)$
(C) $1 + i\cot(\pi/2n)$ (D) none of these

Q.19 If z is a complex number satisfying the equation $|z + i| + |z - i| = 8$, on the complex plane then maximum value of $|z|$ is –



(A) 2 (B) 4
(C) 6 (D) 8

Q.20 If $\sum_{k=0}^{100} i^k = x + iy$, then the values of x and y are

(A) $x = -1, y = 0$ (B) $x = 1, y = 1$
(C) $x = 1, y = 0$ (D) $x = 0, y = 1$

Q.21 a, b, c are three complex numbers on the unit circle $|z| = 1$, such that $abc = a + b + c$. Then $|ab + bc + ca|$ is equal to

(A) 3 (B) 6
(C) 1 (D) 2

Q.22 The points of intersection of the two curves $|Z - 3| = 2$ and $|Z| = 2$ in an Argand plane are

(A) $\frac{1}{2}(7 \pm \sqrt{3})$ (B) $\frac{1}{2}(3 \pm i\sqrt{7})$

(C) $\frac{3}{2} \pm i\sqrt{\frac{7}{2}}$ (D) $\frac{7}{2} \pm i\sqrt{\frac{3}{2}}$

Q.23 The solution of the equation $2z = |z| + 2i$, where z is a complex number, is –

(A) $z = \frac{\sqrt{3}}{3} - i$ (B) $z = \frac{\sqrt{3}}{3} + i$

(C) $z = \frac{\sqrt{3}}{3} \pm i$ (D) None of these

Q.24 The value of the expression

$$\left(1+\frac{1}{\omega}\right)\left(1+\frac{1}{\omega^2}\right)+\left(2+\frac{1}{\omega}\right)\left(2+\frac{1}{\omega^2}\right)+\left(3+\frac{1}{\omega}\right)\left(3+\frac{1}{\omega^2}\right)+....+\left(n+\frac{1}{\omega}\right)\left(n+\frac{1}{\omega^2}\right)$$

where ω is an imaginary cube root of unity, is –

(A) $\frac{n(n^2-2)}{3}$ (B) $\frac{n(n^2+2)}{3}$

(C) $\frac{n(n^2-1)}{3}$ (D) None of these

Q.25 If a complex number z satisfies $|2z + 10 + 10i| \le 5\sqrt{3} - 5$, then the least principal argument of z is –

(A) $-\frac{11\pi}{12}$ (B) $-\frac{5\pi}{6}$

(C) $-\frac{2\pi}{3}$ (D) $-\frac{3\pi}{4}$

Q.26 Principal argument of the complex number $z = \frac{2(1-i\sqrt{3})(1+i)}{(\sqrt{3}-i)^3(-1+i)^4}$ is –

(A) $\frac{\pi}{4}$ (B) $\frac{-5\pi}{12}$

(C) $\frac{2\pi}{3}$ (D) $-\frac{7\pi}{12}$

Q.27 If $z = \frac{1}{2}(i\sqrt{3} - 1)$, then the value of $(z - z^2 + 2z^3)(2 - z + z^2)$ is –

(A) 3 (B) 7
(C) –1 (D) 5

Q.28 Given f (z) = the real part of a complex number z. For example, f (3 – 4i) = 3. If $a \in N, n \in N$ then the value of $\sum_{n=1}^{6a} \log_2 \left| f\left(\left(1+i\sqrt{3}\right)^n\right) \right|$ has the value equal to –

(A) $18a^2 + 9a$ (B) $18a^2 + 7a$
(C) $18a^2 - 3a$ (D) $18a^2 - a$

Q.29 The solutions of the equation $(1 + i\sqrt{3})^x - 2^x = 0$ form

(A) An A.P. (B) A G.P.
(C) A H.P. (D) None of these

Q.30 If $\left|\frac{z_1}{z_2}\right| = 1$ and $\arg(z_1 z_2) = 0$, then

(A) $z_1 = z_2$ (B) $|z_2|^2 = z_1 z_2$
(C) $z_1 z_2 = 1$ (D) none of these

Q.31 Let $Z_i = r_i(\cos\theta_i + i\sin\theta_i)$ i = 1, 2, 3 and $\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} = 0$. Consider the Δ ABC formed by $\frac{\cos 2\theta_1 + \sin 2\theta_1}{Z_1}, \frac{\cos 2\theta_2 + \sin 2\theta_2}{Z_2}, \frac{\cos 2\theta_3 + \sin 2\theta_3}{Z_3}$

Then the complex number lies –

(A) On the side BC (B) Outside the triangle
(C) Inside the triangle (D) On the side CA

Q.32 If $(1 + x)^n = p_0 + p_1 x + p_2 x^2 + ... + p_n x^n$, then $p_0 - p_2 + p_4 - p_6 + ...$ is equal to

(A) $2^{n/2}\cos n\pi/4$ (B) $2^n \sin n\pi/4$
(C) $2^n \cos n\pi/4$ (D) $2^{n/2} \sin n\pi/4$

Q.33 It is given that complex numbers z_1 and z_2 satisfy $|z_1| = 2$ and $|z_2| = 3$. If the included angle of their corresponding vectors is 60° then $\left|\frac{z_1 + z_2}{z_1 - z_2}\right|$ can be expressed as $\sqrt{N}/7$ where N is natural number then N equals –

(A) 126 (B) 119
(C) 133 (D) 19

Q.34 A regular hexagon is drawn with two of its vertices forming a shorter diagonal at z = – 2 and $z = 1 - i\sqrt{3}$. The other four vertices are

(A) $\pm 2\sqrt{3}, \pm i$ (B) $\pm\sqrt{3}, \pm i$
(C) $\sqrt{3}, \sqrt{3} \pm i, -1 - i\sqrt{3}$ (D) none of these

Q.35 If Z is point on the circle |Z – 1| = 1, then $\frac{Z-2}{Z}$ equals

(A) i tan (arg Z) (B) i cot (arg Z)
(C) i tan (arg (Z – 1)) (D) i cot (arg (Z – 1))

Q.36 If z satisfies $|z+1| < |z-2|$ then $\omega = 3z + 2 + i$ satisfies

(A) $|\omega + 2| < |\omega - 8|$
(B) $|\omega + 1 + i| < |\omega - 8 + i|$
(C) $\text{Re}\left(\frac{1}{2\omega - 7}\right) > 0$
(D) $|\omega + 5| < |\omega - 4|$

Q.37 If from a point P representing the complex number z_1 on the circle |z| = 2, pair of tangents are drawn to the circle |z| = 1, where Q (z_2) and R (z_3) are the points of contact, then which of the following options is incorrect –

(A) orthocentre and circumcentre of Δ PQR will coincide and lie on $|z| = \frac{3}{2}$

(B) $\left(\frac{4}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3}\right)\left(\frac{4}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) = 9$

(C) $\arg\left(\frac{z_2}{z_3}\right)$ is either $-\frac{2\pi}{3}$ or $\frac{2\pi}{3}$

(D) Complex no. $\frac{z_1 + z_2 + z_3}{3}$ will lie on the circle |z| = 1

ASSERTION AND REASON QUESTIONS

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1.
(C) Statement -1 is True, Statement-2 is False.
(D) Statement -1 is False, Statement-2 is True.
(E) Statement -1 is False, Statement-2 is False.

Q.38 **Statement 1 :** If $|z_1| = 30$, $|z_2 - (12 + 5i)| = 6$, then maximum value of $|z_1 - z_2|$ is 49.

Statement 2 : If z_1, z_2 are two complex numbers, then $|z_1 - z_2| \le |z_1| + |z_2|$ and equality holds when origin, z_1 and z_2 are collinear and z_1, z_2 are on the opposite side of the origin.

Q.39 **Statement 1 :** Any complex number z satisfy at least one of the two inequalities $|z+1| \ge \frac{1}{\sqrt{2}}$ or $|z^2 + 1| \ge 1$.

Statement 2 : There are no non-zero real numbers a and b such that $a^2 + b^2 \le 0$.

Q.40 **Statement 1 :** Two lines $a\bar{z} + \bar{a}z + b = 0$, $a_1\bar{z} + \bar{a}_1 z + b_1 = 0$ (where $a, a_1 \in C$, $a, a_1 \ne 0$ and $b, b_1 \in R$) are parallel if and only if $\frac{a}{a_1}$ is real.

Statement 2 : Two lines $a\bar{z} + \bar{a}z + b = 0$, $a_1\bar{z} + \bar{a}_1 z + b_1 = 0$ (where $a, a_1 \in C$, $a, a_1 \ne 0$ and $b, b_1 \in R$) are perpendicular if and only if $\frac{a}{a_1}$ is purely imaginary.

Q.41 **Statement 1 :** a, b, c are three non-zero real numbers such that a + b + c = 0 and z_1, z_2, z_3 are three complex numbers such that $az_1 + bz_2 + cz_3 = 0$, then z_1, z_2 and z_3 lie on a circle.

Statement 2 : If z_1, z_2 and z_3 are collinear then $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$

Q.42 **Statement 1 :** Two non-zero complex numbers z_1 and z_2 lie on a straight line through origin if and only if $z_1\bar{z}_2$ is real.

Statement 2 : Two non-zero complex numbers z_1 and z_2 always lie on a straight line passing through origin if and only $\bar{z}_1 z_2$ is real.

MATCH THE COLUMN TYPE QUESTIONS

Q.43 If $z_1, z_2, ... z_{10}$ are the roots of the equation $1 + z + z^2 + ... z^{10} = 0$ match the entries given in column I with one of the entries in column II.

Column I	**Column II**
(a) $(1+z_1)(1+z_2)(1+z_3) .... (1+z_{10})$	(p) 1
(b) $1 + z_1^{100} + z_2^{100} + z_3^{100} + ... + z_{10}^{100}$	(q) –1
(c) $(1-z_1)(1-z_2)(1-z_3) .... (1-z_{10})$	(r) 0
(d) $z_1 \times z_2 \times z_3 \times ...... \times z_{10}$	(s) 11

Code :

(A) a-p, b-r, c-s, d-p (B) a-s, b-q, c-s, d-r

(C) a-r, b-q, c-s, d-p (D) a-r, b-s, c-p, d-q

Q.44 Match the column –

Column I	**Column II**
(a) Locus of the point z satisfying the equation $\text{Re}(z^2) = \text{Re}\ (z + \bar{z})$	(p) A parabola
(b) Locus of the point z satisfying the equation $\|z - z_1\| + \|z - z_2\| = \lambda$, $\lambda \in R^+$ and $\lambda \nless \|z_1 - z_2\|$	(q) A straight line
(c) Locus of the point z satisfying the equation $\left\|\frac{2z - i}{z + 1}\right\| = m$ where $i = \sqrt{-1}$ and $m \in R^+$	(r) An ellipse
(d) If $\|\bar{z}\| = 25$ then the points representing the complex no. $-1 + 75\bar{z}$ will be on a	(s) A rectangular hyperbola
	(t) A circle

Code :

(A) a-s, b-qr, c-qt, d-t (B) a-ps, b-q, c-s, d-t

(C) a-r, b-pqr, c-s, d-p (D) a-qr, b-s, c-p, d-r

Q.45 Match the equation in z, in column I with the corresponding values of arg (z) in column II

Column I	**Column II**
(a) $z^2 - z + 1 = 0$	(p) $-2\pi/3$
(b) $z^2 + z + 1 = 0$	(q) $-\pi/3$
(c) $2z^2 + 1 + i\sqrt{3} = 0$	(r) $\pi/3$
(d) $2z^2 + 1 - i\sqrt{3} = 0$	(s) $2\pi/3$

Code :

(A) a-r, b-ps, c-s, d-p (B) a-pqr, b-ps, c-ps, d-qr

(C) a-qr, b-ps, c-qs, d-pr (D) a-pr, b-qs, c-ps, d-pr

PASSAGE BASED QUESTIONS

Passage 1- (Q.46-Q.48)

Let $f(x) = \frac{1}{x - i}$, where $x \in R$ and let $f(\alpha), f(\beta), f(\gamma), f(\delta)$ be four points on the Argand plane. Now answer the following questions

Q.46 The maximum value of $|f(\alpha) - f(\beta)|$ is –

(A) $|\alpha - \beta|$ (B) $\left|\frac{1}{\alpha} - \frac{1}{\beta}\right|$

(C) 1 (D) 2

Q.47 If a triangle is formed by joining the points $f(\alpha), f(\beta), f(\gamma)$ then maximum value of the area of triangle is –

(A) $3\sqrt{3}$ (B) $\frac{3\sqrt{3}}{4}$ (C) $\frac{3\sqrt{3}}{16}$ (D) None

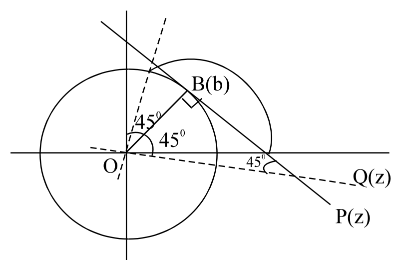
Q.48 Points $f(\alpha), f(\beta), f(\gamma), f(\delta)$ are chosen such that they form a square, the length of square is –

(A) 1/2 (B) $1/\sqrt{2}$

(C) 1 (D) None of these

Passage 2- (Q.49-Q.51)

Let z be a complex number lying on a circle $|z| = \sqrt{2}\,a$ and $b = b_1 + ib_2$ (any complex number), then



Let P(z) be any point on the tangent at B(b), then $OB \perp PB$

$$\Rightarrow \frac{z - b}{|z - b|} = \frac{b - 0}{|b - 0|} e^{i\pi/2}$$

$$\Rightarrow z\bar{b} - b\bar{b} = \bar{z}b + b\bar{b} \Rightarrow z\bar{b} + \bar{z}b = 2|b|^2$$

$\because$ b lie on $z = \sqrt{2}\,a$ $\quad\therefore |b| = \sqrt{2}\,a$

Q.49 The length of perpendicular from z_0 (any point on the circle) on the tangent at 'b' is

(A) $\frac{|z_0\bar{b} + \bar{z}_0 b - a^2|}{2\sqrt{2}\,a}$ (B) $\frac{|z_0\bar{b} + \bar{z}_0 b - 2a^2|}{2\sqrt{2}\,a}$

(C) $\frac{|z_0\bar{b} + \bar{z}_0 b - 3a^2|}{2\sqrt{2}\,a}$ (D) $\frac{|z_0\bar{b} + \bar{z}_0 b - 4a^2|}{2\sqrt{2}\,a}$

Q.50 The equation of tangent at point 'b' is

(A) $z\bar{b} + \bar{z}b = a^2$ (B) $z\bar{b} + \bar{z}b = 2a^2$

(C) $z\bar{b} + \bar{z}b = 3a^2$ (D) $z\bar{b} + \bar{z}b = 4a^2$

Q.51 The equation of straight line parallel to the tangent and passing through centre circle is

(A) $z\bar{b}+\bar{z}b=0$ (B) $2z\bar{b}+\bar{z}b=\lambda$

(C) $2z\bar{b}+3\bar{z}b=0$ (D) $z\bar{b}+\bar{z}b=\lambda$

Passage 3- (Q.52-Q.54)

The complex slope of a line passing through two points represented by complex numbers z_1 and z_2 is defined by $\frac{z_2-z_1}{\bar{z}_2-\bar{z}_1}$ and we shall denote by ω. If z_0 is complex number and c is a real number, then $\bar{z}_0 z+z_0\bar{z}+c=0$ represents a straight line. Its complex slope is $-\frac{z_0}{\bar{z}_0}$.

Now consider two lines

$\alpha\bar{z}+\bar{\alpha}z+i\beta=0$ .... (i) and $a\bar{z}+\bar{a}z+b=0$ .... (ii)

where α, β and a, b are complex constants and let their complex slopes be denoted by ω_1 and ω_2 respectively –

Q.52 If the lines are inclined at an angle of 120° to each other, then –

(A) $\omega_2\bar{\omega}_1=\omega_1\bar{\omega}_1$ (B) $\omega_2\bar{\omega}_1^2=\omega_1\bar{\omega}_2^2$

(C) $\omega_1^2=\omega_2^2$ (D) $\omega_1+2\omega_2=0$

Q.53 Which of the following must be true –

(A) a must be pure imaginary

(B) β must be pure imaginary

(C) a must be real

(D) b must be imaginary

Q.54 If line (i) makes an angle of 45° with real axis, then $(1+i)\left(-\frac{2\alpha}{\bar{\alpha}}\right)$ is –

(A) $2\sqrt{2}$ (B) $2\sqrt{2}i$

(C) $2(1-i)$ (D) $-2(1+i)$

EXERCISE - 3 (NUMERICAL VALUE BASED QUESTIONS)

NOTE : The answer to each question is a NUMERICAL VALUE.

Q.1 The smallest positive integral value of n for which the complex number $\left(1+\sqrt{3}\,i\right)^{n/2}$ is real, is

Q.2 Let z be a complex number of constant non zero modulus such that z^2 is purely imaginary, then the number of possible values of z is

Q.3 Suppose that w is the imaginary $(2009)^{th}$ roots of unity. If $(2^{2009}-1)\sum_{r=1}^{2008}\frac{1}{2-w^r}=(a)(2^b)+c$ where a, b, c ∈ N, then find the least value of (a + b + c).

Q.4 For x ∈ (0, π/2) and $\sin x=\frac{1}{3}$, if $\sum_{n=0}^{\infty}\frac{\sin(nx)}{3^n}=\frac{a+b\sqrt{b}}{c}$ then find the value of $(a+b+c)$, where a, b, c are positive integers.

(You may Use the fact that $\sin x=\frac{e^{ix}-e^{-ix}}{2i}$)

Q.5 The polynomial $f(x)=x^4+ax^3+bx^2+cx+d$ has real coefficients and $f(2i)=f(z+i)=0$. The value of $(a+b+c+d)$ equals

Q.6 The number of solutions of the equation $z^2+z=0$ where z is a complex number, is :

Q.7 If z = (3 + 7*i*) (p + *i*q) where p, q ∈ I – {0}, is purely imaginary then minimum value of $|z|^2$ is

Q.8 Number of values of x (real or complex) simultaneously satisfying the system of equations $1+z+z^2+z^3+.....+z^{17}=0$ and $1+z+z^2+z^3+.....+z^{13}=0$ is

Q.9 Number of complex numbers z satisfying $z^3=\overline{z}$ is

Q.10 The complex number z satisfies $z+|z|=2+8i$. The value of $|z|$ is

Q.11 The minimum value of $|1+z|+|1-z|$ where z is a complex number is

Q.12 If a, b, c are integers, not all simultaneously equal and ω is cube root of unity $(\omega\neq 1)$, then minimum value of $|a+b\omega+c\omega^2|$ is

Q.13 Let z = x + iy be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $z\overline{z}^3+\overline{z}z^3=350$ is

Q.14 If z is any complex number satisfying $|z-3-2i|\le 2$, then the minimum value of $|2z-6+5i|$ is

Q.15 Let $\omega=e^{\frac{i\pi}{3}}$, and a, b, c, x, y, z be non-zero complex numbers such that $a+b+c=x$; $a+b\omega+c\omega^2=y$; $a+b\omega^2+c\omega=z$. Then the value of $\frac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}$ is

Q.16 For any integer k, let $\alpha_k=\cos\left(\frac{k\pi}{7}\right)+i\sin\left(\frac{k\pi}{7}\right)$, where $i=\sqrt{-1}$. The value of the expression

$$\frac{\sum_{k=1}^{12}|\alpha_{k+1}-\alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1}-\alpha_{4k-2}|}$$ is –

EXERCISE - 4 [PREVIOUS YEARS AIEEE / JEE MAIN QUESTIONS]

Q.1 Let z and w are two non zero complex number such that $|z| = |w|$, and Arg (z) + Arg (w) = π then -**[AIEEE 2002]**
(A) $z = w$ (B) $z = \bar{w}$
(C) $\bar{z} = \bar{w}$ (D) $z = -\bar{w}$

Q.2 If $|z - 2| \geq |z - 4|$ then correct statement is-**[AIEEE 2002]**
(A) $R(z) \geq 3$ (B) $R(z) \leq 3$
(C) $R(z) \geq 2$ (D) $R(z) \leq 2$

Q.3 If ω is an imaginary cube root of unity then $(1+\omega-\omega^2)(1+\omega^2-\omega)$ equals- **[AIEEE 2002]**
(A) 0 (B) 1
(C) 2 (D) 4

Q.4 If z and ω are two non- zero comlex numbers such that $|z\omega| = 1$, and Arg (z) – Arg (ω) = $\pi/2$, then $\bar{z}\,\omega$ is equal to- **[AIEEE 2003]**
(A) $-i$ (B) 1
(C) -1 (D) i

Q.5 Let z_1 and z_2 be two roots of the equation $z^2 + az + b = 0$, z being complex. Further assume that the origin, z_1 and z_2 form an equilateral triangle. Then **[AIEEE 2003]**
(A) $a^2 = 4b$ (B) $a^2 = b$
(C) $a^2 = 2b$ (D) $a^2 = 3b$

Q.6 If $\left(\frac{1+i}{1-i}\right)^x = 1$, then **[AIEEE 2003]**
(A) x = 2n + 1, where n is any positive integer
(B) x = 4n, where n is any positive integer
(C) x = 2n, where n is any positive integer
(D) x = 4n + 1, where n is any positive integer

