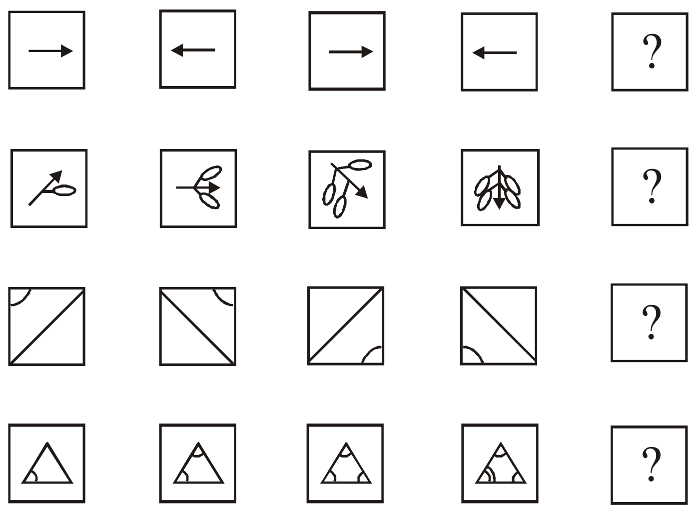
5

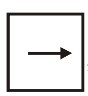
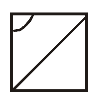
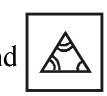
ARITHMETIC PROGRESSION

INTRODUCTION

In practical life you must have observed many things follow a certain pattern, such as the petals of a sunflower, the holes of a honeycomb, the grains on a maize cob, the spirals on a pineapple on a pipe cone etc.

In our day-to-day life, we see patterns of geometric figures on clothes, pictures, posters etc. They make the learners motivated to form such new patterns. This becomes a topic of interest and knowledge to predict the next figure in a pattern.Consider the following patterns :



Can you predict the next figures in (i), (ii), (iii) and (iv) ? A little careful study of the above patterns shows that the next figures in (iv) are , ,  and  respectively.

Think about these and try to find the reasons for those.

Likewise number patterns are also faced by learners in their study. Number patterns play an important role in the field of Mathematics. Let us study the following number patterns :

(i) 2, 4, 6, 8, 10, ... (ii) $1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, \ldots\ldots\ldots$ (iii) 10, 7, 4, 1, –2, ... (iv) 2, 4, 8, 16, 32, ...

(v) $4, \frac{1}{2}, \frac{1}{16}, \frac{1}{128}, \ldots\ldots.$ (vi) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots.$ (vii) 1, 11, 111, 1111, 11111, ...

It is an interesting study to find whether some specific names have been given to some of the above number patterns and the methods of finding some next terms of the given patterns.

Observing various patterns various sequences were defined to solve various summation problems.

A sequence is a function whose domain is the natural numbers. Instead of using the f (x) notation, however, a sequence is listed using the a_n notation. There are infinite sequences whose domain is the set of all positive integers, and there are finite sequences whose domain is the set of the first n positive integers.

When you define a sequence, you must write the general term (nth term or a_n). There are sometimes more than one sequence that is possible if just the first few terms are given.

Among various sequences A.P.(Arithmetic progression),G.P.(Geometric progression) and H.P(Harmonic progression) are most common.

Idea on A.P. was from the mathematician Carl Friedrich Gauss, who, as a young boy, stunned his teacher by adding up

$1 + 2 + 3 + ... + 99 + 100$ within a few minutes. Here's how he did it:

He counted 101 terms in the series, of which 50 is the middle term. He also realised that adding the first and last numbers, 1 and 100, gives, 101; and adding the second and second last numbers, 2 and 99, gives 101, as well as $3 + 98 = 101$ and so on.

Thus he concluded that there are 50 sets of 101 and the middle term is 50. So the sum of the series is:

$$50 (1 + 100) + 50 = 5050.$$

This can be rewritten as: $100/2 (1 + 100) = 5050$

DEFINING A SEQUENCE

There are two common ways to define a sequence by specifying the general term.

The first is to use a form that only depends on the number of the term, n. To find the first five terms when you know the general term, simply substitute the values 1, 2, 3, 4, and 5 into the general form for n and simplify.

Consider the sequence defined by the general term $a_n = 3n - 2$

The first five terms are found by plugging in 1, 2, 3, 4, and 5 for n.