Q.7 Let z, w be complex numbers such that $\bar{z} + i\bar{w} = 0$ and arg zw = π. Then arg z equals- **[AIEEE 2004]**
(A) $\pi/4$ (B) $\pi/2$
(C) $3\pi/4$ (D) $5\pi/4$

Q.8 If $z = x - iy$ and $z^{1/3} = p + iq$, then $\frac{\left(\frac{x}{p}+\frac{y}{q}\right)}{(p^2+q^2)}$ is equal to-
(A) 1 (B) -1 **[AIEEE 2004]**
(C) 2 (D) -2

Q.9 If $|z^2 - 1| = |z|^2 + 1$, then z lies on- **[AIEEE 2004]**
(A) the real axis (B) the imaginary axis
(C) a circle (D) an ellipse

Q.10 If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then arg z_1 – arg z_2 is equal to - **[AIEEE 2005]**
(A) $\pi/2$ (B) $-\pi$
(C) 0 (D) $-\pi/2$

Q.11 If $w = \frac{z}{z - \frac{1}{3}i}$ and $|w| = 1$, then z lies on **[AIEEE 2005]**
(A) an ellipse (B) a circle
(C) a straight line (D) a parabola

Q.12 If the cube roots of unity are 1, ω, ω^2 then the roots of the equation $(x - 1)^3 + 8 = 0$, are - **[AIEEE-2005]**
(A) $-1, -1 + 2\omega, -1 - 2\omega^2$ (B) $-1, -1, -1$
(C) $-1, 1 - 2\omega, 1 - 2\omega^2$ (D) $-1, 1 + 2\omega, 1 + 2\omega^2$

Q.13 If $z^2 + z + 1 = 0$, where z is a complex number, then the value of
$$\left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+\left(z^3+\frac{1}{z^3}\right)^2+....+\left(z^6+\frac{1}{z^6}\right)^2$$
is – **[AIEEE 2006]**
(A) 54 (B) 6
(C) 12 (D) 18

Q.14 The value of $\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11}+i\cos\frac{2k\pi}{11}\right)$ is –**[AIEEE 2006]**
(A) 1 (B) -1
(C) $-i$ (D) i

Q.15 If $|z + 4| \leq 3$, then the maximum and minimum value of $|z + 1|$ are - **[AIEEE 2007]**
(A) 4, 1 (B) 4, 0
(C) 6, 1 (D) 6, 0

Q.16 The conjugate of a complex number is $\frac{1}{i-1}$. Then that complex number is- **[AIEEE 2008]**
(A) $\frac{1}{i+1}$ (B) $\frac{-1}{i+1}$ (C) $\frac{1}{i-1}$ (D) $\frac{-1}{i-1}$

Q.17 If $\left|Z - \frac{4}{z}\right| = 2$, then the maximum value of $|Z|$ is equal to
(A) $\sqrt{5}+1$ (B) 2 **[AIEEE 2009]**
(C) $2+\sqrt{2}$ (D) $\sqrt{3}+1$

Q.18 The number of complex numbers z such that $|z - 1| = |z + 1| = |z - i|$ equals – **[AIEEE 2010]**
(A) 1 (B) 2
(C) ∞ (D) 0

Q.19 Let α, β be real and z be a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line Re z = 1, then it is necessary that **[AIEEE 2011]**
(A) $\beta \in (0, 1)$ (B) $\beta \in (-1, 0)$
(C) $|\beta| = 1$ (D) $\beta \in (1, \infty)$

Q.20 If $\omega\ (\neq 1)$ is a cube root of unity, and $(1 + \omega)^7 = A + B\omega$. Then (A, B) equals **[AIEEE 2011]**
(A) (0, 1) (B) (1, 1)
(C) (1, 0) (D) (–1, 1)

Q.21 If $z \neq 1$ and $\frac{z^2}{z-1}$ zs real, then the point represented by the complex number z lies : **[AIEEE 2012]**
(A) either on the real axis or on a circle passing through the origin.
(B) on a circle with centre at the origin.

(C) either on the real axis or on a circle not passing through the origin.
(D) on the imaginary axis.

Q.22 If z is a complex number of unit modulus and argument θ, then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals – **[JEE MAIN 2013]**

(A) $-\theta$ (B) $\frac{\pi}{2}-\theta$
(C) θ (D) $\pi-\theta$

Q.23 If z is a complex number such that $|z| \ge 2$, then the minimum value of $\left|z+\frac{1}{2}\right|$ **[JEE MAIN 2014]**

(A) is equal to 5/2
(B) lies in the interval (1, 2)
(C) is strictly greater than 5/2
(D) is strictly greater than 3/2 but less than 5/2

Q.24 A complex number z is said to be unimodular if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1-2z_2}{2-z_1\bar{z}_2}$ is is unimodular and z_2 is not unimodular. Then the point z_1 lies on a – **[JEE MAIN 2015]**

(A) Straight line parallel to y-axis
(B) Circle of radius 2.
(C) Circle of radius $\sqrt{2}$.
(D) Straight line parallel to x-axis.

Q.25 A value of θ for which $\frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, is **[JEE MAIN 2016]**

(A) π/6 (B) $\sin^{-1}(\sqrt{3}/4)$
(C) $\sin^{-1}(1/\sqrt{3})$ (D) π/3

Q.26 Let ω be a complex number such that $2\omega + 1 = z$ where $z=-\sqrt{3}$. If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to

(A) –1 (B) 1 **[JEE MAIN 2017]**
(C) –z (D) z

Q.27 If α, β ∈ C are the distinct roots, of the equation $x^2-x+1=0$, then $\alpha^{101}+\beta^{107}$ is equal to:
(A) 1 (B) 2 **[JEE MAIN 2018]**
(C) –1 (D) 0

Q.28 Let $A=\left\{\theta\in\left(-\frac{\pi}{2},\pi\right):\frac{3+2i\sin\theta}{1-2i\sin\theta}\text{ is purely imaginary}\right\}$ then the sum of the element in A is **[JEE MAIN 2019 (JAN)]**

(A) 5π/6 (B) 2π/3
(C) 3π/4 (D) π

Q.29 If $z=\frac{\sqrt{3}}{2}+\frac{i}{2}$ $(i=\sqrt{-1})$, then $(1+iz+z^5+iz^8)^9$ is equal to **[JEE MAIN 2019 (APRIL)]**
(A) –1 (B) 1
(C) 0 (D) $(-1+2i)^9$

Q.30 Let z ∈ C be such that $|z| < 1$. If $\omega=\frac{5+3z}{5(1-z)}$, then **[JEE MAIN 2019 (APRIL)]**
(A) 5 Im (ω) < 1 (B) 4 Im (ω) > 5
(C) 5 Re (ω) > 1 (D) 5 Re (ω) > 4

Q.31 If a > 0 and $z=\frac{(1+i)^2}{a-i}$, has magnitude $\sqrt{\frac{2}{5}}$, then $\bar{z}$ is equal to : **[JEE MAIN 2019 (APRIL)]**

(A) $-\frac{3}{5}-\frac{1}{5}i$ (B) $-\frac{1}{5}+\frac{3}{5}i$
(C) $-\frac{1}{5}-\frac{3}{5}i$ (D) $\frac{1}{5}-\frac{3}{5}i$

Q.32 If z and w are two complex numbers such that |zw| = 1 and arg (z) – arg (w) = π/2, then : **[JEE MAIN 2019 (APRIL)]**
(A) $\bar{z}w = i$ (B) $\bar{z}w = -i$
(C) $z\bar{w}=\frac{1-i}{\sqrt{2}}$ (D) $z\bar{w}=\frac{-1+i}{\sqrt{2}}$

Q.33 The equation |z–i| = |z–1|, $i=\sqrt{-1}$, represents: **[JEE MAIN 2019 (APRIL)]**
(A) the line through the origin with slope –1.
(B) a circle of radius 1.
(C) a circle of radius 1/2.
(D) the line through the origin with slope 1.

Q.34 Let z ∈ C with Im (z) = 10 and it satisfies $\frac{2z-n}{2z+n}=2i-1$ for some natural number n. Then: **[JEE MAIN 2019 (APRIL)]**
(A) n = 20 and Re (z) = – 10 (B) n = 20 and Re (z) = 10
(C) n = 40 and Re (z) = – 10 (D) n = 40 and Re (z) = 10

Q.35 If z = x + iy and real part $\left(\frac{z-1}{2z+i}\right)=1$ then locus of z is –
(A) Straight line with slope 2. **[JEE MAIN 2020 (JAN)]**
(B) Straight line with slope –1/2
(C) circle with diameter $\sqrt{5}/2$
(D) circle with diameter 1/2

Q.36 If the equation $x^2 + bx + 45 = 0$, b ∈ R has conjugate complex roots and they satisfy $|z+1| = 2\sqrt{10}$, then **[JEE MAIN 2020 (JAN)]**
(A) $b^2+b=12$ (B) $b^2-b=30$
(C) $b^2-b=36$ (D) $b^2+b=30$

Q.37 Let $\alpha = \frac{-1+i\sqrt{3}}{2}$ & $a = (1+\alpha)\sum_{k=0}^{100}\alpha^{2k}$, $b = \sum_{k=0}^{100}\alpha^{3k}$. If a and b are roots of quadratic equation then quadratic equation is **[JEE MAIN 2020 (JAN)]**

(A) $x^2 - 102x + 101 = 0$ (B) $x^2 - 101x + 100 = 0$

(C) $x^2 + 101x + 100 = 0$ (D) $x^2 + 102x + 100 = 0$

Q.38 Let z be complex number such that $\left|\frac{z-i}{z+2i}\right| = 1$ and $|z| = \frac{5}{2}$. Then the value of $|z+3i|$ is : **[JEE MAIN 2020 (JAN)]**

(A) $\sqrt{10}$ (B) $2\sqrt{3}$

(C) 7/2 (D) 15/4

Q.39 If z be a complex number satisfying $|\text{Re}(z)| + |\text{Im}(z)| = 4$, then $|z|$ cannot be **[JEE MAIN 2020 (JAN)]**

(A) $\sqrt{\frac{17}{2}}$ (B) $\sqrt{10}$

(C) $\sqrt{8}$ (D) $\sqrt{7}$

Q.40 If $z = \left(\frac{3+i\sin\theta}{4-i\cos\theta}\right)$ is purely real and $\theta \in \left(\frac{\pi}{2}, \pi\right)$ arg $(\sin\theta + i\cos\theta)$ is – **[JEE MAIN 2020 (JAN)]**

(A) $-\tan^{-1}(3/4)$ (B) $\pi - \tan^{-1}(3/4)$

(C) $\pi - \tan^{-1}(4/3)$ (D) $\tan^{-1}(4/3)$

ANSWER KEY

EXERCISE - 1

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	A	B	B	A	D	B	C	A	B	B	C	A	B	A	D	A	B	A	A	C	B	B	C	D	A
Q	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
A	D	A	C	B	A	A	C	A	B	B	D	A	D	C	D	D	A	C	B	C	C	C	C	A	B
Q	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75
A	C	C	A	C	D	C	B	A	D	A	B	A	B	A	A	A	C	C	C	C	A	D	D	D	D
Q	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98		
A	B	D	A	A	D	B	D	D	D	D	C	A	C	A	D	C	D	C	C	D	D	B	B		

EXERCISE - 2

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	A	B	B	C	C	C	A	C	D	A	C	A	D	B	D	C	C	C	B	C	C	B	B	B	B
Q	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
A	D	B	D	A	B	C	A	C	D	A	B	A	B	A	B	D	A	A	A	C	C	C	B	D	D
Q	51	52	53	54																					
A	A	B	B	C																					

EXERCISE - 3

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A	6	4	4016	41	9	2	3364	1	5	17	2	1	48	5	3	4

EXERCISE - 4

Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
A	D	A	D	A	D	B	C	D	B	C	C	C	C	C	C	B	A	A	D	B	A	C	B	B	C
Q	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40										
A	C	A	B	A	C	C	B	D	C	C	B	A	C	D	C										

CHAPTER- 5 : COMPLEX NUMBERS

SOLUTIONS TO TRY IT YOURSELF

TRY IT YOURSELF-1

(1) 135 leaves remainder as 3 when it is divided by 4

$\therefore \quad i^{135} = i^3 = -1$

(2) We have, $(a+b) - i(3a+2b) = 5 + 2i$

$\Rightarrow \quad a + b = 5$ and $-(3a+2b) = 2$

$\Rightarrow \quad a = -12, b = 17$

(3) $(x+iy)^{1/3} = a - ib$

$x + iy = (a - ib)^3 = (a^3 - 3ab^2) + i(b^3 - 3a^2b)$

$x = a^3 - 3ab^2,\ y = b^3 - 3a^2b$

$$\frac{x}{a} = a^2 - 3b^2 \text{ and } \frac{y}{b} = b^2 - 3a^2$$

$$\frac{x}{a} - \frac{y}{b} = a^2 - 3b^2 - b^2 - 3a^2 = 4(a^2 - b^2)$$

$\therefore \quad k = 4$

(4) $z^2 - az + a - 1 = 0$

Putting $z = 1 + i$ in the equation, we get $a = 2 + i$

$\Rightarrow \quad z^2 - (2+i)z + 1 + i = 0$ is the equation

$\Rightarrow \quad z = 1$ is the other root.

(5) $$\frac{(1+i)^2}{3-i} = \frac{1+2i+i^2}{3-i} = \frac{2i}{3-i} = \frac{2i}{3-i}\,\frac{3+i}{3+i} = \frac{6i+2i^2}{9-i^2}$$

$$= \frac{-2+6i}{10} = -\frac{1}{5} + \frac{3}{5}i$$

(6) Let $\sqrt{9+40i} = x + iy$. Then, $(x+iy)^2 = 9 + 40i$

$\Rightarrow \quad x^2 - y^2 = 9 \quad$ .... (1)

and $xy = 20 \quad$ .... (2)

Squaring eq. (1) and adding with 4 times the square of (2), we get $x^4 + y^4 - 2x^2y^2 + 4x^2y^2 = 81 + 1600$

$\Rightarrow \quad (x^2+y^2)^2 = 1681 \Rightarrow x^2 + y^2 = 41 \quad$ ...... (3)

From eq. (1) + (3), we get $x^2 = 25 \Rightarrow x = \pm 5$ and ± 4

From eq. (2), we can see that x and y are of same sign.

$\Rightarrow \quad x + iy = (5 + 4i)$ or $-(5 + 4i)$

(7) $$\left(\frac{1}{3}+3i\right)^3 = \left(\frac{1}{3}\right)^3 + (3i)^3 + 3\times\left(\frac{1}{3}\right)^2 \times 3i + 3\times\frac{1}{3}\times(3i)^2$$

$$= \frac{1}{27} + 27i^3 + i + 9i^2 = \frac{1}{27} - 27i + i - 9$$

$[i^3 = -i \text{ and } i^2 = -1]$

$$= \left(\frac{1}{27} - 9\right) - 26i = \frac{-242}{27} - 26i$$

(8) $z = \sqrt{5} + 3i$ then $\bar{z} = \sqrt{5} - 3i$

and $|z| = (\sqrt{5})^2 + (3)^2 = 5 + 9 = 14$

Therefore, the multiplicative inverse of $\sqrt{5} + 3i$ is given

by $\dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2} = \dfrac{\sqrt{5}-3i}{14} = \dfrac{\sqrt{5}}{14} - \dfrac{3}{14}i$.

TRY IT YOURSELF-2

(1) Let $z = (1-i)^{-i}$. Taking log on both sides,

$\log z = -i\log(1-i)$

$$= -i\log\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) = -i\log(\sqrt{2}e^{-i(\pi/4)})$$

$$= -i\left[\frac{1}{2}\log 2 + \log e^{-i\pi/4}\right] = -i\left[\frac{1}{2}\log 2 - \frac{i\pi}{4}\right]$$

$$= -\frac{i}{2}\log 2 - \frac{\pi}{4} \Rightarrow z = e^{-\pi/4}e^{-i(\log 2)/2}$$

$$\Rightarrow \quad \text{Re}(z) = e^{-\pi/4}\cos\left(\frac{1}{2}\log 2\right)$$

(2) $|z| = z + 1 + 2i$

$\Rightarrow \quad \sqrt{x^2+y^2} = x + iy + 1 + 2i = x + 1 + (2+y)i$

$\Rightarrow \quad \sqrt{x^2+y^2} = x + 1$ and $0 = 2 + y$ or $y = -2$

$\Rightarrow \quad \sqrt{x^2+4} = x + 1$

$\Rightarrow \quad x^2 + 4 = x^2 + 2x + 1 \Rightarrow 2x = 3 \Rightarrow x = 3/2$

$\Rightarrow \quad x + iy = \dfrac{3}{2} - 2i$

(3) Given, $3 + ix^2y = \overline{x^2 + y + 4i}$

$-3 + ix^2y = x^2 + y - 4i \Rightarrow -3 = x^2 + y \quad$ ..... (1)

and $\quad x^2y = -4 \quad$ ..... (2)

$\therefore \quad -3 = x^2 - \dfrac{4}{x^2} \quad$ [Putting $y = -4/x^2$ from (2) in (1)]

$\Rightarrow \quad x^4 + 3x^2 - 4 = 0 \Rightarrow (x^2+4)(x^2-1) = 0$

(4) $|z_1| = 1 \Rightarrow z_1\bar{z}_1, |z_2| = 2 \Rightarrow z_2\bar{z}_2 = 4,$

$|z_3| = 3 \Rightarrow z_3\bar{z}_3 = 9$

Also, $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$

$\Rightarrow \quad |z_1z_2z_3\bar{z}_3 + z_1z_2z_3\bar{z}_2 + z_1\bar{z}_1z_2z_3| = 12$

$\Rightarrow \quad |z_1z_2z_3|\,|\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$

$\Rightarrow \quad |z_1||z_2||z_3|\,|\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12$

$\Rightarrow \quad 6|\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 12 \Rightarrow |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 2$

$\Rightarrow \quad |z_1 + z_2 + z_3| = 2$

(5) $|z_1 + z_2| \le |z_1| + |z_2| = |24 + 7i| + 6 = 25 + 6 = 31$

Also, $|z_1 + z_2| = |z_1 - (-z_2)| \ge ||z_1| - |z_2||$

$\Rightarrow \quad |z_1 + z_2| \ge |25 - 6| = 19$

Hence, the least value of $|z_1 + z_2|$ is 19 and the greatest value is 25.

(6) $$\text{amp}\left(\frac{1+\sqrt{3}i}{\sqrt{3}+i}\right) = \text{amp}(1+\sqrt{3}i) - \text{amp}(\sqrt{3}+i)$$

$$= \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

(7) $\arg(z_1z_2) = \arg(z_1) + \arg(z_2) = 170° + 70° = 240°$
Thus, z_1z_2 lies in third quadrant. Hence, its principal argument is – 120°

(8) We have, $z = -1 - i\sqrt{3}$

Let $-1 - i\sqrt{3} = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts, we get

$r\cos\theta = -1$ .... (1)

and $r\sin\theta = -\sqrt{3}$ .... (2)

Squaring and adding eq. (1) and (2), we get

$r^2(\cos^2\theta + \sin^2\theta) = 1 + 3$

$\Rightarrow r^2 = 4 \Rightarrow r = 2 \Rightarrow \text{Modulus} = |z| = r = 2$

$(-1, -\sqrt{3})$ lies in the third quadrant so it principal argument line in third quadrant.

Also, dividing (2) by (1), we get

$$\tan\theta = \sqrt{3} \Rightarrow \theta = \tan^{-1}(\sqrt{3}) = \tan\left(\frac{-2\pi}{3}\right)$$

$$\Rightarrow \text{Argument} = \theta = -\frac{2\pi}{3}$$

Hence, the modulus and arguments of the complex number $-1 - i\sqrt{3}$ are 2 and $-\frac{2\pi}{3}$ respectively.

(9) We have, $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$ .... (1)

Comparing (1), with $ax^2 + bx + c = 0$, we get

$a = \sqrt{3}, b = -\sqrt{2}$ and $c = 3\sqrt{3}$

Here, $b^2 - 4ac = (-\sqrt{2})^2 - 4(\sqrt{3})(3\sqrt{3}) = 2 - 36 = -34$

$$\therefore \quad x = \frac{-(-\sqrt{2}) \pm \sqrt{-34}}{2\sqrt{3}} = \frac{\sqrt{2} \pm i\sqrt{34}}{2\sqrt{3}}$$

(10) **(A).** $\arg(-z) - \arg(z) = \arg(-z/z) = \arg(-1) = \pi$

TRY IT YOURSELF-3

(1) Let $x = \sqrt{-2 + 2\sqrt{-2 + 2\sqrt{-2 + \ldots\infty}}}$

$$\Rightarrow x^2 = -2 + \sqrt{2\sqrt{-2 + 2\sqrt{-2 + \ldots\infty}}}$$

$$\Rightarrow x^2 = -2 + \sqrt{2}x \Rightarrow x^2 + 2 = \sqrt{2}x$$

$$\Rightarrow (x^2 + 2)^2 = 2x^2 \Rightarrow x^4 + 2x^2 + 4 = 0$$

$$\Rightarrow x^2 = \frac{-2 + \sqrt{-12}}{2} = -1 + \sqrt{3}i = 2\omega^2$$

$$\Rightarrow x = \pm\sqrt{2}\omega$$

(2) $z + z^{-1} = 1 \Rightarrow z^2 - z + 1 = 0 \Rightarrow z = -\omega \text{ or } -\omega^2$

For $z = -\omega$, $z^{100} + z^{-100} = (-\omega)^{100} + (-\omega)^{-100}$

$$= \omega + \frac{1}{\omega} = \omega + \omega^2 = -1$$

For $z = -\omega^2$, $z^{100} + z^{-100} = (-\omega^2)^{100} + (-\omega^2)^{-100}$

$$= \omega^{200} + \frac{1}{\omega^{200}} = \omega^2 + \frac{1}{\omega^2} = \omega^2 + \omega = -1$$

(3) $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8)$
$= (1 - \omega)(1 - \omega^2)(1 - \omega)(1 - \omega^2)$
$= (1 - \omega)^2(1 - \omega^2)^2 = (1 - 2\omega + \omega^2)(1 - 2\omega^2 + \omega^4)$
$= (1 - 2\omega + \omega^2)(1 - 2\omega^2 + \omega)$
$= (-3\omega)(-3\omega^2) = 9\omega^3 = 9$

(4) **(A).** $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{\frac{i\pi}{2}}$

$$i^i = \left(e^{\frac{i\pi}{2}}\right)^i = e^{-\frac{\pi}{2}} \Rightarrow z = (i)^{(i)^{(i)}} = i^{e^{-\frac{\pi}{2}}} \Rightarrow |z| = 1$$

(5) **(A).** We have, $z^3 + 2z^2 + 2z + 1 = 0$
$(z^3 + 1) + 2z(z + 1) = 0$; $(z + 1)(z^2 + z + 1) = 0$
$z = -1, \omega, \omega^2$.
Since, $z = -1$ does not satisfy $z^{1985} + z^{100} + 1 = 0$
while $z = \omega, \omega^2$ satisfy it, hence sum is $\omega + \omega^2 = -1$.

(6) **(D).** Let $z = (1)^{1/n} = \cos(2k\pi + i\sin 2k\pi)^{1/n}$

$$z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, k = 0, 1, 2, \ldots\ldots, n - 1$$

Let, $z_1 = \cos\left(\frac{2k_1\pi}{n}\right) + i\sin\left(\frac{2k_2\pi}{n}\right)$ and

$$z_2 = \cos\left(\frac{2k_2\pi}{n}\right) + i\sin\left(\frac{2k_2\pi}{n}\right)$$

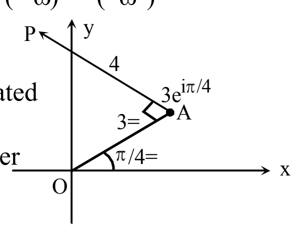
be the two values of z s.t. they subtend ∠ of 90° at

origin. $\Rightarrow \frac{2k_1\pi}{n} - \frac{2k_2\pi}{n} = \pm\frac{\pi}{2} \Rightarrow 4(k_1 - k_2) = \pm n$

As k_1 and k_2 are integers and $k_1 \neq k_2$; $n = 4m, m \Rightarrow I$

(7) **(B).** $(1 + \omega^2)^n = (1 + \omega^4)^n \Rightarrow (-\omega)^n = (-\omega^2)^n$
$\Rightarrow (\omega)^n = 1 \Rightarrow n = 3.$

(8) **(D).** Let OA = 3, so that the complex number associated with A is $3e^{i\pi/4}$.
If z is the complex number associated with P, then



$$\frac{z - 3e^{i\pi/4}}{0 - 3e^{i\pi/4}} = \frac{4}{3}e^{-i\pi/2} = -\frac{4i}{3}$$

$$\Rightarrow 3z - 9e^{i\pi/4} = 12ie^{i\pi/4} \Rightarrow z = (3 + 4i)e^{i\pi/4}.$$

(9) **3.** On taking $\omega = e^{\frac{i\pi}{3}}$. Expression is in terms of a, b, c

So lets assume $\omega = e^{\frac{i2\pi}{3}}$,

then the solution is following

$a + b + c = x$; $a + b\omega + c\omega^2 = y$; $a + b\omega^2 + c\omega = z$

$$\frac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}=\frac{x\bar{x}+y\bar{y}+z\bar{z}}{|a|^2+|b|^2+|c|^2}$$

$$=\frac{(a+b+c)(\bar{a}+\bar{b}+\bar{c})+(a+b\omega+c\omega^2)(\bar{a}+\bar{b}\omega^2+c\omega)+(a+b\omega^2+c\omega)(\bar{a}+\bar{b}\omega+\bar{c}\omega^2)}{|a|^2+|b|^2+|c|^2}$$

$$=\frac{3(|a|^2+|b|^2+|c|^2)}{|a|^2+|b|^2+|c|^2}=3$$

TRY IT YOURSELF-4

(1) $\bar{z}=\bar{a}+\dfrac{r^2}{z-a}\Rightarrow \bar{z}-\bar{a}=\dfrac{r^2}{z-a}$

$\Rightarrow (z-a)(\bar{z}-\bar{a})=r^2 \Rightarrow |z-a|^2=r^2 \Rightarrow |z-a|=r$

Hence, locus of z is circle having center a and radius r.

(2) $|3z-2|+|3z+2|=4$

$\Rightarrow \left|z-\dfrac{2}{3}\right|+\left|z+\dfrac{2}{3}\right|=\dfrac{4}{3}$ ...... (1)

If P (z) be any point A ≡ (2/3, 0), B ≡ (–2/3, 0) then (1) represents PA + PB = 4

Clearly, AB = 4/3 ⇒ PA + PB = AB ⇒ P is any point on the line segment AB.

(3) $\left|\dfrac{z-2}{z-3}\right|=2 \Rightarrow |z-2|^2=4|z-3|^2$

$\Rightarrow |x-2+iy|^2=4|x-3+iy|^2$

$\Rightarrow (x-2)^2+y^2=4[(x-3)^2+y^2)]$

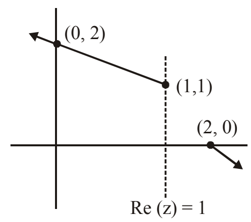
$\Rightarrow 3x^2+3y^2-24x+4x+36-4=0$

$\Rightarrow x^2+y^2-\dfrac{20}{3}x+\dfrac{32}{3}=0$

This represents a circle with centre [(10/3, 0)] and radius by $\sqrt{\dfrac{100}{9}-\dfrac{32}{3}}=\sqrt{\dfrac{4}{9}}=\dfrac{2}{3}$

(4) **(D).** The given quation is written as

$$\arg(z-(1+i))=\begin{cases}3\pi/4, & \text{when } x\le 2\\ -\pi/4, & \text{when } x>2\end{cases}$$



Therefore, the locus is a set of two rays.

(5) **(B).** $2\left|z-\dfrac{1}{2}\right|=|z-1| \quad \therefore \quad \dfrac{|z-1|}{\left|z-\dfrac{1}{2}\right|}=2$

(6) **(A).** $|z_1|=|z_2|=|z_3|=1$

Hence, the circumcentre of triangle is origin. Also, centroid $\dfrac{z_1+z_2+z_3}{3}=0$, which coincides with the circumcentre. So, the triangle is equilateral. Since radius is 1, length of side is a = $\sqrt{3}$. Therefore, the area of the triangle is $(\sqrt{3}/4)\,a^2=(3\sqrt{3}/4)$.

(7) **(D).** Given $z=\dfrac{3}{2+\cos\theta+i\sin\theta}$

$\cos\theta+i\sin\theta=\dfrac{3}{z}-2=\dfrac{3-2z}{z}$

$1=\dfrac{|3-2z|}{|z|}$ [Taking modulus]

$\Rightarrow \dfrac{\left|z-\dfrac{3}{2}\right|}{|z|}=\dfrac{1}{2}$. Hence, locus of z is a circle.

(8) **(A).** The point $(\sqrt{2}-1,-\sqrt{2})$ and $(\sqrt{2}-1,\sqrt{2})$ are equidistant from the point (–1, 0).

The shaded area belongs to the region outside the sector of circle $|z+1|=2$, lying between the line rays $\arg(z+1)=\pi/4$ and $\arg(z+1)=-\pi/4$.

CHAPTER- 5 : COMPLEX NUMBERS

EXERCISE-1

(1) **(A).** $[i]^{198} = [i^2]^{99} = [-1]^{99} = -1$

(2) **(B).** $i^n + i^{n+1} + i^{n+2} + i^{n+3}$
$= i^n [1 + i + i^2 + i^3]$
$= i^n [1 + i - 1 - i] = i^n [0] = 0$

(3) **(B).** Given $\frac{3+2i\sin\theta}{1-2i\sin\theta} \times \frac{1+2i\sin\theta}{1+2i\sin\theta}$

$= \frac{3+6i\sin\theta + 2i\sin\theta - 4\sin^2\theta}{1+4\sin^2\theta}$

$= \frac{3-4\sin^2\theta + 8i\sin\theta}{1+4\sin^2\theta}$

If it is purely real then

$\frac{8\sin\theta}{1+4\sin^2\theta} = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = n\pi$

(4) **(A).** Let $z = x + iy$ then

$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy}$

$= \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$

$= \frac{x^2-1+iy(x-1)+iy(x+1)+y^2}{(x+1)^2+y^2}$

$= \frac{(x^2-1+y^2)+i[2xy]}{(x+1)^2+y^2}$

If it is purely Imaginary

$\frac{x^2-1+y^2}{(x+1)^2+y^2} = 0 \Rightarrow x^2+y^2-1 = 0 \Rightarrow x^2+y^2 = 1$

which is the equation of a circle.

(5) **(D).** Let $z = x + iy$, then $|z-4| < |z-2|$
$\Rightarrow (x-4)^2 + y^2 < (x-2)^2 + y^2$
$\Rightarrow -4x < -12 \Rightarrow x > 3 \Rightarrow R(z) > 3$

(6) **(B).** $\sqrt{-2} \times \sqrt{-3} = \sqrt{2}i \times \sqrt{3}i = \sqrt{6}(i)^2 = -\sqrt{6}$

(7) **(C).** Here $x + iy = (a - ib)^3 = (a^3 - 3ab^2) + i(-3a^2b + b^3)$
$\Rightarrow x = a^3 - 3ab^2,\ y = b^3 - 3a^2b$

$\Rightarrow \frac{x}{a} - \frac{y}{b} = (a^2 - 3b^2) - (b^2 - 3a^2) = 4(a^2 - b^2) \Rightarrow k = 4$

(8) **(A).** Let $z = x + iy$ ......(i)

Given $|z + i| = |z - i|$

or $|x + iy + i| = |x + iy - i|$

or $|x + i(y+1)| = |x + i(y-1)|$

or $\sqrt{x^2 + (y+1)^2} = \sqrt{x^2 + (y-1)^2}$

or $x^2 + (y+1)^2 = x^2 + (y-1)^2$

or $y^2 + 2y + 1 = y^2 - 2y + 1$ or $4y = 0$ or $y = 0$

Hence from (i), we get $z = x$, where x is any real number.

(9) **(B).** $3 - 4i$ i.e., $(3,-4)$ lie in fourth quadrant in complex plane, after turned anticlockwise through 180^o this will lie in II quadrant, therefore, the number will be $-3+4i$, now after stretching it 2.5 times i.e., multiplying by 2.5, the required complex number will be $\frac{-15}{2} + 10i$.

(10) **(B).** $\frac{1-ix}{1+ix} = a - ib \Rightarrow \frac{(1-ix)(1-ix)}{(1+ix)(1-ix)} = a - ib$

$\Rightarrow \frac{1-x^2-2ix}{1+x^2} = a - ib \Rightarrow \frac{1-x^2}{1+x^2} = a$ and $\frac{2x}{1+x^2} = b$

Now we can write x as

$x = \frac{\frac{2x}{1+x^2}}{\frac{2}{1+x^2}} = \frac{\frac{2x}{1+x^2}}{\frac{1-x^2}{1+x^2}+1} = \frac{b}{1+a} = \frac{2b}{1+1+2a}$

$= \frac{2b}{1+(a^2+b^2)+2a} = \frac{2b}{(1+a)^2+b^2}$

(11) **(C).** $\frac{(1+i)^n}{(1-i)^{n-2}} = (1+i)^n (1-i)^{2-n}$ given +ve with n= 1

$(1+i)(1-i) = 2$

(12) **(A).** $\left|\frac{1+i\sqrt{3}}{\left(1+\frac{1}{i+1}\right)^2}\right| = \left|\frac{1+i\sqrt{3}}{\frac{(i+2)^2}{(i+1)^2}}\right| = \frac{\sqrt{1+3}}{1+4} \times (1+1) = \frac{2\times 2}{5}$

(13) **(B).** $\text{amp}\left(\frac{a+ib}{a-ib}\right) = \text{amp}(a+ib) - \text{amp}(a-ib)$

$= \tan^{-1}\left(\frac{b}{a}\right) - \tan^{-1}\left(-\frac{b}{a}\right)$

$= \tan^{-1}\left[\frac{2(b/a)}{1-(b^2/a^2)}\right] = \tan^{-1}\left(\frac{2ab}{a^2-b^2}\right)$

(14) **(A).** Given $|z_1| = |z_2| = ... = |z_n| = 1$ .....(1)

Now $\left|\frac{1}{z_1} + \frac{1}{z_2} + ... + \frac{1}{z_n}\right| = \left|\frac{\bar{z}_1}{z_1\bar{z}_1} + \frac{\bar{z}_2}{z_2\bar{z}_2} + ... \frac{\bar{z}_n}{z_n\bar{z}_n}\right|$

$= \left|\frac{\bar{z}_1}{|z_1|^2} + \frac{\bar{z}_2}{|z_2|^2} + ... \frac{\bar{z}_n}{|z_n|^2}\right| = |\bar{z}_1 + \bar{z}_2 + ... + \bar{z}_n|$

from (1)

$= |z_1 + z_2 + ... + z_n|$ $(\because |\bar{z}| = |z|)$

(15) **(D).** $z^{-1} = \dfrac{\bar{z}}{|z|^2} = \dfrac{(1/2)-i}{(1/2)^2+1} = \dfrac{2}{5} - \dfrac{4}{5}i = \left(\dfrac{2}{5}, -\dfrac{4}{5}\right)$

(16) **(A).** Multiply above and below by conjugate of denominator and put real part equal to zero.

$$= \frac{\tan\theta - i\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)}{1+2i\sin\frac{\theta}{2}} \times \frac{1-2i\sin\frac{\theta}{2}}{1-2i\sin\frac{\theta}{2}}$$

$\therefore\ \tan\theta - 2\sin\dfrac{\theta}{2}\left(\sin\dfrac{\theta}{2}+\cos\dfrac{\theta}{2}\right) = 0$

$\Rightarrow \dfrac{\sin\theta}{\cos\theta} - (1-\cos\theta) - \sin\theta = 0$

$\Rightarrow \sin\theta\left(\dfrac{1-\cos\theta}{\cos\theta}\right) - (1-\cos\theta) = 0$

$\Rightarrow (1-\cos\theta)(\tan\theta - 1) = 0$

$\cos\theta = 1 \Rightarrow \theta = 2n\pi$ and $\tan\theta = 1 \Rightarrow \theta = n\pi + \dfrac{\pi}{4}$

(17) **(B).** Let $z_1 = a+ib$ and $z_2 = c+id$ $(b \neq 0, d \neq 0)$.
Then $z_1 + z_2$ and $z_1 z_2$ are real
$\Rightarrow b + d = 0$ and $ad + bc = 0$
$\Rightarrow d = -b$ and $c = a$ ($\because b \neq 0, d \neq 0$) $\Rightarrow z_1 = \bar{z}_2$

(18) **(A).** $|z_1 + z_2| = |z_1 - z_2|$

$\Rightarrow \left|\dfrac{z_1}{z_2}+1\right| = \left|\dfrac{z_1}{z_2}-1\right| \Rightarrow \dfrac{z_1}{z_2}$ lies on $\perp$ bisector of 1 and – 1

$\Rightarrow \dfrac{z_1}{z_2}$ lies on imaginary axis $\Rightarrow \dfrac{z_1}{z_2}$ is purely imaginary

$\Rightarrow \arg\left(\dfrac{z_1}{z_2}\right) = \pm\dfrac{\pi}{2}$; $|\arg(z_1) - \arg(z_2)| = \dfrac{\pi}{2}$

(19) **(A).** Expression

$= (az_1 - bz_2)\overline{(az_1 - bz_2)} + (bz_1 + az_2)\overline{(bz_1 + az_2)}$

$= (az_1 - bz_2)(a\bar{z}_1 - b\bar{z}_2) + (bz_1 + az_2)(b\bar{z}_1 + a\bar{z}_2)$

$= a^2|z_1|^2 + b^2|z_2|^2 + b^2|z_1|^2 + a^2|z_2|^2$

$= (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

(20) **(C).** Let $z = 1 - \cos\theta - i\sin\theta = r(\cos\phi + i\sin\phi)$

$\therefore\ \tan\phi = -\dfrac{\sin\theta}{1-\cos\theta}$

$= \dfrac{-2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = -\cot(\theta/2)$

$= -\tan\left(\dfrac{\pi}{2}-\dfrac{\theta}{2}\right)$ or $\tan\phi = \tan\left(\dfrac{\theta}{2}-\dfrac{\pi}{2}\right)$

$\therefore\ \text{amp}(z) = \dfrac{\theta}{2} - \dfrac{\pi}{2}$

(21) **(B).** $|z| = \dfrac{|\cos(\pi/3) - i\sin(\pi/3)||\sqrt{3}+i|}{|i-1|} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

Again amp(z) = amp $\{\cos(\pi/3) - i\sin(\pi/3)\}$
$+ \text{amp}(\sqrt{3}+i) - \text{amp}(-1+i)$

$= -\dfrac{\pi}{3} + \dfrac{\pi}{6} - \left(\pi - \dfrac{\pi}{4}\right) = -\dfrac{11\pi}{12}$

$z = \sqrt{2}\left\{\cos\left(-\dfrac{11\pi}{12}\right) + i\sin\left(-\dfrac{11\pi}{12}\right)\right\}$

$= \sqrt{2}\left\{\cos\left(\dfrac{13\pi}{12}\right) + i\sin\left(\dfrac{13\pi}{12}\right)\right\}$

(22) **(B).** $\because\ |z_1 + z_2|^2 = |z_1|^2\ |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

$\therefore$ If $\theta_1 - \theta_2 = \pm\dfrac{\pi}{2}$; Then $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$

i.e. $\text{Arg}(z_1) - \text{Arg}(z_2) = \pm\dfrac{\pi}{2}$

$\Rightarrow \text{Arg}\left(\dfrac{z_1}{z_2}\right) = \pm\dfrac{\pi}{2} \Rightarrow \dfrac{z_1}{z_2}$ is purely imaginary

(23) **(C).** Let $\sqrt{-8-6i} = \pm(a+ib)$
$\Rightarrow -8-6i = a^2 - b^2 + 2iab$
$\Rightarrow a^2 - b^2 = -8$ ... (1)
$2ab = -6 \Rightarrow ab = -3$ ... (2)
$(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2$
$= (-8)^2 + (-6)^2 = 64 + 36 = 100$
$\Rightarrow a^2 + b^2 = 10$ ... (3)
From equation, (2) and (3) $a = 1, b = -3$
So, $\sqrt{-8-6i} = \pm(1-3i)$

(24) **(D).** $\sin x + i\cos 2x = \cos x + i\sin 2x$
$\Rightarrow \tan x = 1$ and $\tan 2x = 1$

$\Rightarrow x = n\pi + \dfrac{\pi}{4}$ and $x = \dfrac{n\pi}{2} + \dfrac{\pi}{8}$

$\Rightarrow x \in \left\{\ldots, \dfrac{-7\pi}{4}, \dfrac{-3\pi}{4}, \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}, \ldots\right\}$

$\cap \left\{\ldots, \dfrac{-7\pi}{8}, \dfrac{-3\pi}{8}, \dfrac{\pi}{8}, \dfrac{5\pi}{8}, \dfrac{9\pi}{8}, \ldots\right\}$

$\Rightarrow$ there is no common value of x.

(25) **(A).** $|z - \sqrt{3} + i| = |(z+2i) - (\sqrt{3}+i)|$
$\leq |(z+2i)| + |(\sqrt{3}+i)| \leq 1 + 2 = 3$

$\Rightarrow$ The greatest value of $|z - \sqrt{3} + i|$ is 3.

Again $|z - \sqrt{3} + i| = |(z+2i) - (\sqrt{3}+i)|$
$\geq |\sqrt{3}+i| - |z+2i| \geq 2 - 1 = 1$

Thus least value of $|z - \sqrt{3} + i|$ is 1.