(1) $3(1) - 2 = 1$ (2) $3 (2) - 2 = 4$ (3) $3 (3) - 2 = 7$ (4) $3(4) - 2 = 10$ (5) $3(5) - 2 = 13$

Therefore, the first five terms of the sequence are 1, 4, 7, 10, 13

Now consider the sequence defined by the general term an = 1 / n.

The first five terms are 1/1, 1/2, 1/3, 1/4, and 1/5.

The second way is to recursively define a sequence. A recursive definition uses the current and/or previous terms to define the next term. You can think of a_{k+1} being the next term, a_k being the current term, and a_{k-1} being the previous term.

Consider the sequence where $a_1 = 5$ and $a_{k+1} = 2 a_k - 1$.

You can read that last part as "the next term is one less than twice the current term"

The first five terms are:

(1) 5 (by definition), (2) $2 (5) - 1 = 9$ (twice the first term of 5 minus 1),
(3) $2 (9) - 1 = 17$ (twice the second term of 9 minus 1), (4) $2 (17) - 1 = 33$ (twice the third term of 17 minus 1),
(5) $2 (33) - 1 = 62$ (twice the fourth term of 33 minus 1)

Now consider the sequence defined by $a_1 = 2$, $a_2 = 1$, and $a_{k+2} = 3a_k - a_{k+1}$

You can read that last part as "the next term is 3 times the last term minus the current term"

The first five terms are: **(1)** 2 (by definition), **(2)** 1 (by definition),

(3) $3 (2) - 1 = 5$ (3 times first term minus second term), Note that when k = 1, the sequence gets written as $a_{1+2} = 3a_1 - a_{1+1}$ which becomes $a_3 = 3a_1 - a_2$. Since $a_1 = 2$ and $a_2 = 1$, this is where the $3 (2)- 1 = 5$ comes from.

(4) $3 (1) - 5 = -2$ (3 times second term minus third term),

(5) $3 (5) - (-2) = 17$ (3 times third term minus fourth term)

One famous example of a recursively defined sequence is the Fibonacci Sequence. The first two terms of the Fibonacci Sequence are 1 by definition. Every term after that is the sum of the two preceding terms.

The Fibonacci Sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... $a_{n+1} = a_n + a_{n-1}$.

Fibonacci sequences occur frequently in nature. For example, take a leaf on a stem of many plants (like cherry, elm, or pear trees). Count the number of leaves until you reach one directly in line with the one you selected. The total number of leaves (not including the first one) is usually a Fibonacci number. If the left and right handed spirals on a pine cone, sunflower seed heads, or pineapples are counted, the two numbers are often consecutive Fibonacci numbers.

RECOGNITION OF NUMBER PATTERNS

Suppose you want to purchase a handkerchief whose cost is Rs 5. If you want to purchase two handkerchiefs, then you have to pay Rs 10. Therefore, if the number of handkerchiefs is increased by one successively, the respective cost (in Rs) would increase by 5 every time, i.e., the respective costs of one, two, three, ... handkerchiefs would be 5, 10, 15, ...

Can you recognize the relationship between any consecutive numbers of the above pattern ?

If you observe the pattern carefully, you will find that each successive number, other than the first number, is obtained by adding a constant number to the preceding number. Therefore, these numbers form a number pattern. Each number of the number pattern is called a term.

Some examples of numbers patterns are given below :

(a) 1, 2, 3, 4, ... (b) 4, 2, 0, –2, –4, ... (c) 3, 6, 9, 12, ... (d) 13, 9, 5, 1, –3, –7, ...

(e) $2\frac{1}{4}, 6\frac{1}{4}, 10\frac{1}{4}, 14\frac{1}{4}, ...$ (f) 1, 2, 4, 8, 16, ... (g) $16, 4, 1, \frac{1}{4}, ...$ (h) 5, 25, 125, 625, ...

In number patterns (a), (c) and (e), each successive term, other than the first term, can be obtained by adding a constant number 1, 3 and 4 respectively to the preceding term.

In the number patterns (b) and (d), each successive term, other than the first term, can be obtained by subtracting a constant number 2 and 4 respectively from the preceding term.

But in the number patterns (f), (g) and (h), each term, other than the first term, can be obtained by multiplying the preceding term by a constant number 2, $\frac{1}{4}$ and 5 respectively.

ARITHMETIC PROGRESSION

You are all aware that salaries of employees are often calculated on the basis of their basic salary, plus fixed increments (increases) for each year of service and all other usual allowances.

Suppose a person begins to work for a firm in the scale of Rs 4500-125-7000.

Then in successive years his basic salary (in Rs) will be 4500, 4625, 4750, 4875, 5000 and he will get usual allowances on the basic salary.