(26) **(D).** $\frac{z-1}{z+1}=\frac{(x+iy)-1}{(x+iy)+1}=\frac{(x-1)+iy}{(x+1)+iy}$

$=\frac{\{(x-1)+iy\}\{(x+1)-iy\}}{\{(x+1)+iy\}\{(x+1)-iy\}}$

$=\frac{\{(x^2-1)+y^2\}+i\{y(x+1)-y(x-1)\}}{(x+1)^2+y^2}$

$=\left\{\frac{(x^2-1)+y^2}{(x+1)^2+y^2}\right\}+i\left\{\frac{2y}{(x+1)^2+y^2}\right\}$

$\therefore\ \text{amp}\left(\frac{z-1}{z+1}\right)=\tan^{-1}\left\{\frac{2y}{(x+1)^2+y^2}\div\frac{(x^2-1)+y^2}{(x+1)^2+y^2}\right\}$

$\Rightarrow \frac{\pi}{4}=\tan^{-1}\left\{\frac{2y}{x^2+y^2-1}\right\}\Rightarrow \tan\frac{\pi}{4}=\frac{2y}{x^2+y^2-1}$

$\Rightarrow 1=\frac{2y}{x^2+y^2-1}\Rightarrow x^2+y^2-1=2y$

$\Rightarrow x^2+y^2-2y=1$

(27) **(A).** According to condition, $3-ix^2y=x^2+y+4i$

$\Rightarrow x^2+y=3$ and $x^2y=-4 \Rightarrow x=\pm2, y=-1$

$\Rightarrow (x,y)=(2,-1)$ or $(-2,-1)$

(28) **(C).** Given that $\overline{(x+iy)(1-2i)}=1+i$

$\Rightarrow x-iy=\frac{1+i}{1+2i}\Rightarrow x+iy=\frac{1-i}{1-2i}$.

(29) **(B).** $\left|z+\frac{2}{z}\right|=2\Rightarrow |z|-\frac{2}{|z|}\le 2\Rightarrow |z|^2-2|z|-2\le 0$

$|z|\le\frac{2\pm\sqrt{4+8}}{2}\le 1\pm\sqrt{3}$.

Hence max. value of $|z|$ is $1+\sqrt{3}$

(30) **(A).** Let $z_1=a+ib=(a,b)$ and $z_2=c-id=(c,-d)$

Where $a>0$ and $d>0$ ......(i)

Then $|z_1|=|z_2|\Rightarrow a^2+b^2=c^2+d^2$

Now $\frac{z_1+z_2}{z_1-z_2}=\frac{(a+ib)+(c-id)}{(a+ib)-(c-id)}$

$=\frac{[(a+c)+i(b-d)][(a-c)-i(b+d)]}{[(a-c)+i(b+d)][(a-c)-i(b+d)]}$

$=\frac{(a^2+b^2)-(c^2+d^2)-2(ad+bc)i}{a^2+c^2-2ac+b^2+d^2+2bd}$

$\frac{-(ad+bc)i}{a^2+b^2-ac+bd}$ [using (i)]

$\therefore \frac{(z_1+z_2)}{(z_1-z_2)}$ is purely imaginary.

However if $ad+bc=0$, then $\frac{(z_1+z_2)}{(z_1-z_2)}$ will be equal to zero. According to the conditions of the equation, we can have ad + bc = 0

(31) **(A).** $|z|=1\Rightarrow|x+iy|=1\Rightarrow x^2+y^2=1$

$\omega=\frac{z-1}{z+1}=\frac{(x-1)+iy}{(x+1)+iy}\times\frac{(x+1)-iy}{(x+1)-iy}$

$=\frac{(x^2+y^2-1)}{(x+1)^2+y^2}+\frac{2iy}{(x+1)^2+y^2}=\frac{2iy}{(x+1)^2+y^2}$

$\therefore\ \text{Re}(\omega)=0$.

(32) **(C).** $arg\left(\frac{1-i\sqrt{3}}{1+i\sqrt{3}}\right)=arg(1-i\sqrt{3})-arg(1+i\sqrt{3})$

$=-60^o-60^o=-120^o$ or 240^o.

(33) **(A).** We know that the principal value of θ lies between $-\pi$ and π.

(34) **(B).** $arg\left(\frac{13-5i}{4-9i}\right)=arg(13-5i)-arg(4-9i)$

$=-\tan^{-1}\left(\frac{5}{13}\right)+\tan^{-1}\frac{9}{4}=\frac{\pi}{4}$

(35) **(B).** $\frac{1+2i}{1-(1-i)^2}=\frac{1+2i}{1-(1-1-2i)}=\frac{1+2i}{1+2i}=1+0i$

Modulus = 1

Amplitude $\theta=\tan^{-1}\frac{0}{1}=0$.

(36) **(D).** Given $z_1=1+2i$, $z_2=3+5i$ and $\bar{z}_2=3-5i$

$=\frac{13+i}{3+5i}\times\frac{3-5i}{3-5i}=\frac{44-62i}{34}$

Then $\text{Re}\left(\frac{\bar{z}_2 z_1}{z_2}\right)=\frac{44}{34}=\frac{22}{17}$.

(37) **(A).** $x+iy=\sqrt{\frac{a+ib}{c+id}}\Rightarrow x-iy=\sqrt{\frac{a-ib}{c-id}}$

Also $x^2+y^2=(x+iy)(x-iy)=\sqrt{\frac{a^2+b^2}{c^2+d^2}}$

$\Rightarrow (x^2+y^2)^2=\frac{a^2+b^2}{c^2+d^2}$

(38) **(D).** $\sqrt{a+ib}=x+yi\Rightarrow\left(\sqrt{a+ib}\right)^2=(x+yi)^2$

$\Rightarrow a=x^2-y^2, b=2xy$ and hence

$\sqrt{a-ib}=\sqrt{x^2-y^2-2xyi}=\sqrt{(x-yi)^2}=x-iy$

Note: In the question, it should have been given that $a,b,x,y\in R$.

(39) **(C).** $\because$ $az^2+bz+c=0$ .... (1)

and z_1, z_2 (roots of (1)) are such that $\text{Im}(z_1z_2)\neq 0$

z_1 and z_2 are not conjugates of each other
complex roots of (1) are not conjugate of each other
coefficient a, b, c cannot all be real.
at least one of a, b, c, be is imaginary.

(40) **(D).** $3 + i x^2y$ and $x^2 + y + 4i$ are conjugate
then $x^2y = -4$ and $x^2 + y = 3$
$\Rightarrow x^2 = 4, y = -1 \Rightarrow x^2 + y^2 = 5$

(41) **(D).** $\arg(z - i + 2) = \frac{\pi}{6} \Rightarrow \tan\frac{\pi}{6} = \frac{y-1}{x+2}$

$\Rightarrow x - \sqrt{3}y = -(\sqrt{3}+2), x > -2, y > 1$ .... (1)

$(z + 4 - 3i) = -\frac{\pi}{4} \Rightarrow \tan\left(-\frac{\pi}{4}\right) = \frac{y-3}{x+4}$

$\Rightarrow y + x = -1, x > -4, y < 3$ .... (2)

so, there is no point of intersection.

(42) **(A).** $|z| + |z-1| + |2z-3| = |z| + |z-1| + |3-2z|$
$\geq |z + z - 1 + 3 - 2z| = 2$
$\therefore |z| + |z-1| + |2z-3| \geq 2 \therefore \lambda = 2$
then $2[x] + 3 = 3[x - \lambda] = 3[x-2]$
$2[x] + 3 = 3([x] - 2)$
or $[x] = 9$ then $y = 2.9 + 3 = 21$
$\therefore [x+y] = [x+21] = [x] + 21 = 9 + 21 = 30$

(43) **(C).** $\because iz^2 = \bar{z}$
Taking modulus of both sides

$|iz^2| = |\bar{z}| \Rightarrow |i||z^2| = |z|$

$\Rightarrow |z^2| = |z| \Rightarrow |z| = 0$ or 1

(44) **(B).** $|z+4| \leq 3 \Rightarrow -3 \leq z+4 \leq +3$

$\Rightarrow -6 \leq z+1 \leq 0 \Rightarrow 0 \leq -(z+1) \leq 6$

$\Rightarrow 0 \leq |z+1| \leq 6$

Hence greatest and least values of $|z+1|$ are 6 and 0 respectively.

(45) **(C).** Conjugate of $(x+iy)(1-2i) = 1+i$

$\therefore (x+iy)(1-2i) = 1-i \quad \therefore \quad x+iy = \frac{1-i}{1-2i}$

(46) **(C).** $\frac{1+2i}{1-(1-i)^2} = \frac{1+2i}{1-1-i^2+2i} = \frac{1+2i}{1+2i} = 1 + i\cdot 0$

$\therefore$ Modulus $= 1$, Amplitude $= \tan^{-1}|0/1| = 0$

(47) **(C).** $|\sqrt{3}+i| = \sqrt{3+1} = 2$; $|3i+4| = \sqrt{9+16} = 5$

$|8+6i| = \sqrt{64+36} = 10 \qquad \therefore |Z| = \frac{2^3 \times 5^2}{10^2} = 2$

(48) **(C).** $\because A \equiv (1, 2); B \equiv (-3, 1); C \equiv (-2, -3); D \equiv (2, -2)$
$\therefore AB^2 = 16 + 1 = 17, BC^2 = 1 + 16 = 17$
$CD^2 = 16 + 1 = 17, AC^2 = 9 + 25 = 34$
$BD^2 = 25 + 9 = 34.$
Now since AB = BC = CD and AC = BD
$\therefore$ ABCD is square.

(49) **(A).** Let $z = x + iy$ then

$\left|\frac{z-3i}{z+3i}\right| = 1 \Rightarrow |z-3i| = |z+3i|$

$\Rightarrow |x+iy-3i| = |x+iy+3i|$

$\Rightarrow \sqrt{x^2+(y-3)^2} = \sqrt{x^2+(y+3)^2} \quad \Rightarrow 12y = 0$

$\Rightarrow y = 0$, which is equation of x-axis

(50) **(B).** $|z - i\,\text{Re}(z)| = |z - \text{Im}(z)|$
Let $z = x + iy$, then
$|x + iy - ix| = |x + iy - y|$
i.e. $x^2 + (y-x)^2 = (x-y^2) + y^2$
i.e. $x^2 = y^2$ i.e. $y = \pm x$

(51) **(C).** $\left|\frac{z_1 - z_3}{z_2 - z_3}\right| = \left|\frac{1}{2} - i\frac{\sqrt{3}}{2}\right| = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1.$

so, $|z_1 - z_3| = |z_2 - z_3|$

$\text{amp}\left(\frac{z_1-z_3}{z_2-z_3}\right) = \tan^{-1}\left(\frac{-\sqrt{3}/2}{1/2}\right) = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

or $\text{amp}\left(\frac{z_2-z_3}{z_1-z_3}\right) = \frac{\pi}{3}$ or $\angle z_2z_3z_1 = 60^o$

$\therefore$ The triangle has two sides equal and the angle between the equal sides = 60°. So, it is equilateral.

(52) **(C).** $\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{3} \Rightarrow \tan^{-1}\left[\frac{(x-2)+iy}{(x+2)+iy}\right] = \frac{\pi}{3}$

$\Rightarrow \sqrt{(x-2)^2+y^2} = \tan(\pi/3)[\sqrt{(x+2)^2+y^2}]$

Squaring both sides,

$\Rightarrow (x-2)^2 + y^2 = 3[x+2]^2 + y^2]$

$\Rightarrow x^2 + y^2 + 4 - 4x = 3x^2 + 3y^2 + 12x + 12$

$\Rightarrow 2x^2 + 2y^2 + 16x + 8 = 0 \Rightarrow x^2 + y^2 + 8x + 4 = 0$

which is a equation of circle.

(53) **(A).** $z_1, z_2, 0$ are vertices of an equilateral triangle, so we have
$z_1{}^2 + z_2{}^2 + 0^2 = z_1z_2 + z_2.0 + 0.z_1$ (a property)
$\Rightarrow z_1{}^2 + z_2{}^2 = z_1z_2 \Rightarrow z_1{}^2 + z_2{}^2 - z_1z_2 = 0$

(54) **(C).** $|w| = 1 \Rightarrow |z - (1/5)i| = |z|$
$\Rightarrow |z - (1/5)i|^2 = |z|^2 \Rightarrow |x + iy - 1/5i|^2 = |x + iy|^2$
$\Rightarrow x^2 + (y - 1/5)^2 = x^2 + y^2 \Rightarrow -2/5y + 1/25 = 0$
$\Rightarrow 10y = 1$, which is a line.

(55) **(D).** $\log_{\sqrt{3}}\frac{|z|^2 - |z| + 1}{2 + |z|} < 2$

$\Rightarrow \frac{|z|^2 - |z| + 1}{2 + |z|} < (\sqrt{3})^2$

$\Rightarrow |z|^2 - |z| + 1 < 6 + 3|z| \Rightarrow |z|^2 - 4|z| - 5 < 0$
$\Rightarrow (|z| - 5)(|z| + 1) \Rightarrow (|z| - 5) < 0$
since $|z| + 1 > 0 \Rightarrow |z| < 5$
Hence z lies inside the circle $|z| = 5$

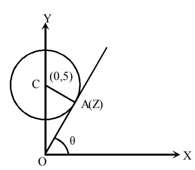
(56) **(C).** Since z_1, z_2, z_3, are vertices of an equilateral triangle, so $z_1^2+z_2^2+z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$ ...(1)
Further the circumcenter of an equilateral triangle is same as its centroid, so
$z_0 = (z_1+z_2+z_3)/3$
$\Rightarrow 9z_0^2 = z_1^2+z_2^2+z_3^2+2(z_1z_2+z_2z_3+z_3z_1)$
$= z_1^2+z_2^2+z_3^2+2(z_1^2+z_2^2+z_3^2)$
$\therefore z_1^2+z_2^2+z_3^2 = 3z_0^2$.

(57) **(B).** Let the given points be A, B, C, D respectively.
Then ABCD is a parallelogram, so $\overrightarrow{AB} = \overrightarrow{DC}$
$\Rightarrow z_2 - z_1 = z_3 - z_4 \Rightarrow z_1 + z_3 = z_2 + z_4$

(58) **(A).** Given points are A(3, 4), B(5, –2) and C (– 1, 16).
Now slope of AB $= \dfrac{-2-4}{5-3} = -3$
slope of BC $= \dfrac{16+2}{-1-5} = -3 \therefore$ slope of AB = slope of BC
$\Rightarrow$ A, B, C are collinear.

(59) **(D).** The required complex number is point of contact C (0, 25) is the centre of the circle and radius is 15.
Now $|z| = OP = \sqrt{OC^2 - PC^2} = \sqrt{625-225} = 20$
amp (z) = θ = ∠XOP = ∠OCP
$\therefore \cos\theta = \dfrac{PC}{OC} = \dfrac{15}{25} = \dfrac{3}{5}$ and $\sin\theta = \dfrac{OP}{OC} = \dfrac{20}{25} = \dfrac{4}{5}$
$\therefore z = 20\left(\dfrac{3}{5}+\dfrac{4}{5}i\right) = 12+16i$

(60) **(A).** We have OC = 5, CA = 1



$\theta = \angle AOX = \min.\text{amp } z, \therefore \angle AOC = 90^o - \theta$
$\Rightarrow \sin(90^o - \theta) = \dfrac{1}{5} \Rightarrow \cos\theta = \dfrac{1}{5}$
$\therefore z = OA\cos\theta + iOA\sin\theta$
$\Rightarrow z = \sqrt{5^2-1}\left(\dfrac{1}{5}\right) + i\sqrt{5^2-1}\sqrt{1-\dfrac{1}{5^2}}$
$= \dfrac{2\sqrt{6}}{5}(1+i\,2\sqrt{6})$.

(61) **(B).** Let z_1, z_2, z_3 be three complex numbers in A.P.
Then $2z_2 = z_1 + z_3$.
Thus the complex number z_2 is the mid-point of the line joining the points z_1 and z_3 So the three points z_1, z_2 and z_3 are in a straight line.

(62) **(A).** $BD = 2AC \Rightarrow 2DM = 2(2AM)$
or $DM = 2AM$ or $DM^2 = 4AM^2$
or $5 = 4[(x-2)^2 + (y+1)^2]$ .....(i)
Again slope of $DM = -2$ and slope of AM is $\dfrac{y+1}{x-2}$
AM is perpendicular to DM
$\therefore -2\left(\dfrac{y+1}{x-2}\right) = -1 \Rightarrow x - 2 = 2(y+1)$ .....(ii)
Hence from (i) and (ii), we get
$\therefore y = -\dfrac{1}{2}, -\dfrac{3}{2}$ and $x = 3, 1$

(63) **(B).** The two circles are $C_1(0,0), r_1 = 12$, $C_2(3,4), r_2 = 5$ and it passes through origin, the centre of C_1.
$C_1C_2 = 5 < r_1 - r_2 = 7$. Hence circle C_2 lies inside circle C_1. Therefore minimum distance between them is
$AB = C_1B - C_1A = r_1 - 2r_2 = 12 - 10 = 2.$

(64) **(A).** Let P (Z), A (0, 0), B (1, 0)
$\therefore |Z| + |Z-1| = PA + PB$ will be mininimum when p lies on line segment AB
$\therefore \min(|Z|+|Z-1|) = AB = 1$

(65) **(A).** Let $Z = x + iy$
$\therefore Z + |Z| = 0$
$\Rightarrow x + iy + \sqrt{x^2+y^2} = 0$
Equating real and imaginary parts, we get
Imaginary part : y = 0
Real parts $= x + \sqrt{x^2+y^2} = 0 \Rightarrow x + \sqrt{x^2} = x + |x| = 0$
$\therefore |x| = -x$
Hence, Z lies on x-axis : $x \le 0$

(66) **(A).** $|z-1|^2 = |z+2i|^2$
$(x+iy-1)(x-iy-1) = (x+iy+2i)$

(67) **(C).** $1+i\sqrt{3} = 2\left(\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}\right) = 2e^{i\frac{\pi}{3}}$,
$1 - i = \sqrt{2}\left(\dfrac{1}{\sqrt{2}} - i\dfrac{1}{\sqrt{2}}\right) = \sqrt{2}e^{-i\frac{\pi}{4}}$
$\therefore \left(1+i\sqrt{3}\right)^6 = 2^6 e^{2i\pi} = 2^6, (1-i)^8 = 2^4 e^{-2i\pi} = 2^4$
Given expression $= 2^6 + 2^4 = 80$.

(68) **(C).** $\omega^{35} + \omega^{25} = \omega^2 + \omega = -1$
and $\omega^{10} + \omega^{23} = \omega + \omega^2 = -1$
$\therefore$ the given expression is
$\sin\left(-\dfrac{\pi}{2}\right) + \cos\left(-\dfrac{5\pi}{4}\right) = -1 - \dfrac{1}{\sqrt{2}} = -\left(\dfrac{2+\sqrt{2}}{2}\right)$

(69) **(C).** The given equation is $\frac{z^5-1}{z-1}=0$ which means that

z_1, z_2, z_3, z_4 are four out of five roots of unit except 1.

$$z_1^4+z_2^4+z_3^4+z_4^4+1^4=0 \Rightarrow \left|\sum_{i=1}^{4} z_i^4\right|=1$$

(70) **(C).** $x_1.x_2.x_3....x_\infty$

$$=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3^2}+i\sin\frac{\pi}{3^2}\right)\left(\cos\frac{\pi}{3^3}+i\sin\frac{\pi}{3^3}\right)....$$

$$=\cos\left(\frac{\pi}{3}+\frac{\pi}{3^2}+\frac{\pi}{3^3}+.....\infty\right)+i\sin\left(\frac{\pi}{3}+\frac{\pi}{3^2}+\frac{\pi}{3^3}+....\infty\right)$$

$$=\cos\left(\frac{\pi/3}{1-\frac{1}{3}}\right)+i\sin\left(\frac{\pi/3}{1-\frac{1}{3}}\right)=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=i.$$

(71) **(A).** $x_1x_2x_3.......\infty$

$$=\cos\left(\frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\pi}{2^3}+.........\right)+i\sin\left(\frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\pi}{2^3}+.........\right)$$

$$=\cos\left(\frac{\pi/2}{1-1/2}\right)+i\sin\left(\frac{\pi/2}{1-1/2}\right)$$

$$=\cos\pi+i\sin\pi=-1+i.0=-1.$$

(72) **(D).** $z^3+\frac{3(\bar{z})^2}{z}=0$

Let $z=re^{i\theta}\Rightarrow r^3e^{i3\theta}+3re^{-i2\theta}=0$

Since r cannot be zero $\Rightarrow re^{i5\theta}=-3$

which will hold for r = 3 and 5 disinct values of θ

There are five solutions.

(73) **(D).** $\left(\sin\frac{2\pi k}{7}-i\cos\frac{2\pi k}{7}\right)$

$$=-i\left(\cos\frac{2\pi k}{7}+i\sin\frac{2\pi k}{7}\right)=-i\,e^{\frac{2\pi ki}{7}}$$

$$\therefore \sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}-i\cos\frac{2\pi k}{7}\right)=-i\,e^{\frac{2\pi i}{7}}\left\{\frac{1-e^{\frac{12\pi i}{7}}}{1-e^{\frac{2\pi i}{7}}}\right\}$$

$$=-i\left\{\frac{e^{\frac{2\pi i}{7}}-1}{1-e^{\frac{2\pi i}{7}}}\right\}=i \qquad (\because e^{2\pi i}=1)$$

(74) **(D).** Given, complex function $z=i\log(2-\sqrt{3})$.

The given equation may be written as

$$e^{iz}=e^{i^2\log(2-\sqrt{3})}=e^{-\log(2-\sqrt{3})}=e^{\log(2-\sqrt{3})^{-1}}$$

or $e^{iz}=(2+\sqrt{3})$. Similarly, $e^{-iz}=(2-\sqrt{3})$.

We know that

$$\cos z=\frac{e^{iz}+e^{-iz}}{2}=\frac{(2+\sqrt{3})+(2-\sqrt{3})}{2}=2.$$

(75) **(D).** Let $z=-1+i\sqrt{3}$, $r=\sqrt{1+3}=2$

$$\theta=\tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)=\frac{2\pi}{3} \qquad \therefore z=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$$

$$\therefore (z)^{20}=\left[2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)\right]^{20}$$

$$=2^{20}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)^{20}=2^{20}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{20}.$$

(76) **(B).** Vertices are $0=0+i0$, $z=x+iy$

and $ze^{i\alpha}=(x+iy)(\cos\alpha+i\sin\alpha)$

$=(x\cos\alpha-y\sin\alpha)+i(y\cos\alpha+x\sin\alpha)$

$$\therefore \text{Area}=\frac{1}{2}\begin{vmatrix}0 & 0 & 1\\ x & y & 1\\ (x\cos\alpha-y\sin\alpha) & (y\cos\alpha+x\sin\alpha) & 1\end{vmatrix}$$

$$=\frac{1}{2}[xy\cos\alpha+x^2\sin\alpha-xy\cos\alpha+y^2\sin\alpha]$$

$$=\frac{1}{2}\sin\alpha(x^2+y^2)=\frac{1}{2}|z|^2\sin\alpha \quad [\because |z|=\sqrt{x^2+y^2}].$$

(77) **(D).** $z^3=\bar{z}\,i\,|z| \Rightarrow |z|=1$ or $|z|=0$

Thus, z = 0 is a solution.

If $|z|=1$. Let $z=e^{i\theta}$ then $e^{i3\theta}=e^{-i\theta}i$

$\Rightarrow e^{i4\theta}=i$

$$\Rightarrow 4\theta=\frac{\pi}{2},\frac{5\pi}{2},\frac{9\pi}{2},\frac{13\pi}{2}$$

$$\therefore \theta=\frac{\pi}{8},\frac{5\pi}{8},\frac{9\pi}{8},\frac{13\pi}{8} \text{ are solutions.}$$

$\therefore$ In all there are 5 solutions.

(78) **(A).** $\because \left(\frac{1-i}{2}\right)^x=1$; $\left(\frac{\sqrt{2}\text{cis}(-\pi/4)}{2}\right)^x=1$

$$\text{cis}\left(-\frac{\pi x}{4}\right)=(\sqrt{2})^x$$

Clearly equation is satisfied by x = 0 only.

(79) **(A).** The first equation can be written as

$(z+1)(z^2+z+1)=0$. Its roots are -1, ω and ω^2

Now, let $f(z)=z^{1985}+z^{100}+1$

We have $f(-1)=(-1)^{1985}+(-1)^{100}+1\neq 0$

Therefore -1 is not a root of the equation $f(z)=0$

Again $f(\omega)=\omega^{1985}+\omega^{100}+1$

$$=(\omega^3)^{661}\omega^2+(\omega^3)^{33}\omega+1=\omega^2+\omega+1=0$$

Therefore ω is a root of the equation $f(z)=0$.
Similarly, we can show that $f(\omega^2)=0$
Hence ω and ω^2 are the common roots.

(80) **(D).** Here $1^{1/3}=1,\omega,\omega^2$

$\therefore$ For the equation $(x-2)^3+27=0$

$\Rightarrow (x-2)^3=-27=-3^3$

$\Rightarrow x-2=-3(1)^{1/3}=-3(1,\omega,\omega^2)=-3,-3\omega,3\omega^2$

$\Rightarrow x=-1,2-3\omega,2-3\omega^2$.

(81) **(B).** $-1+i\sqrt{3}=2\left(\text{cis}\frac{2\pi}{3}\right)$. Therefore,

$$(-1+i\sqrt{3})^{2010}=2^{2010}\left(\text{cis}\frac{2\pi}{3}\right)^{2010}=2^{2010}$$

pure real itself is real part.

[Observe that 2010 is multiple of 3 and $\left(\text{cis}\frac{2\pi}{3}\right)^{2010}=1$

(82) **(D).** Put n = 1
$GE=(1-\omega+\omega^2)(1-\omega^2+\omega)=(-2\omega)(-2\omega^2)=4\omega^3=4$

(83) **(D).** 2 is a root of $\alpha^2-\alpha+1=0$

$$\alpha=\frac{-1\pm i\sqrt{3}}{2}=\omega \text{ or } \omega^2 \quad \therefore \alpha^{2011}=\omega^{2011}=\omega=\alpha$$

(84) **(D).** $2x=-1+\sqrt{3}i$; $x=\frac{-1+\sqrt{3}i}{2}=\omega$

$LHS=(1-\omega^2+\omega)^6-(1-\omega+\omega^2)^6$
$=(-2\omega^2)^6-(-2\omega)^6=64-64=0$

(85) **(D).** G. E. $=(-2\omega)(-2\omega^2)=4\omega^3=4$

(86) **(C).** $\frac{1}{1-\cos\theta+i\sin\theta}=\frac{1}{2\sin^2\frac{\theta}{2}+2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$$=\frac{1}{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}+i\cos\frac{\theta}{2}\right)}=\frac{\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$

Real part = 1/2

(87) **(A).** $i^i=\left(e^{i\frac{\pi}{2}}\right)^i=e^{-\frac{\pi}{2}}=$ a purely real quantity.

(88) **(C)** $\left|Z+\frac{1}{Z}\right|\geq|Z|-\left|-\frac{1}{Z}\right|\geq 3-\frac{1}{3}=\frac{8}{3}$

(89) **(A).** $\left|\frac{z-5i}{z+5i}\right|=1$

$\Rightarrow |z-5i|^2=|z+5i|^2\Rightarrow x^2+(y-5)^2=x^2+(y+5)^2$
$\Rightarrow y=0$

(90) **(D).** Let $w=-i+\frac{15}{z}$, then $i+w=\frac{15}{z}$

$\therefore |i+w|=\frac{15}{|z|}=3$

is a circle with centre at (0, – 1) and radius = 3

(91) **(C).** Suppose x is a real root.
Then $x^3+ix-1=0\Rightarrow x^3-1=0$ and $x=0$.
There is no real number satisfying these two equations.

(92) **(D)** Z describes a circle of radius 2 with its centra at 4 + 3i. |Z| is its distance from Z = 0. If follows that the ends of the diameter through Z = 0 will be the positions of Z having maximum and minimum values of |Z|. The centre being at a distance of 5 units from Z = 0, the maximum and minimum values of |Z| are 7 and 3.

(93) **(C).** $w=\frac{1-iz}{1+iz}=\frac{-i(z+i)}{z-i}$

$\therefore |w|=|-i|\left|\frac{z+i}{z-i}\right|=\left|\frac{z+i}{z-i}\right|=2$ $\qquad\therefore$ z lies on a circle

(94) **(C).** $\left|\frac{\alpha\bar{\beta}+\bar{\alpha}\beta}{|\alpha\beta|}\right|=\leq\frac{|\alpha\bar{\beta}|+|\bar{\alpha}\beta|}{|\alpha\beta|}=2$

$\therefore$ Maximum value = 2

(95) **(D).** $\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right|\leq\left|\frac{z_1}{|z_1|}\right|+\left|\frac{z_2}{|z_2|}\right|\leq\frac{|z_1|}{|z_1|}+\frac{|z_2|}{|z_2|}\leq 2$

$\therefore (|z_1|+|z_2|)\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right|\leq 2(|z_1|+|z_2|)$.

(96) **(D).** The given equation is $[Z-(3-i)]\,\overline{[Z-(3-i)]}=16$ and represents a circle with radius 4 and centre at 3 – i. All the points Z on the circle are solutions.

(97) **(B).** The equation can be rewritten

$Z\bar{Z}-Z(1-i)-\bar{Z}(1+i)+(1+i)=0$

i.e., $[Z-(1+i)][\bar{Z}-(1-i)]=0$ giving

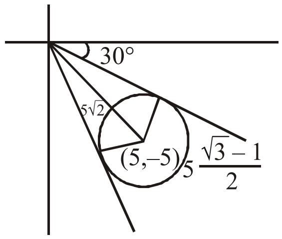
$Z=1+i$ and $\bar{Z}=1-i$.

(98) **(B).** The given in equality is equivalent to
$(2Z-3i)(2\bar{Z}+3i)<(3Z-2i)(3\bar{Z}+2i)$
which reduces to $|Z|^2>1$.

EXERCISE-2

(1) **(A).** $|z-5+5i|\leq 5\frac{(\sqrt{3}-1)}{2}$ is a circle centre at (5 – 5i)

and radius $=\frac{5(\sqrt{3}-1)}{2}$



Distance of centre from the origin = $5\sqrt{2}$

$\therefore$ least principal argument of z is equal to

$$-\left(\frac{\pi}{4}+\sin^{-1}\frac{\sqrt{3}-1}{2\sqrt{2}}\right) = -\left(\frac{\pi}{4}+\frac{\pi}{12}\right) = -\frac{\pi}{3}$$

(2) **(B).** If $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

$\Rightarrow iz = ir(\cos\theta + i\sin\theta) = -r\sin\theta + ir\cos\theta$

or $e^{iz} = e^{(-r\sin\theta + ir\cos\theta)} = e^{-r\sin\theta}\, e^{i r\cos\theta}$

or $|e^{iz}| = |e^{-r\sin\theta}|\,|e^{r i\cos\theta}|$

$= e^{-r\sin\theta}[\cos^2(r\cos\theta) + \sin^2(r\cos\theta)]^{1/2} = e^{-r\sin\theta}$

(3) **(B).** $\left|\sqrt{2}Z_1 + i\sqrt{3}Z_2\right|^2 + \left|\sqrt{3}Z_1 + i\sqrt{2}Z_2\right|^2$

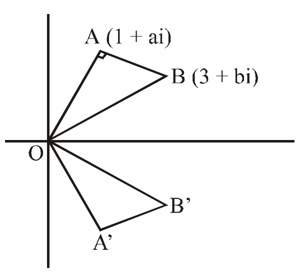
$$= \left(\sqrt{2}Z_1 + i\sqrt{3}Z_2\right)\left(\sqrt{2}\bar{Z}_1 - i\sqrt{3}\bar{Z}_2\right) + \left(\sqrt{3}\bar{Z}_1 + i\sqrt{2}Z_2\right)\left(\sqrt{3}Z_1 - i\sqrt{2}\bar{Z}_2\right)$$

$$= 5\left(|Z_1|^2 + |Z_2|^2\right) > 5\,.\,2\sqrt{|Z_1|^2|Z_2|^2} = 10|Z_1Z_2|,$$

since AM > GM for $|Z_1| \neq |Z_2|$

(4) **(C).** Since $\angle OAB = \frac{\pi}{2}$ and OA = AB, $(3 + bi) - (1 + ai)$

$= (-1 - ai)i\,2 + (b - a)i = a - i$



Comparison gives a = 2 and b = 1.

Another Figure is also possible.

This gives a = – 2 and b = – 1.

(5) **(C).** $\begin{vmatrix} 1 & Z_1 & \bar{Z}_1 \\ 1 & Z_2 & \bar{Z}_2 \\ 1 & Z_3 & \bar{Z}_3 \end{vmatrix} = \begin{vmatrix} 1 & 2x_1 & \bar{Z}_1 \\ 1 & 2x_2 & \bar{Z}_2 \\ 1 & 2x_3 & \bar{Z}_3 \end{vmatrix} = 2\begin{vmatrix} 1 & x_1 & -iy_1 \\ 1 & x_2 & -iy_2 \\ 1 & x_3 & -iy_3 \end{vmatrix}$

$$= -2i\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 0$$

Implies that the points are collinear.

(6) **(C).** Then n roots are given by $Z_r + a = Z_r\, e^{i\frac{2r\pi}{n}}$, $r = 0, 1, 2, \ldots\ldots, n - 1$.

$$Z_r = \frac{-a}{1-\cos\frac{2r\pi}{n} - i\sin\frac{2r\pi}{n}} = \frac{-2}{2\sin\frac{r\pi}{n}\left(\sin\frac{r\pi}{n} - i\cos\frac{r\pi}{n}\right)}$$

$$= \frac{-a}{2\sin\frac{r\pi}{n}}\left(\sin\frac{r\pi}{n} + i\cos\frac{r\pi}{n}\right) = \frac{-a}{2}\left(1 + i\cot\frac{r\pi}{n}\right)$$

$\therefore$ $\text{Re}(Z_r) = \frac{-a}{2}$ for all r, i.e., all the roots lie on

$$\text{Re}\left(Z + \frac{a}{2}\right) = 0$$

Which is a straight line parallel to Im Z-axis.

(7) **(A)** $|Z - mi| = m + 5$ represent a circle with mi or B (0, m) as centre and radius m + 5.

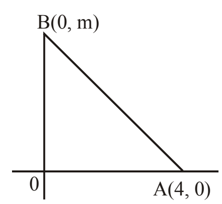
$|Z - 4| < 3$ represent the interior of a circle with centre 4 or A (4, 0) and radius 3.

If there is to be at least one z satisfying both the two circles should intersect.

(i.e,) $r_1 - r_- < d < r_1 + r_2$

$m + 5 - 3 < \sqrt{m^2 + 16} < m + 5 + 3$

$\ldots\ldots\ldots\ {}^2 + 4m + 4 < m^2 + 169 < m^2 + 16m + 64$



$\therefore$ m < 3 and m > – 3

$\therefore$ $m \in (-3, 3)$

(8) **(C).** $\frac{3 + 2i\sin\theta}{1 - 2i\sin\theta}$ will be purely imaginary, if the real part

vanishes, i.e., $\frac{3 - 4\sin^2\theta}{1 + 4\sin^2\theta} = 0 \Rightarrow 3 - 4\sin^2\theta = 0$ (only if

θ be real) $\Rightarrow \sin\theta = \pm\frac{\sqrt{3}}{2} = \sin\left(\pm\frac{\pi}{3}\right)$

$$\Rightarrow \theta = n\pi + (-1)^n\left(\pm\frac{\pi}{3}\right) = n\pi \pm \frac{\pi}{3}$$

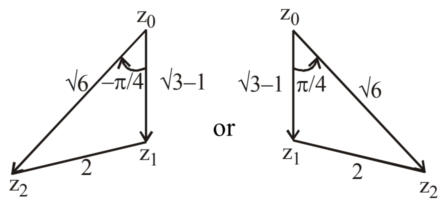
(9) **(D).** As $|Z|^2 = Z\bar{Z}$, the given inequality can be written

$[(\sqrt{3} + i)Z - (\sqrt{2} - i)\bar{Z}][(\sqrt{3} - i)\bar{Z} - (\sqrt{2} + i)Z]$

$+ [(\sqrt{2} + i)Z + (\sqrt{3} - i)\bar{Z}][(\sqrt{2} - i)\bar{Z} + (\sqrt{3} + i)Z] < 28$

$\Rightarrow 3Z\bar{Z} + 4Z\bar{Z} + 3Z\bar{Z} + 4Z\bar{Z} < 28 \Rightarrow |Z|^2 < 2$

(10) **(A).** $\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{6+(4-2\sqrt{3})-4}{2\sqrt{6}(\sqrt{3}-1)} = \dfrac{1}{\sqrt{2}}$



$\Rightarrow \angle A = \dfrac{\pi}{4} \quad \therefore\ z_2 - z_0 = \dfrac{\sqrt{6}}{\sqrt{3}-1}\text{cis}\left(\pm\dfrac{\pi}{4}\right)(z_1 - z_0)$

$2^4(z_2-z_0)^4 = [\sqrt{6}\,(\sqrt{3}+1)]^4 \text{cis}\,(\pm\pi)\,(z_1-z_0)^4$

(11) **(C).** Let $z = a + ib \Rightarrow \bar{z} = a - ib$

Hence, we have $z^{2008} = \bar{z}$

$\therefore\ |z|^{2008} = |\bar{z}| = |z|$

$|z|[|z|^{2007} - 1] = 0$

$|z| = 0$ or $|z| = 1$, if $|z| = 0 \Rightarrow z = 0 \Rightarrow (0, 0)$

if $|z| = 1$; $z^{2009} = z\bar{z} = |z|^2 = 1$

$\Rightarrow$ 2009 values of z $\Rightarrow$ Total = 2010

(12) **(A).** Let z_1, z_2 are the two roots with $|z_1| = 1$

$\because\ z_1z_2 = \dfrac{c}{a} \Rightarrow |z_2| = \left|\dfrac{c}{a}\right|\dfrac{1}{|z_1|} = 1 \Rightarrow z_1\bar{z}_1 = z_2\bar{z}_2 = 1$

$\because\ z_1 + z_2 = -\dfrac{b}{a}$ and $|b| = |a| \Rightarrow |z_1+z_2|^2 = 1$

$\Rightarrow (z_1+z_2)(\bar{z}_1+\bar{z}_2) = 1$

$\Rightarrow (z_1+z_2)\left(\dfrac{1}{z_1}+\dfrac{1}{z_2}\right) = 1 \Rightarrow (z_1+z_2)^2 = z_1z_2$

$\Rightarrow \left(-\dfrac{b}{a}\right)^2 = \dfrac{c}{a} \Rightarrow b^2 = ac$

(13) **(D).** We have $\left|\dfrac{z_1}{2}+\dfrac{z_2}{3}+\dfrac{z_3}{4}+\dfrac{z_4}{5}\right|$

$= \dfrac{k}{60}|z_1z_2z_3z_4|\left|\dfrac{1}{z_1}+\dfrac{1}{z_2}+\dfrac{1}{z_3}+\dfrac{1}{z_4}\right|$

Now, $z_1\bar{z}_1 = 2$, $z_2\bar{z}_2 = 3$, $z_3\bar{z}_3 = 4$ and $z_4\bar{z}_4 = 5$

So, $k = \dfrac{60}{|z_1z_2z_3z_4|} = \dfrac{60}{\sqrt{2}\sqrt{3}\sqrt{4}\sqrt{5}} = \sqrt{30}\,|z_4z_1z_2|$

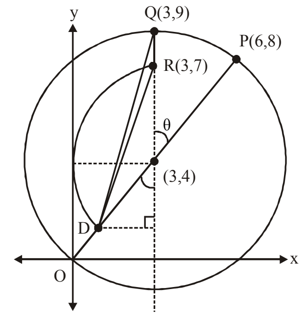
Note for objective takes,

$z_1 = \sqrt{2}, z_2 = \sqrt{3}, z_3 = 2, z_4 = \sqrt{5}$

(14) **(B).** Point on $C_1 : |z - 3 - 4i| = 5$

where $|z|$ is maximum is $P \equiv 6 + 8i$

Let complex number coresponding to point Q be z_2



Taking rotation of 6 + 8i about 3 + 4i, we get

$\dfrac{z_2-(3+4i)}{6+8i-(3+4i)} = e^{i\tan^{-1}\frac{3}{4}}$

$z_2 = (3+4i)+(3+4i)\left(\cos\left(\tan^{-1}\dfrac{3}{4}\right)\right)+i\sin\left(\tan^{-1}\dfrac{3}{4}\right)$

$= 3+4i+(3+4i)\left(\dfrac{4}{5}+i\dfrac{3}{5}\right) = 3+4i+\dfrac{1}{5}(3+4i)(4+3i)$

$= 3+9i$

$\therefore$ Complex number corresponding to R, $z_3 = 3 + 7i$.

(15) **(D).** $z_1 + z_2 + z_3 = 0$

$z_1 = \cos\theta_1 + i\sin\theta_1$

$z_2 = \cos\theta_2 + i\sin\theta_2$

$z_3 = \cos\theta_3 + i\sin\theta_3$

$\therefore\ \cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0 = \sin\theta_1 + \sin\theta_2 + \sin\theta_3$

$\Sigma\cos^2\theta_1 + 2\Sigma\cos\theta_1\cos\theta_2 = 0$

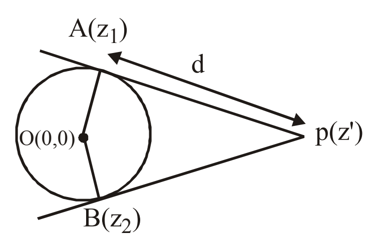
$\Sigma\sin^2\theta_1 + 2\Sigma\sin\theta_1\sin\theta_2 = 0$

$2\Sigma\,(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) = -3$

i.e. $\Sigma\cos(\theta_1-\theta_2) = -\dfrac{3}{2}$

(16) **(C).** $AP = \dfrac{d}{r}AO.e^{i\pi/2}$; $z' - z_1 = \dfrac{d}{r}(-z_1 i)$ .... (1)

$BP = \dfrac{d}{r}BO.e^{-i\pi/2}$; $z' - z_2 = \dfrac{d}{r}z_2 i$ .... (2)



Now from eq. (1) and (2), we get

$\dfrac{z'-z_1}{z'-z_2} = -\dfrac{z_1}{z_2} \Rightarrow z' = \dfrac{2z_1z_2}{z_1+z_2}$

(17) **(C).** $x^2 - 2x\cos\theta + 1 = 0,$

$$\therefore\ x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2},\ \cos\theta \pm i\sin\theta$$

Let $x = \cos\theta + i\sin\theta$

$\therefore\ x^{2n} - 2x^n\cos n\theta + 1 = \cos 2n\theta + i\sin 2n\theta$
$\quad - 2(\cos n\theta + i\sin n\theta)\cos n\theta + 1$
$= \cos 2n\theta + 1 - 2\cos^2 n\theta + i(\sin 2n\theta - 2\sin n\theta\cos n\theta)$
$= 0 + i\,0 = 0$

(18) **(C).** We have, $1 + \omega + \omega^2 + ... + \omega^{n-1} = \dfrac{1-\omega^n}{1-\omega}$

But $\omega^n = \cos\left(\dfrac{n\pi}{n}\right) + i\sin\left(\dfrac{n\pi}{n}\right) = \cos\pi + i\sin\pi = -1$

and $1 - \omega = 2\sin^2\dfrac{\pi}{2n} - 2i\sin\dfrac{\pi}{2n}\cos\dfrac{\pi}{2n}$

$$= -2i\sin\left(\frac{\pi}{2n}\right)\left[\cos\frac{\pi}{2n} + i\sin\frac{\pi}{2n}\right]$$

Thus, $1 + \omega + \omega^2 + ... + \omega^{n-1}$

$$= \frac{2[\cos(\pi/2n) - i\sin(\pi/2n)]}{-2i\sin(\pi/2n)} = 1 + i\cot(\pi/2n)$$

(19) **(B).** If $|z+i| + |z-i| = 8,$
$PF_1 + PF_2 = 8\ \therefore\ |z|_{max} = 4\ \Rightarrow (B)$

(20) **(C).** $\sum_{k=0}^{100} i^k = x + iy,\ \Rightarrow 1 + i + i^2 + ... + i^{100} = x + iy$

Given series is G.P.

$$\Rightarrow \frac{1.(1-i^{101})}{1-i} = x + iy \Rightarrow \frac{1-i}{1-i} = x + iy \Rightarrow 1 + 0i = x + iy$$

Equating real and imaginary parts, we get the required result.

(21) **(C)** $a\bar{a} = b\bar{b} = c\bar{c} = 1 \qquad \therefore\ \bar{a} = \dfrac{1}{a}$ etc.

$$|abc| = |a+b+c|\,|\bar{a} + \bar{b} + \bar{c}| = \left|\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right| = \left|\frac{\sum ab}{abc}\right|$$

$\therefore\ |\sum ab| = |abc|\,|abc| = (|a||b||c|)^2 = 1$

(22) **(B).** $|Z| = 2$ implies $Z\bar{Z} = 4$ and $|Z-3| = 2$ implies
$Z\bar{Z} - 3Z - 3\bar{Z} + 9 = 4.$
The points of intersection are given by

$Z + \bar{Z} = 3.\ Z = \dfrac{3}{2} + ia$ gives $Z\bar{Z} = \dfrac{9}{4} + \alpha^2 = 4$ so than

$\alpha^2 = \dfrac{7}{4}$. The points intersection are $\dfrac{1}{2}(3 \pm i\sqrt{7})$

(23) **(B).** $2(x+iy) = \sqrt{x^2+y^2} + 2i$

$2x = \sqrt{x^2+y^2}$ and $2y = 2$ i.e. $y = 1$

$4x^2 = x^2 + 1$ i.e., $3x^2 = 1$ i.e. $x = \pm\dfrac{1}{\sqrt{3}}$

$x = \dfrac{1}{\sqrt{3}}\ (\because x \ge 0) \qquad \therefore\ z = \dfrac{1}{\sqrt{3}} + i = \dfrac{\sqrt{3}}{3} + i$

(24) **(B).** $\left(1+\dfrac{1}{\omega}\right)\left(1+\dfrac{1}{\omega^2}\right) + \left(2+\dfrac{1}{\omega}\right)\left(2+\dfrac{1}{\omega^2}\right) + \left(3+\dfrac{1}{\omega}\right)\left(3+\dfrac{1}{\omega^2}\right) + ... + \left(n+\dfrac{1}{\omega}\right)\left(n+\dfrac{1}{\omega^2}\right)$

$$\left(r+\frac{1}{\omega}\right)\left(r+\frac{1}{\omega^2}\right) = (r+\omega^2)(r+\omega)$$

$= r^2 + (\omega+\omega^2)r + 1 = (r^2 - r + 1)$

$$= \sum_{r=1}^{n}(r^2 - r + 1) = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + n$$

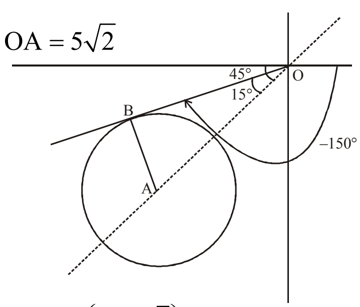
$$= \frac{n}{6}[2n^2 + 3n + 1 - 3n - 3 + 6] = \frac{n}{6}(2n^2+4) = \frac{n(n^2+2)}{3}$$

(25) **(B).** Point B has least principal argument

$AB = \dfrac{5(\sqrt{3}-1)}{2},\quad OA = 5\sqrt{2}$

$\angle AOB = \dfrac{\pi}{12}$

$\therefore\ Arg(z) = -\dfrac{5\pi}{6}$



(26) **(D).** $z = \dfrac{2(1-i\sqrt{3})(1+i)}{(\sqrt{3}-i)^3(-1+i)^4} = \dfrac{2\left(\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2}\right)\left(\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i\right)2\sqrt{2}}{8\left(\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i\right)^3 .4\left(-\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)^4}$

$$= \frac{1}{4\sqrt{2}}\ \frac{\text{cis}\left(-\frac{\pi}{3}\right)\text{cis}\frac{\pi}{4}}{\text{cis}\left(-\frac{3\pi}{6}\right)\text{cis}\left(4.\frac{3\pi}{4}\right)}$$

$$= \frac{1}{4\sqrt{2}}\text{cis}\left(-\frac{\pi}{3} + \frac{\pi}{4} + \frac{\pi}{2} - 3\pi\right) = \frac{1}{4\sqrt{2}}\text{cis}\left(-\frac{31\pi}{12}\right)$$

$= \dfrac{1}{4\sqrt{2}}\text{cis}\left(-\dfrac{7\pi}{12}\right)\ \therefore$ Princpal value of z is $-\dfrac{7\pi}{12}$

(27) **(B).** $z = \frac{-1+i\sqrt{3}}{2}$ is a cube root of unity.

$\therefore\ (z-z^2+2z^3)(2-z+z^2)$
$= (z-z^2+2)(2-z+z^2) = (2+z-z^2)(2-(z-z^2))$
$= 4-(z-z^2)^2 = 4-(z^2+z^4-2z^3)$
$= 4-(z^2+z-2) = 4-(z^2+z+1-3) = 4+3 = 7$

(28) **(D).** $(1+i\sqrt{3})^n = \left[2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right]^n = 2^n\left(\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}\right)$

$f((1+i\sqrt{3})^n) = \text{real part of } z = 2^n\cos\frac{n\pi}{3}$

$\therefore\ \sum_{n=1}^{6a}\log_2\left|2^n\cos\frac{n\pi}{3}\right| = \sum_{n=1}^{6a} n+\log_2\left|\cos\frac{n\pi}{3}\right|$

$= \frac{6a(6a+1)}{2}+\underbrace{(-1-1+0-1-1+0)}_{\text{a such term}}$

$= 3a(6a+1)-4a = 18a^2-a$

(29) **(A).** Rewriting the equation, $\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^x = 1$ and

$e^{i\frac{\pi}{3}x} = e^{i2r\pi}$

$r = 0, \pm1, \pm2, \ldots$ giving the solutions $x = 6r, r = 0, \pm 2, \ldots$ which form an A.P. with common difference 6.

(30) **(B).** Let $z_1 = r_1(\cos\theta_1+i\sin\theta_1)$.

Then $\left|\frac{z_1}{z_2}\right| = 1 \Rightarrow |z_1| = |z_2| \Rightarrow |z_1| = |z_2| = r_1$

Now $\arg(z_1 z_2) = 0 \Rightarrow \arg(z_1)+\arg(z_2) = 0$

$\Rightarrow \arg(z_2) = -\theta_1$

$z_2 = r_1(\cos(-\theta_1)+i\sin(-\theta_1)) = r_1(\cos\theta_1-i\sin\theta_1) = \bar{z}_1$

$\Rightarrow \bar{z}_2 = \left(\bar{\bar{z}}_1\right) = z_1 \Rightarrow |z_2|^2 = z_1 z_2$

(31) **(C)** $\frac{1}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3} = 0 \Rightarrow \sum\frac{1}{(\cos\theta_1+i\sin\theta_1)} = 0$

$\Rightarrow \sum\frac{\cos\theta_1+i\sin\theta_1}{r_1} = 0 \Rightarrow \sum\frac{\cos\theta_1+i\sin\theta_1}{r_1} = 0$

$\Rightarrow \sum\frac{(\cos\theta_1+i\sin\theta_1)^2}{(\cos\theta_1+i\sin\theta_1)} = 0 \Rightarrow \sum\frac{(\cos 2\theta_1+i\sin 2\theta_1)}{Z_1} = 0$

$\Rightarrow \frac{1}{3}\sum\frac{(\cos 2\theta_1+i\sin 2\theta_1)}{Z_1} = 0$

(32) **(A).** $(1+i)^n = 2^{n/2}(\cos n\pi/4+i\sin n\pi/4)$ .... (1)

putting $x = i$ in the given relation, we have

$(1+i)^n = p_0+p_1 i+p_2 i^2+p_3 i^3+\ldots\ldots+p_n i^n$

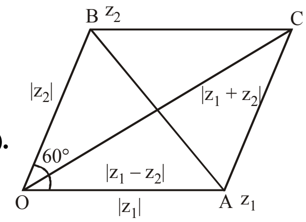
$= p_0+p_1 i-p_2-p_3 i+p_4+p_5 i-\ \ldots\ldots$

$= (p_0-p_2+p_4-\ldots..)+i(p_1-p_3+p_5-\ldots..)$ .... (2)

Equating real parts of (1) and (2), we get

$p_0-p_2+p_4-\ldots\ldots\ldots = 2^{n/2}\cos n\pi/4$

(33) **(C).**



Using cosine rule,

$|z_1+z_2| = \sqrt{|z_1|^2+|z_2|^2-2|z_1||z_2|\cos 120°]}$

$= \sqrt{4+9+2\times 3} = \sqrt{19}$

and $|z_1-z_2| = \sqrt{|z_1|^2+|z_2|^2-2|z_1||z_2|\cos 60°}$

$= \sqrt{4+9-6} = \sqrt{7}$

$\therefore\ \left|\frac{z_1+z_2}{z_1-z_2}\right| = \sqrt{\frac{19}{7}} = \frac{\sqrt{133}}{7} \Rightarrow N = 133$

(34) **(D).** A regular hexagon is circumscribed by a circle with its centre at the centre of the hexagon and radius equal to the length of a side. The sides subtend an angle of $\pi/3$ at the centre. The length of a shorter diagonal $= 2\sqrt{3}$.

Length of a side is therefore $\sqrt{3}\ \sec\frac{\pi}{6} = 2 =$ radius of the circle.

Centre is $Z = 0$ and the other vertices are $2, \pm 1+i\sqrt{3}$ and $-1-i\sqrt{3}$.

(35) **(A).** $|Z-1| = 1 \Rightarrow Z-1 = e^{i\theta}$

$\Rightarrow \frac{Z-2}{Z} = \frac{e^{i\theta}-1}{e^{i\theta}+1} = \frac{\cos\theta-1+i\sin\theta}{\cos\theta+1+i\sin\theta}$

$= \frac{2\sin\frac{\theta}{2}\left(i\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)} = i\tan\frac{\theta}{2} = i\tan(\arg Z)$

$(\because \arg Z = \arg(1+\cos\theta+i\sin\theta)$

$= \arg\left(2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\right) = \frac{\theta}{2}$

(36) **(B).** Given $|z+1| < |z-2|$ and $\omega = 3z+2+i$

$\therefore\ \omega+\bar{\omega} = 3z+2+i+3z+2\bar{z}-i$

$\therefore\ \omega+\bar{\omega} = 3(z+\bar{z})+4$ ...... (1)

Now $|z+1|^2 < |z-2|^2$

$(z+1)(\bar{z}+1) < (z-2)(\bar{z}-2) \Rightarrow z+\bar{z} < 1$ ...... (2)

from (1) & (2) $\frac{\omega+\overline{\omega}-4}{3}<1 \Rightarrow \omega+\overline{\omega}<7$...... (3)

$|\omega+1+i|<|\omega-8+i|$

$|\omega+1+i|^2<|\omega-8+i|^2$

$\Rightarrow (\omega+1+i)(\overline{\omega}+1-i)<(\omega-8+i)(\overline{\omega}-8-i)$

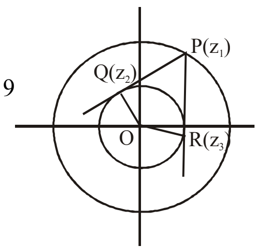
$\Rightarrow \omega+\overline{\omega}<7$ which is true from (3)

(37) **(A).** PQR is equilateral triangle so orthocentre, circumcentre and centroid will coincide and lies on $|z|=1$,

$\left|\frac{z_1+z_2+z_3}{3}\right|=1 \Rightarrow |z_1+z_2+z_3|^2=9$

$(z_1+z_2+z_3)(\overline{z}_1+\overline{z}_2+\overline{z}_3)=9$

$\Rightarrow\left(\frac{4}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)\left(\frac{4}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right)=9$



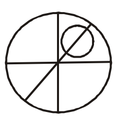
$\arg\left(\frac{z_2}{z_3}\right)=\angle QOR=120°$

(38) **(B).** $C_1C_2=13$

$r_1=30,\ r_2=6$

$C_1C_2<r_1-r_2$

$\therefore$ The circle $|z-(12+5i)=6$ lies with in the circle $|z|=30$



$\therefore\ \max|z_1-z_2|=30+13+6=49$

$\therefore$ Statement-1 is true.

Statement-2 $|z_1-z_2|\le|z_1|+|z_2|$ is always true.

Equality sign holds if z_1, z_2 origin are collinear and z_1 and z_2 lies on opposite sides of the origin.

$\therefore$ Statement-2 is true.

(39) **(A).** Suppose by contradiction

$|z+1|<\frac{1}{\sqrt{2}}$ or $|1+z^2|<1$.

Let $z=a+ib,\ z^2=a^2-b^2+2iab$

$|z+1|<\frac{1}{\sqrt{2}} \Rightarrow (1+a)^2+b^2<\frac{1}{2}$

$\Rightarrow 2(a^2+b^2)+4a+1<0$ .... (i)

$|z^2+1|<1 \Rightarrow (1+a^2-b^2)^2+4a^2b^2<1$

$\Rightarrow (a^2+b^2)^2+2(a^2-b^2)<0$ .... (ii)

Adding (i) and (ii) gives

$(a^2+b^2)^2+(2a+1)^2<0$, which is impossible for $a, b\in R$

(40) **(B).** Let $a=\alpha+i\beta$ and $a_1=\alpha_1+i\beta_1$

Now, the two lines are given by

$2(\alpha x+\beta y)+b=0$ .... (1)

and $2(\alpha_1x+\beta_1y)+b_1=0$ .... (2)

The lines (1) and (2) are parallel if and only if

$-\frac{\alpha}{\beta}=\frac{\alpha_1}{\beta_1} \Leftrightarrow \frac{\alpha}{i\beta}-\frac{\alpha_1}{i\beta_1}$

$\Leftrightarrow \frac{\alpha+i\beta}{\alpha-i\beta}=\frac{\alpha_1+i\beta_1}{\alpha_1-i\beta_1} \Leftrightarrow \frac{a}{\overline{a}}=\frac{a_1}{\overline{a}_1} \Leftrightarrow \frac{a}{a_1}=\overline{\left(\frac{a}{a_1}\right)} \Leftrightarrow \frac{a}{a_1}$

is real

Next, (1) and (2) are perpendicular to each other if and only if $\left(-\frac{\alpha}{\beta}\right)\left(-\frac{\alpha_1}{\beta_1}\right)=-1 \Leftrightarrow \frac{\alpha}{i\beta}=\frac{-\beta_1}{i\alpha_1}$

$\Leftrightarrow \frac{\alpha+i\beta}{\alpha-i\beta}=\frac{-\beta_1+i\alpha_1}{-\beta_1-i\alpha_1}=\frac{i(\alpha_1+i\beta_1)}{(-i)(\alpha_1-i\beta_1)}$

$\Leftrightarrow \frac{a}{\overline{a}}=-\frac{a_1}{\overline{a}_1} \Leftrightarrow \frac{a}{a_1}$ is purely imaginary

(41) **(D).** $a+b+c=0 \quad \Rightarrow c=-(a+b)$

$\therefore az_1+bz_2+cz_3=0$

$\Rightarrow az_1+bz_2-(a+b)z_3=0$

$\Rightarrow z_3=\frac{az_1+bz_2}{a+b}$

$\Rightarrow z_3$ divides the segment joining z_1 and z_2 in the ratio $b:a \Rightarrow z_1, z_2$ and z_3 are collinear.

(42) **(A).** z_1, z_2 will lie on a straight line through the origin if the origin O divides the join of z_1, z_2 in some ratio.

$\Rightarrow 0=\frac{z_1+kz_2}{1+k}$ for some $k\in R$.

$\Rightarrow \frac{z_1}{z_2}=-k\in R \Rightarrow z_1\overline{z}_2=k|z_2|^2\in R$

Next $z_1\overline{z}_2\in R \Rightarrow \overline{z}_1z_2\in R$

(43) **(A).**

(a) $1, z_1, z_2, \ldots\ldots z_{10}$ are the 11th roots of unity

$\therefore 1+z+z^2+\ldots\ldots z^{10}=(z-z_1)(z-z_2)\ldots\ldots\ldots\ldots.(z-z_{10})$

for $z=-1,\ 1=(-1-z_1)(-1-z_2)\ldots\ldots\ldots.(-1-z_{10})$

$=(1+z_1)(1+z_2)\ldots\ldots\ldots.(1+z_{10})$

(b) $1+z_1^{100}+z_2^{100}+z_3^{100}+\ldots\ldots+z_{10}^{100}$

$=1+z_1+z_2+\ldots\ldots+z_{10}=0$

(c) $1+z+z^2+\ldots\ldots+z^{10}$

$=(z-z_1)(z-z_2)(z-z_3)\ldots\ldots(z-z_{10})$

$\therefore 11=(1-z_1)(1-z_2)(1-z_3)\ldots\ldots\ldots\ldots.(1-z_{10})$

(d) $z_1, z_2, \ldots\ldots z_{10}$ are the roots of $z^{11}-1=0$

$\therefore$ product of roots $1, z_1, z_2, \ldots\ldots z_{10}=(-1)^{11}(-1)=1$

(44) **(A).**

(a) Put $z=x+iy$

$\therefore \mathrm{Re}(x+iy)^2=\mathrm{Re}(x+iy+x-iy)$

$x^2-y^2=2x$ or $x^2-y^2-2x=0$

Rectangular hyperbola, eccentricity $=\sqrt{2}$

(b) For ellipse $\lambda>|z_1-z_2|$ and for straight line $\lambda=|z_1-z_2|$

(c) $\because \left|\dfrac{2z-i}{z+1}\right| = m \Rightarrow \left|\dfrac{z-\frac{i}{2}}{z+1}\right| = \dfrac{m}{2}$

For $m=2$, $\left|\dfrac{z-\frac{i}{2}}{z+1}\right| = 1 \Rightarrow \left|z-\dfrac{i}{2}\right| = |z+1|$

i.e., a straight line and for $m \neq 2$, locus is circle.

(d) Let $z = x+iy$

$\Rightarrow\ x^2+y^2 = 25^2$

$-1+75\bar{z} = 75x-1+i\,75y = h+ik$

$\Rightarrow \left(\dfrac{h+1}{75}\right)^2 + \left(\dfrac{k}{75}\right)^2 = 25^2$

$\Rightarrow$ Locus of (h, k) is a circle.

(45) (C).

(a) $z = \dfrac{1 \pm \sqrt{-3}i}{2} = \dfrac{1+i\sqrt{-3}}{2}$ or $\dfrac{1-i\sqrt{-3}}{2}$; amp $z = \dfrac{\pi}{3}$

or amp $z = -\dfrac{\pi}{3} \Rightarrow$ qr

(b) $z = \dfrac{-1 \pm \sqrt{3}i}{2} = \dfrac{-1+i\sqrt{3}}{2}$ or $\dfrac{-1-i\sqrt{3}}{2}$;

amp $z = \dfrac{2\pi}{3}$ or $-\dfrac{2\pi}{3} \Rightarrow$ ps

(c) $2z^2 = -1-i\sqrt{3} \Rightarrow z^2 = \dfrac{-1-i\sqrt{3}}{2} = \cos\left(-\dfrac{2\pi}{3}\right) + i\sin\left(-\dfrac{2\pi}{3}\right)$

$z = \cos\left(\dfrac{2m\pi-(2\pi/3)}{2}\right) + i\sin\left(\dfrac{2m\pi-(2\pi/3)}{2}\right)$

$m=0,\ z = \cos\left(-\dfrac{\pi}{3}\right) + i\sin\left(-\dfrac{\pi}{3}\right)$; $m=1$,

$z = \cos\left(\dfrac{2\pi}{3}\right) + i\sin\left(\dfrac{2\pi}{3}\right)$

$\Rightarrow$ amp $z = -\dfrac{\pi}{3}$ or $\dfrac{2\pi}{3} \Rightarrow$ qs

(d) $2z^2+1-i\sqrt{3} = 0$

$z^2 = \dfrac{-1+i\sqrt{3}}{2} = \cos\left(\dfrac{2\pi}{3}\right) + i\sin\left(\dfrac{2\pi}{3}\right)$;

$z = \cos\left(\dfrac{2m\pi+(2\pi/3)}{2}\right) + i\sin\left(\dfrac{2m\pi+(2\pi/3)}{2}\right)$

$m=0,\ z = \cos\left(\dfrac{\pi}{3}\right) + i\sin\left(\dfrac{\pi}{3}\right)$; $m=1$,

$\cos\left(\dfrac{4\pi}{3}\right) + i\sin\left(\dfrac{4\pi}{3}\right)$

or $\cos\left(-\dfrac{2\pi}{3}\right) + i\sin\left(-\dfrac{2\pi}{3}\right) \Rightarrow$ pr

(46) (C), (47) (C), (48) (B).

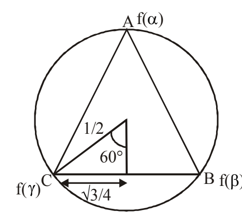
$\because f(\alpha) = \dfrac{1}{\alpha-i} \times \dfrac{\alpha+i}{\alpha+i} = \dfrac{\alpha}{\alpha^2+1} + i\dfrac{1}{\alpha^2+1}$

$\Rightarrow$ Real part $x = \dfrac{\alpha}{\alpha^2+1}$, $y = \dfrac{1}{\alpha^2+1}$

$\Rightarrow \dfrac{x}{y} = \alpha$, then $x = \dfrac{(x/y)}{(x/y)^2+1} \Rightarrow x^2+y^2 = y$

$\Rightarrow (x-0)^2 + \left(y-\dfrac{1}{2}\right)^2 = \left(\dfrac{1}{2}\right)^2 \Rightarrow f(\alpha)$ lies on the circle.

$\therefore \max|f(\alpha)-f(\beta)| =$ diameter of the circle $= 2 \cdot \dfrac{1}{2} = 1$



If $f(\alpha), f(\beta), f(\gamma)$, lies on circle, then Δ ABC for maximum area will be an equilateral triangle

$\Rightarrow R = \dfrac{abc}{4\Delta} \Rightarrow \dfrac{1}{2} = \dfrac{(\sqrt{3}/2)^3}{4\Delta} \Rightarrow \Delta = \dfrac{3\sqrt{3}}{16}$ (units)^2

If $f(\alpha), f(\beta), f(\gamma), f(\delta)$ forms a square then its area

$= \dfrac{1}{2}$ (diagonal)$^2 = \dfrac{1}{2}(1)^2 = \dfrac{1}{2}$ (units)^2 and side $= \dfrac{1}{\sqrt{2}}$ unit

(49) (D). Length of perpendicular from z_0 on the tangent at B is,

$\dfrac{|z_0\bar{b}+\bar{z}_0 b-4a^2|}{2|b|} \Rightarrow \dfrac{|z_0\bar{b}+\bar{z}_0 b-4a^2|}{2\sqrt{2}\,a}$

(50) (D). $\because$ b lie on $z = \sqrt{2}\,a$

$\therefore\ b| = \sqrt{2}\,a \quad |z\bar{b}+\bar{z}b = 4a^2$

(51) (A). The equation of straight line parallel to $z\bar{b}+\bar{z}b = \lambda$, which passes through origin is

$\lambda = 0$ or $z\bar{b}+\bar{z}b = 0$ is a straight line parallel to tangent at 'b' and passing through centre.

(52) (B). $\omega_1 = \omega_2 e^{i\frac{4\pi}{3}} \Rightarrow \omega_1^3 = \omega_2^3$

$\Rightarrow \omega_1^3\bar{\omega}_1^2\bar{\omega}_2^2 = \omega_2^3\bar{\omega}_1^2\bar{\omega}_2^2 \Rightarrow \omega_2\bar{\omega}_1^2 = \omega_1\bar{\omega}_2^2$

(53) **(B).** Since $i\beta$ is real
$\therefore$ β pure imaginary.

(54) **(C).** $-\frac{\alpha}{\bar{\alpha}} = e^{\pm\frac{i\pi}{2}} = \pm i \;\therefore\; (1+i)\left(-\frac{2\alpha}{\bar{\alpha}}\right) = \pm 2(-1+i)$

EXERCISE-3

(1) **6.** $\left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^{n/2}$ is real

$2^{n/2}\left[\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right]$ is real

hence $\sin\frac{n\pi}{6} = 0 \;\therefore\; \frac{n\pi}{6} = k\pi \;\therefore\; n = 6k$

smallest positive n is 6

(2) **4.** Let $z = x + iy$, $x, y \in R$ and $x^2 + y^2 = 2$ (say)
$\therefore z^2$ is purely imaginary
$(x+iy)^2$ is purely imaginary
$x^2 - y^2 + 2xyi = 0 + ki \quad k \in R - \{0\}$
$\therefore x^2 = y^2$ and $2xy = k$ [If $k = 0$ then $x = 0$ and $y = 0$]
let $k > 0$ say 2
$\therefore xy = 1 \Rightarrow y = 1/x$
$x^4 = 1 \quad \Rightarrow x^2 = 1 \Rightarrow x = 1$ or -1
$\therefore y = 1$ or -1
$\therefore z$ is $1 + i$ or $-1 - i$
if $k < 0$ say -2 then $xy = -1$; $\quad y = -1/x$
$x^4 = 1 \Rightarrow x^2 = 1$
$x = 1$ or -1
$y = -1$ or 1
$\therefore z$ is $1 - i$ or $-1 + i$
$\therefore$ there are four values of z which are $\pm 1 \pm i$

(3) **4016.** Let x be the $(2009)^{th}$ root of unit $\neq 1$, then
$x^{2009} - 1 = (x-1)(x-w) \,.......\, (x - w^{2008})$
Taking log on both sides, we get
$\ln(x^{2009} - 1) = \ln(x-1) + \ln(x-w) + \ln(x-w^2) \,.....\, + \ln(x - w^{2008})$
$\therefore$ On differentiate both the side w.r.t. x, we get

$$\frac{(2009)x^{2008}}{x^{2009}-1} = \frac{1}{x-1} + \sum_{r=1}^{2008}\frac{1}{x-w^r} \qquad .... (1)$$

Putting $x = 2$ in eq. (2), we get

$$\Rightarrow 1 + \sum_{r=1}^{2008}\frac{1}{2-w^r} = \frac{2009\,(2^{2008})}{2^{2009}-1}$$

Multiplying both sides of above equation by $(2^{2009} - 1)$, we get

$$\therefore (2^{2009}-1)\sum_{r=1}^{2008}\frac{1}{2-w^r} = 2009\,.\,2^{2008} - 2^{2009} + 1$$

$= 2^{2008}(2009 - 2) + 1 = 2^{2008}.2007 + 1 = [(a)(2^b) + c]$

$\therefore a = 2007, b = 2008, c = 1$
Hence, $a + b + c = 4016$

(4) **41.** $\sum_{n=0}^{\infty}\frac{\sin(nx)}{3^n}$ put $\sin(nx) = \frac{e^{nix} - e^{nix}}{2i}$

$$\therefore \sum_{n=0}^{\infty}\frac{\sin(nx)}{3^n} = \frac{1}{2i}\sum_{n=0}^{\infty}\frac{e^{nix} - e^{nix}}{3^n}$$

$$= \frac{1}{2i}\left[\sum_{n=0}^{\infty}\left(\frac{e^{ix}}{3}\right)^n - \sum_{n=0}^{\infty}\left(\frac{e^{-ix}}{3}\right)^n\right]$$

$$= \frac{1}{2i}\left[\frac{1}{1-\frac{e^{ix}}{3}} - \frac{1}{1-\frac{e^{-ix}}{3}}\right]$$

$$= \left[\frac{3}{3-e^{ix}} - \frac{3}{3-e^{-ix}}\right] = \frac{3}{2i}\left[\frac{(3-e^{-ix}) - (3-e^{ix})}{9 - 3(e^{ix}+e^{-ix}) + 1}\right]$$

$$= \frac{3}{2i}\left[\frac{2i\sin x}{10 - 6\cos x}\right] = \frac{3\sin x}{2(5 - 3\cos x)} = \frac{1}{2\left(5 - 3\sqrt{1-(1/9)}\right)}$$

$$= \frac{1}{2(5 - 2\sqrt{2})} = \frac{5 + 2\sqrt{2}}{34}$$

$\Rightarrow a = 5, b = 2, c = 37 \Rightarrow a + b + c = 5 + 2 + 37 = 41$

(5) **9.** If a polynomial has real coefficients then roots occur in complex conjugate and
$\therefore$ roots are $2i, -2i, 2+i, 2-i$
hence $f(x) = (x+2i)(x-2i)(x-2-i)(x-2+i)$
$f(1) = (1+2i)(1-2i)(1-2-i)(1-2+i)$
$f(1) = 5 \times 2 = 10$
Also $f(1) = 1 + a + b + c + d$
$\therefore 1 + a + b + c + d = 10 \Rightarrow a + b + c + d = 9$

(6) **2.** $z(z+1) = 0 \Rightarrow z = 0$ or $z = -1$

(7) **3364.** $z = (3p - 7q) + i(3q + 7p)$
for purely imaginary $3p = 7q \Rightarrow p = 7$ or $q = 3$
(for least value)
$|z| = |3 + 7i|\,|p + iq| \Rightarrow |z|^2 = 58(p^2 + q^2) = 58[7^2 + 9] = 58^2$

(8) **1.** $1 - z^{18} = 0$; $1 - z^{14} = 0 \quad \Rightarrow z^{14} = 1$ or $z^{18} = 1$
since one is extraneous root $z = -1$ is the common root.

(9) **5.** $z = 0$; $z = \pm 1$; $z = \pm i$;
$z^3 = \bar{z} \;\Rightarrow |z|^3 = |\bar{z}| = |z|$
hence $|z| = 0$ or $|z|^2 = 1$
again $z^4 = z\,\bar{z} = |z|^2 = 1 \Rightarrow z^4 = 1$

$\Rightarrow$ no. of roots are 5

Note that the equation $z^n = \bar{z}$ will have $(n+2)$ solutions.

(10) **17.** Let $z = a + bi$.

$|z|^2 = a^2 + b^2$.

So, $z + |z| = 2 + 8i$

$a + bi + \sqrt{a^2 + b^2} = 2 + 8i$

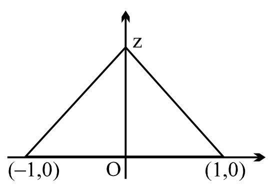
$a + \sqrt{a^2 + b^2} = 2,\ b = 8\ ;\ a + \sqrt{a^2 + 64} = 2$

$a^2 + 64 = (2 - a)^2 = a^2 - 4a + 4,$

$4a = -60,\ a = -15.$ Thus, $a^2 + b^2 = 225 + 64 = 289$

$\therefore |z| = \sqrt{a^2 + b^2} = \sqrt{289} = 17$

(11) **2.** distance of z (1,0) & (–1, 0), will be minimum with z is at 'O'



$y \le |z| + 1 + |z| + 1 = 2 + 2|z| = 2$ where $z = 0$

(12) **1.** $|a + b\omega + c\omega^2| = \sqrt{\left(a - \frac{b}{2} - \frac{c}{2}\right)^2 + \frac{3}{4}(c-b)^2}$

$= \sqrt{\frac{1}{2}((a-b)^2 + (b-c)^2 + (c-a)^2)}$

This is minimum when $a = b$ and

$(b - c)^2 = (c - a)^2 = 1 \Rightarrow$ The minimum value is 1.

(13) **48.** $z\bar{z}(z^2 + \bar{z}^2) = 350$

$\Rightarrow 2(x^2 + y^2)(x^2 - y^2) = 350 \Rightarrow (x^2 + y^2)(x^2 - y^2) = 175$

Since $x, y \in I$, the only possible case which gives integral solution, is

$x^2 + y^2 = 25$ .... (1)

$x^2 - y^2 = 7$ ....(2)

From (1) and (2) $x^2 = 16\ ;\ y^2 = 9$

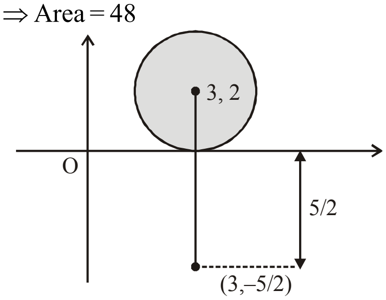
$\Rightarrow x = \pm 4\ ;\ y = \pm 3 \Rightarrow$ Area = 48

(14) **5.** $|2z - 6 + 5i|$

$= 2\left|z - \left(3 - \frac{5i}{2}\right)\right|$

For minimum

$= 2 \times \frac{5}{2} = 5$



(15) **3.** On taking $\omega = e^{\frac{i\pi}{3}}$. Expression is in terms of a, b, c

So lets assume $\omega = e^{\frac{i2\pi}{3}}$,

then the solution is following

$a + b + c = x\ ;\ a + b\omega + c\omega^2 = y\ ;\ a + b\omega^2 + c\omega = z$

$$\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = \frac{x\bar{x} + y\bar{y} + z\bar{z}}{|a|^2 + |b|^2 + |c|^2}$$

$$= \frac{(a+b+c)(\bar{a}+\bar{b}+\bar{c}) + (a + b\omega + c\omega^2)(\bar{a} + \bar{b}\omega^2 + c\omega) + (a + b\omega^2 + c\omega)(\bar{a} + \bar{b}\omega + \bar{c}\omega^2)}{|a|^2 + |b|^2 + |c|^2}$$

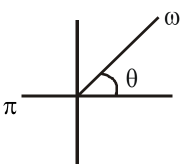
$$= \frac{3(|a|^2 + |b|^2 + |c|^2)}{|a|^2 + |b|^2 + |c|^2} = 3$$

(16) **4.** $\alpha_k = \cos\frac{2k\pi}{14} + i\sin\frac{2k\pi}{14} = e^{i\frac{2k\pi}{14}}$

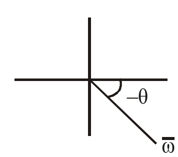
$$\frac{\sum_{k=1}^{12}\left|e^{\frac{i2(k+1)\pi}{14}} - e^{\frac{i2k\pi}{14}}\right|}{\sum_{k=1}^{3}\left|e^{\frac{i(4k-1)\pi}{14}} - e^{\frac{i(4k-2)\pi}{14}}\right|} = \frac{\sum_{k=1}^{12}\left|e^{\frac{i2\pi}{14}} - 1\right|}{\sum_{k=1}^{3}\left|e^{\frac{i2\pi}{14}} - 1\right|} = \frac{12}{3} = 4$$

EXERCISE-4

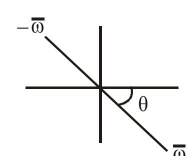
(1) **(D).** $|z| = |\omega|$ and $\text{Arg}(z) + \text{Arg}\,\omega = \pi$



Let $\arg\omega = \theta \Rightarrow \arg\bar{\omega} = -\theta$



$\Rightarrow \arg -\bar{\omega} = -\theta + \pi$ .... (i)



But $\arg(z) + \arg\omega = \pi \Rightarrow \arg z = \pi - \arg\omega = \pi - \theta$ ..... (ii)

From (i) and (ii), $z = -\bar{\omega}$ $\{\because |z| = |\omega| \Rightarrow |z| = |-\bar{\omega}|\}$

Alternate : Let $\arg\omega = \theta \Rightarrow \arg z = \pi - \theta$

Let $|z| = |\omega| = r$ $\{\because \omega = r[\cos\theta + i\sin\theta]\}$

and $z = r[\cos(\pi - \theta) + \sin(r - \theta)]$ $= r[-\cos\theta + i\sin\theta]$

$= -r[\cos\theta - i\sin\theta]$; $z = -\bar{\omega}$

(2) **(A).** $|z-2| \geq |z-4|$
Let $z = x + iy \Rightarrow |x + iy - 2| \geq |x + iy - 4|$
$\Rightarrow |(x-2) + iy)| \geq |(x-4) + iy|$
$\Rightarrow |x - 2 + iy)|^2 \geq |(x-4) + iy|^2$
$\Rightarrow (x-2)^2 + y^2)| \geq |(x-4)^2 + y^2$
$\Rightarrow x^2 + 4 - 4x \geq x^2 + 16 - 8x \Rightarrow 4x \geq 12 \Rightarrow x \geq 3 \Rightarrow \text{Re}(z) \geq 3$

(3) **(D).** ω is cube root of unity then $(1+\omega-\omega^2)(1+\omega^2-\omega)$
$\{\because 1 + \omega + \omega^2 = 0 \Rightarrow 1 + \omega = -\omega^2$
$1 + \omega^2 = -\omega$ and $\omega^3 = 1\}$
$(-\omega^2 - \omega^2)(-\omega - \omega) = (-2\omega^2)(-2\omega) = 4\omega^3 = 4$

(4) **(A).** $\because |z\omega| = 1 \Rightarrow |z||\omega| = 1$

$\Rightarrow |z| = \dfrac{1}{|\omega|}$ .... (1) and let $\arg(\omega) = \theta$

$\therefore \arg(z) = \dfrac{\pi}{2} + \theta \quad \therefore$ We know that $\dfrac{z_2}{z_1} = \dfrac{|z_2|}{|z_1|} e^{i\alpha}$

(where α is the angle between them)

$\Rightarrow \dfrac{z}{\omega} = \dfrac{|z|}{|\omega|} e^{i\pi/2} \Rightarrow \dfrac{z}{\omega} = \dfrac{1}{|\omega|^2} i \qquad \{\because |z| = \dfrac{1}{|\omega|}\}$

$\Rightarrow z = \dfrac{i\omega}{|\omega|^2} \Rightarrow \bar{z} = \dfrac{\overline{i\omega}}{|\omega|^2} = \dfrac{\bar{i}\,\bar{\omega}}{|\omega|^2} = -\dfrac{i\bar{\omega}}{|\omega|^2}$

$\{\because \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ and $\bar{i} = -i\}$

Again $\bar{z}\omega = \dfrac{-i\bar{\omega}\omega}{|\omega|^2} = \dfrac{-i|\omega|^2}{|\omega|^2} = -i \quad \{\because z\bar{z} = |z|^2\}$

(5) **(D).** z_1, z_2 are roots of equation $z^2 + az + b = 0$
$z_1 + z_2 = -a$ .... (1) and $z_1.z_2 = b$ .... (2)
We know if z_1, z_2, z_3 form an equilateral triangle then
$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$
$\therefore$ In question z_1, z_2 and origin form an equilateral triangle $\therefore z_1^2 + z_2^2 + 0^2 = z_1z_2 + z_2.0 + 0.z_1$
$\Rightarrow z_1^2 + z_2^2 = z_1z_2 \Rightarrow (z_1+z_2)^2 - 2z_1z_2 = z_1z_2$
$\Rightarrow (z_1+z_2)^2 = 3z_1z_2 \Rightarrow (-a)^2 = 3b \quad$ {from (1) and (2)}
$\Rightarrow a^2 = 3b$

(6) **(B).** $\left(\dfrac{1+i}{1-i}\right)^x = 1 \Rightarrow \left(\dfrac{1+i}{1-i} \times \dfrac{1+i}{1+i}\right)^x = 1$

$\Rightarrow \left(\dfrac{1+i^2+2i}{1+1}\right)^x = 1 \Rightarrow \left[\dfrac{2i}{2}\right]^x = 1 \Rightarrow i^x = 1$

$\Rightarrow$ x must be multiple of 4
$\therefore$ x = 4n where n is any positive integer

(7) **(C).** $\bar{z} + i\bar{\omega} = 0$
and $\arg z\omega = \pi$ then $\arg(z) = ?$ $\qquad \because \bar{z} + i\bar{\omega} = 0$

$\Rightarrow \bar{z} = -i\bar{\omega} \Rightarrow \bar{\bar{z}} = \overline{-i\bar{\omega}} = \overline{-c}.\bar{\bar{\omega}}$

$\Rightarrow z = i\omega \Rightarrow \omega = \dfrac{z}{i} \quad \because \arg z\omega = \pi \Rightarrow \arg\left(\dfrac{z.z}{i}\right) = \pi$

$\Rightarrow \arg\dfrac{z^2}{i} = \pi \Rightarrow \arg z^2 - \arg i = \pi$

$2\arg z - \pi/2 = \pi \Rightarrow 2\arg z = \dfrac{3\pi}{2} \Rightarrow \arg z = \dfrac{3\pi}{4}$

(8) **(D).** $z = x - iy$ and $z^{1/3} = p + iq$
$z = (p+iq)^3$; $z = p^3 + (iq)^3 + 3(p)(iq)(p+iq)$
$\Rightarrow x - iy = p^3 - 3pq^2 + i(3p^2q - q^3)$
On comparing, $x = p^3 - 3pq^2$ and $-y = 3p^2q - q^3$

$\Rightarrow \dfrac{x}{p} = p^2 - 3q^2$ and $\dfrac{-y}{q} = 3p^2 - q^2$ and $\dfrac{y}{q} = q^2 - 3p^2$

On adding, $\dfrac{x}{p} + \dfrac{y}{q} = p^2 - 3q^2 - 3p^2 + q^2$

$\dfrac{x}{p} + \dfrac{y}{q} = -2p^2 - 2q^2$

$\Rightarrow \dfrac{x}{p} + \dfrac{y}{q} = -2(p^2+q^2) \Rightarrow \dfrac{\frac{x}{p} + \frac{y}{q}}{p^2+q^2} = -2$

(9) **(B).** $|z^2 - 1| = |z|^2 + 1$. Let $z = x + iy$
$\because |(x+iy)^2 - 1| = |x+iy|^2 + 1$
$\Rightarrow |x^2 - y^2 + 2ixy - 1| = x^2 + y^2 + 1$
$\Rightarrow |(x^2 - y^2 - 1) + 2ixy| = x^2 + y^2 + 1$

$\Rightarrow \sqrt{(x^2 - y^2 - 1)^2 + 4x^2y^2} = x^2 + y^2 + 1$

Squaring both side
$\Rightarrow (x^2 - y^2 - 1)^2 + 4x^2y^2 = (x^2 + y^2 + 1)^2$
$\Rightarrow x^4 + y^4 + 1 - 2x^2y^2 + 2y^2 - 2x^2 + 4x^2y^2$
$= x^4 + y^4 + 1 + 2x^2y^2 + 2y^2 + 2x^2$
$\Rightarrow 2x^2y^2 - 2x^2 = 2x^2y^2 + 2x^2 \Rightarrow 4x^2 = 0 \Rightarrow x = 0$
$\Rightarrow$ z is purely imaginary $\Rightarrow$ z lies on imaginary axes

(10) $|z_1 + z_2| = |z_1| + |z_2|$
{Let $\arg z_1 = \theta_1$ and $\arg z_2 = \theta_2$}
$|z_1 + z_2|^2 = (|z_1| + |z_2|)^2$
$|z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$
$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$
$\cos(\theta_1 - \theta_2) = 1$
$\theta_1 - \theta_2 = 0$ or $2n\pi$; $n \in I$
$\arg z_1 - \arg z_2 = 0$ or $2n\pi$; $n \in I$

(11) **(C).** $\omega = \dfrac{z}{z - \frac{1}{3}i}$ and $|\omega| = 1$

$\because \omega = \dfrac{z}{z - \frac{1}{3}i} \Rightarrow |\omega| = \left|\dfrac{z}{z - \frac{1}{3}i}\right|$

$\Rightarrow 1 = \frac{|z|}{\left|z - \frac{1}{3}i\right|} \Rightarrow \left|z - \frac{i}{3}\right| = |z| \Rightarrow \left|z - \frac{i}{3}\right| = |z - 0|$

{ $\because$ z is equidistant from i/3 & 0}
$\Rightarrow$ Locus of z is perpendicular
Bisector of line joining i/3 and 0 { $\because$ if $|z - z_1| = |z - z_2| \Rightarrow$ z lies on $\perp$ bisector of line joining z_1 and z_2}

(12) **(C).** 1, ω, ω^2 are cube roots of unity
$(x-1)^3 + 8 = 0 \Rightarrow (x-1)^3 = -8$
$\Rightarrow (x-1) = (-8)^{1/3}$ { $\because$ if $x = (-1)^{1/3} \Rightarrow x = -1, -\omega$ and $-\omega^2$}
$\Rightarrow x - 1 = -2, -2\omega$ or $-2\omega^2 \Rightarrow x - 1 = -2 \Rightarrow x = -1$
$x - 1 = -2\omega \Rightarrow x = -2\omega + 1$
$x - 1 = -2\omega^2 \Rightarrow x = -2\omega^2 + 1$

(13) **(C).** $z^2 + z + 1 = 0$

$z = \frac{-1 \pm \sqrt{1 - 4.1.1}}{2.1}$ $\left\{ \because \omega = \frac{-1 + i\sqrt{3}}{2}, \omega^2 = \frac{-1 - i\sqrt{3}}{2} \right.$

$= \frac{-1 \pm i\sqrt{3}}{2}$

$\Rightarrow z = \omega$ or $\omega^2 \Rightarrow \omega^3 = 1$ and $1 + \omega + \omega^2 = 0$

$\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2 + ... + \left(z^6 + \frac{1}{z^6}\right)^2$

$\left(z + \frac{1}{z}\right)^2 = \left(\omega + \frac{1}{\omega}\right)^2 = (\omega + \omega^2)^2 = (-1)^2 = 1$ .... (1)

$\left(z^2 + \frac{1}{z^2}\right)^2 = \left(\omega^2 + \frac{1}{\omega^2}\right)^2 = (\omega^2 + \omega)^2 = (-1)^2 = 1$...... (2)

$\left(z^3 + \frac{1}{z^3}\right)^2 = \left(\omega^3 + \frac{1}{\omega^3}\right)^2 = (1+1)^2 = (2)^2 = 4$ .... (3)

$\left(z^4 + \frac{1}{z^4}\right)^2 = \left(\omega^4 + \frac{1}{\omega^4}\right)^2 = \left(\omega + \frac{1}{\omega}\right)^2$
$= (\omega + \omega^2)^2 = (-1)^2 = 1$ .... (4)

$\left(z^5 + \frac{1}{z^5}\right)^2 = \left(\omega^5 + \frac{1}{\omega^5}\right)^2 = \left(\omega^2 + \frac{1}{\omega^2}\right)^2$
$= (\omega^2 + \omega)^2 = (-1)^2 = 1$ .... (5)

$\left(z^6 + \frac{1}{z^6}\right)^2 = \left(\omega^6 + \frac{1}{\omega^6}\right)^2 = (1+1)^2 = (2)^2 = 4$......... (6)

From (1), (2), (3), (4), (5), (6)

$\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2 + \left(z^4 + \frac{1}{z^4}\right)^2$
$+ \left(z^5 + \frac{1}{z^5}\right)^2 + \left(z^6 + \frac{1}{z^6}\right)^2$
$= 1 + 1 + 4 + 1 + 1 + 4 = 12$

(14) **(C).** $\sum_{k=1}^{10} \left(\sin\frac{2k\pi}{11} + i\cos\frac{2k\pi}{11}\right)$

$= \sum_{k=1}^{10} i\left(\cos\frac{2k\pi}{11} - i\sin\frac{2k\pi}{11}\right)$ { $\because e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$}

$= i\sum_{k=1}^{10} e^{i\left(\frac{-2k\pi}{11}\right)} \Rightarrow i\left[e^{\frac{-i2\pi}{11}} + e^{\frac{-i4\pi}{11}} + e^{\frac{-i6\pi}{11}} + ... + e^{\frac{-i20\pi}{11}}\right]$

$= i\left[e^{\frac{-i2\pi}{11}} \frac{[1 - e^{\frac{-i2\pi}{11}})^{10}}{1 - e^{-2i\pi/11}}\right]$

{ $\because$ sum of n terms of G.P. is $\frac{a(1 - r^n)}{1 - r}$, where a = first term, r = common ratio}

$i\left[\frac{e^{\frac{-i2\pi}{11}}(1 - e^{-20\pi/11})}{1 - e^{-i2\pi/11}}\right] = i\left[\frac{e^{\frac{-i2\pi}{11}} - e^{-i\,22\pi/11})}{1 - e^{-i2\pi/11}}\right]$

$= i\left[\frac{e^{-i2\pi/11} - e^{-i2\pi})}{1 - e^{-i2\pi/11}}\right] = i\left[\frac{e^{-i2\pi/11} - 1}{1 - e^{-i2\pi/11}}\right] = -i$

(15) **(C).** $|z + 4| \le 3$
$\because |z_1 + z_2| \le |z_1| + |z_2|$
$\therefore |z + 4 - 3| \le |z + 4| + |-3|$
$\Rightarrow |z + 1| \le 3 + 3$ { $\because |z + 4| \le 3 \Rightarrow$ max. $|z + 4| = 3$}
$\Rightarrow |z + 1| \le 6$

(16) **(B).** Let complex no. is z its conjuate is $\bar{z}$

$\therefore \bar{z} = \frac{1}{i - 1} \Rightarrow \bar{\bar{z}} = \frac{1}{-i - 1} \Rightarrow z = -\left(\frac{1}{1 + i}\right)$

(17) **(A).** $||Z_1| - |Z_2|| \le |Z_1 - Z_2|$

$\Rightarrow |Z| - \frac{4}{|Z|} \le 2 \Rightarrow |Z|^2 - 2|Z| - 4 \le 0 \Rightarrow |Z|_{max} = \sqrt{5} + 1$

(18) **(A).** Let $z = x + iy$
$|z - 1| = |z + 1| \Rightarrow \text{Re } z = 0 \Rightarrow x = 0$
$|z - 1| = |z - i| \Rightarrow x = y$
$|z + 1| = |z - i| \Rightarrow y = -x$
Only (0, 0) will satisfy all conditions.
$\Rightarrow$ Number of complex number z = 1

(19) **(D).** Let roots be $p + iq$ and $p - iq$, $p, q \in R$
Root lie on line Re (z) = 1 $\Rightarrow p = 1$
Product of roots $= p^2 + q^2 = \beta = 1 + q^2$
$\Rightarrow \beta \in (1, \infty)$, $(q \ne 0, \because$ roots are distinct)

(20) **(B).** $(1 + \omega)^7 = A + B\omega$
$(-\omega^2)^7 = A + B\omega$
$-\omega^{14} = A + B\omega$; $-\omega^2 = A + B\omega$

$1+\omega = A + B\omega \quad \therefore \ (A, B) = (1, 1)$

(21) **(A).** $\dfrac{z^2}{z-1} = \dfrac{\bar{z}^2}{\bar{z}-1}$; $z\bar{z}z - z^2 = z\bar{z}\,\bar{z} - \bar{z}^2$

$|z|^2 (z-\bar{z}) - (z-\bar{z})(z+\bar{z}) = 0$

$(z-\bar{z})(|z|^2 - (z+\bar{z})) = 0$

Either $z = \bar{z} \Rightarrow$ real axis

or $|z|^2 = z + \bar{z} \Rightarrow z\bar{z} - z - \bar{z} = 0$

represents a circle passing through origin.

(22) **(C).** $|z| = 1$, $\arg z = \theta$, $z = e^{i\theta}$

$$\bar{z} = \frac{1}{z} \ ; \ \arg\left(\frac{1+z}{1+\frac{1}{z}}\right) = \arg(z) = \theta$$

(23) **(B).** $|z| \ge 2$

$$\left|z + \frac{1}{2}\right| \ge \left||z| - \left|\frac{1}{2}\right|\right| \ge \left|2 - \frac{1}{2}\right| \ge \frac{3}{2}$$

Hence, minimum distance between z and (–1/2, 0) is 3/2.

(24) **(B).** $\left(\dfrac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right) = 1$; $\left(\dfrac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right)\left(\dfrac{\bar{z}_1 - 2\bar{z}_2}{2 - \bar{z}_1 z_2}\right) = 1$

$z_1\bar{z}_1 - 2z_1\bar{z}_2 - 2z_2\bar{z}_1 + 4z_2\bar{z}_2$

$= 4 - 2\bar{z}_1 z_2 - 2z_1\bar{z}_2 + z_1\bar{z}_1 z_2\bar{z}_2$

$z_1\bar{z}_1 + 4z_2\bar{z}_2 = 4 + z_1\bar{z}_1 z_2\bar{z}_2$

$z\bar{z}_1(1 - z_2\bar{z}_2) - 4(1 - z_2\bar{z}_2) = 0$

$(z\bar{z}_1 - 4)(1 - z_2\bar{z}_2) = 0 \Rightarrow z_1\bar{z}_1 = 4$

$|z| = 2$ i.e. z lies on circle of radius 2.

(25) **(C).** $\text{Re}((2 + 3i\sin\theta)(1 + 2i\sin\theta)) = 2 - 6\sin^2\theta = 0$

$\Rightarrow \sin^2\theta = 1/3$

(26) **(C).** $2\omega + 1 = z$; $\omega = \dfrac{\sqrt{3}i - 1}{2}$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k \ ; \ R_1 \to R_1 + R_2 + R_3$$

$$\begin{vmatrix} 3 & 0 & 0 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$$

$= 3[\omega(-\omega^2 - 1) - \omega^4] \equiv 3[-\omega^3 - \omega - \omega] = 3[-1 - 2\omega]$

$= -3[2\omega + 1] = -3z = 3k \Rightarrow k = -z$

(27) **(A).** $x^2 - x + 1 = 0$

$\Rightarrow \ x = \dfrac{1 \pm \sqrt{-3}}{2} = -\omega, -\omega^2$

(where ω and ω^2 are non-real cube roots of unity)

$\Rightarrow \ \alpha = -\omega$ and $\beta = -\omega^2$

$\Rightarrow \ (-\omega)^{101} + (-\omega^2)^{107}$

$= -(\omega^{101} + \omega^{214}) = -(\omega^2 + \omega) = 1$

(28) **(B).** Given $z = \dfrac{3 + 2i\sin\theta}{1 - 2i\sin\theta}$ is purely imaginary.

So, real part becomes zero.

$$z = \left(\frac{3 + 2i\sin\theta}{1 - 2i\sin\theta}\right) \times \left(\frac{1 + 2i\sin\theta}{1 + 2i\sin\theta}\right)$$

$$z = \frac{(3 - 4\sin^2\theta) + i(8\sin\theta)}{1 + 4\sin^2\theta}$$

Now, $\text{Re}(z) = 0$; $\dfrac{3 - 4\sin^2\theta}{1 + 4\sin^2\theta} = 0$; $\sin^2\theta = \dfrac{3}{4}$

$\sin\theta = \pm\dfrac{\sqrt{3}}{2} \Rightarrow \theta = -\dfrac{\pi}{3}, \dfrac{\pi}{3}, \dfrac{2\pi}{3} \quad \because \ \theta \in \left(-\dfrac{\pi}{2}, \pi\right)$

Then sum of the elements in A is $-\dfrac{\pi}{3} + \dfrac{\pi}{3} + \dfrac{2\pi}{3} = \dfrac{2\pi}{3}$

(29) **(A).** $z = \dfrac{\sqrt{3}}{2} + \dfrac{i}{2} = \cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}$

$$z^5 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = \frac{-\sqrt{3} + i}{2}$$

$$z^8 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\left(\frac{1 + i\sqrt{3}}{2}\right)$$

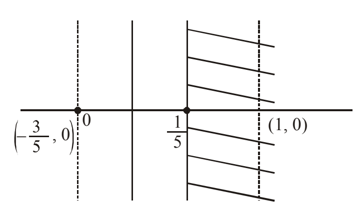
$$(1 + iz + z^5 + iz^8)^9 = \left(1 + \frac{i\sqrt{3}}{2} - \frac{1}{2} - \frac{\sqrt{3}}{2} + \frac{i}{2} - \frac{i}{2} + \frac{\sqrt{3}}{2}\right)^9$$

$$= \left(\frac{1 + i\sqrt{3}}{2}\right)^9 = \cos 3\pi + i\sin 3\pi = -1$$

(30) **(C).** $|z| < 1$

$5\omega(1 - z) = 5 + 3z$

$5\omega - 5\omega z = 5 + 3z$



$z = \dfrac{5\omega - 5}{3 + 5\omega}$; $|z| = \left|\dfrac{5\omega - 5}{3 + 5\omega}\right| < 1$

$5|\omega - 1| < |3 + 5\omega|$

$5|\omega - 1| < 5\left|\omega + \dfrac{3}{5}\right|$; $|\omega - 1| < 5\left|\omega - \left(-\dfrac{3}{5}\right)\right|$

(31) **(C).** Given $a > 0$

$$z = \frac{(1+i)^2}{a-i} = \frac{2i\,(a+i)}{a^2+1}$$

Also, $|z| = \sqrt{\frac{2}{5}} \Rightarrow \frac{2}{\sqrt{a^2+1}} = \sqrt{\frac{2}{5}} \Rightarrow a = 3$

So, $\bar{z} = \frac{-2i\,(3-i)}{10} = \frac{-1-3i}{5}$

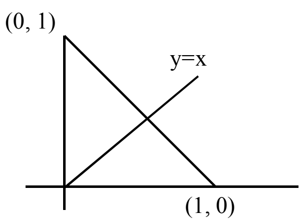
(32) **(B).** $|z|.|w| = 1$

$z = re^{i\,(\theta+\pi/2)}$ and $w = \frac{1}{r}e^{i\theta}$

$\bar{z}\cdot w = e^{-i\,(\theta+\pi/2)}\cdot e^{i\theta} = e^{-i\,(\pi/2)} = -i$

$z\cdot \bar{w} = e^{i\,(\theta+\pi/2)}\cdot e^{-i\theta} = e^{i\,(\pi/2)} = i$

(33) **(D).**



$|z-i| = |z-1|$; $y = x$

(34) **(C).** Put $z = x + 10i$

$\therefore$ $2\,(x+10i) - n = (2i-1)\,.\,[2\,(x+10i)+n]$

Compare real and imginary coefficients

$x = -10, n = 40$

(35) **(C).** $z = x + iy$

$$\left(\frac{z-1}{2z+i}\right) = \frac{(x-1)+iy}{2\,(x+iy)+i}$$

$$= \frac{(x-1)+iy}{2x+(2y+1)\,i} \times \frac{2x-(2y+1)\,i}{2x-(2y+1)\,i}$$

$$\text{Re}\left(\frac{z-1}{2z+i}\right) = \frac{2x\,(x-1)+y\,(2y+1)}{(2x)^2+(2y+1)^2} = 1$$

$\Rightarrow$ $2x^2 + 2y^2 - 2x + y = 4x^2 + 4y^2 + 4y + 1$

$\Rightarrow$ $2x^2 + 2y^2 + 2x + 3y + 1 = 0$

$\Rightarrow$ $x^2 + y^2 + x + \frac{3}{2}y + \frac{1}{2} = 0$

Circle with centre $\left(-\frac{1}{2}, -\frac{3}{4}\right)$

$$r = \sqrt{\frac{1}{4} + \frac{9}{16} - \frac{1}{2}} = \sqrt{\frac{4+9-8}{16}} = \frac{\sqrt{5}}{4}$$

(36) **(B).** Let $z = \alpha \pm i\beta$ be roots of the equation

So $2\alpha = -b$ and $\alpha^2 + \beta^2 = 45$,

$(\alpha+1)^2 + \beta^2 = 40.$ So $(\alpha+1)^2 - \alpha^2 = -5$

$\Rightarrow$ $2\alpha + 1 = -5 \Rightarrow 2\alpha = -6$, so $b = 6$

Hence, $b^2 - b = 30$

(37) **(A).** $\alpha = \omega$, $b = 1 + \omega^3 + \omega^6 + \ldots\ldots = 101$

$a = (1+\omega)\,(1 + \omega^2 + \omega^4 + \ldots\ldots\omega^{198} + \omega^{200})$

$$= (1+\omega)\frac{(1-(\omega^2)^{101})}{1-\omega^2} = \frac{(1+\omega)\,(1-\omega)}{1-\omega^2} = 1$$

Equation : $x^2 - (101+1)\,x + (101)\times 1 = 0$

$\Rightarrow$ $x^2 - 102x + 101 = 0$

(38) **(C).** $\left|\frac{z-i}{z+2i}\right| = 1 \Rightarrow |z-i| = |z+2i|$

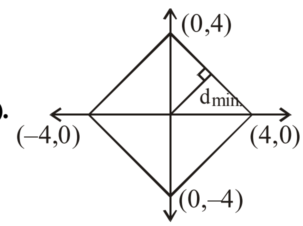
$\Rightarrow$ z lies on perpendicular bisector of (0, 1) and (0, –2).

$\Rightarrow$ $\text{Im}\, z = -1/2$

Let $z = x - \frac{i}{2}$; $|z| = 5/2 \Rightarrow x^2 = 6$

$$\therefore\quad |z+3i| = \left|x + \frac{5i}{2}\right| = \sqrt{x^2 + \frac{25}{4}} = \sqrt{6 + \frac{25}{4}} = \frac{7}{2}$$

(39) **(D).**



$z = x + iy$ $\qquad |x| + |y| = 4$

$|z| = \sqrt{x^2+y^2} \Rightarrow |z|_{min} = \sqrt{8}$

$|z|_{max} = 4 = \sqrt{16}$

So $|z|$ cannot be $\sqrt{7}$

(40) **(C).** $z = \frac{3+i\sin\theta}{4-i\cos\theta} \times \frac{4+i\sin\theta}{4+i\cos\theta}$

As z is purely real

$\Rightarrow$ $3\cos\theta + 4\sin\theta = 0 \Rightarrow \tan\theta = -3/4$

$$\arg(\sin\theta + i\cos\theta) = \pi + \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$= \pi + \tan^{-1}\left(-\frac{4}{3}\right) = \pi - \tan^{-1}\left(-\frac{4}{3}\right)$$