The first term of this pattern is 4500 and each successive term, other than the first term, can be obtained by adding a constant number (here increment in Rs) 125 to the preceding number. We can say that the terms of the pattern progress arithmetically. Such a number pattern is called an Arithmetic Progression (abbreviated as A.P). The constant number is called common difference. Thus Arithmetic Progression is defined as a series in which difference between any two consecutive terms is constant throughout the series. This constant difference is called Common difference . If 'a' is the first term and 'd' is the common difference, then an AP can be written as

a + (a + d) + (a + 2d) + (a + 3d) + ...

Probably the simplest arithmetic progression is of natural numbers : 1, 2, 3, 4, 5, 6, ...

The multiplication tables are all familiar A.P's,

2, 4, 6, 8, 10, ...
3, 6, 9, 12, 15, ...
4, 8, 12, 16, 20, ...

An arithmetic progression can start from any number, positive or negative for example :

5, 8, 11, 14, 17, ...

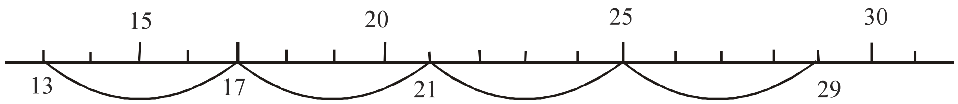
$2\frac{1}{4}, 6\frac{1}{4}, 10\frac{1}{4}, 14\frac{1}{4}, ...$

–7, –2, 3, 8, ...

Rational numbers can serve as the common difference, as in the A.P : $9, 11\frac{1}{2}, 14, 16\frac{1}{2}, 19, ...$

and negative numbers, as in the A.P, 10, 7, 4, 1, –2, –5, ...

An arithmetic progression can be represented on the number line by a series of points placed the same distance apart. For example, the progression 13, 17, 21, 25, ... is shown in Fig.



Finite A.P. : Number of terms are finite.

Example :

(a) The minimum temperatures (in degree celsius) recorded for a week in the month of January in a city, arranged in ascending order are – 3.1, – 3.0, – 2.9, – 2.8, – 2.7, – 2.6, – 2.5

(b) The cash prizes (in Rs) given by a school to the toppers of Classes I to XII are, respectively, 200, 250, 300, 350, . . ., 750.

Infinite A.P. : Do not have last term.

Example : (i) 100, 70, 40, 10, . . . **(ii)** – 3, –2, –1, 0, . . . **(iii)** –1.0, –1.5, –2.0, –2.5, . . .

Common Difference : Since this difference is common to all consecutive pairs of terms, it is called the common difference. It is denoted by d. If the difference in consecutive terms is not constant, then the sequence is not arithmetic. The common difference can be found by subtracting two consecutive terms of the sequence.

The formula for the common difference of an arithmetic sequence is: $d = a_{n+1} - a_n$

It can be positive, negative or zero.

Example 1 :

For the AP : $\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, ...$ write the first term a and the common difference d.

Sol. Here, $a = \frac{3}{2}$, $d = \frac{1}{2} - \frac{3}{2} = -1$

Remember that we can find d using any two consecutive terms, once we know that the numbers are in AP.

Example 2 :

Write the first four terms of the AP, when the first term a and the common difference d are given as follows :

(i) a = – 4, d = 0 (ii) a = 1.5, d = 1.5

Sol. (i) The first four terms are

$-4, -4 + 0, - + 2 \times 0, -4 + 3 \times 0$ i.e., $-4, -4, -4, -4,$

(ii) The first four terms are

$1.5, 1.5 + 1.5, 1.5 + 2 \times 1.5, 1.5 + 3 \times 1.5$ i.e., $1.5, 3, 4.5, 6$

Example 3 :

For the following APs, write the first term and the common difference :

(i) 0.2, 0.4, 0.6, 0.8 ,............ (ii) –5, –1, 3, 7, ...

Sol. (i) Here, first term (a) = 0.2 and common difference (d) = $a_2 - a_1 = 0.4 - 0.2 = 0.2$

(ii) Here, first term (a) = –5 and common difference (d) = $a_2 - a_1 = (-1) - (-5) = 4$

SELF CHECK

Q.1 In which of the following situations, does the list of numbers involved make an arithmetic progression, and why?

(i) The taxi fare after each km when the fare is Rs 15 for the first km and Rs 8 for each additional km.

(ii) List of integer.

(iii) The amount of air present in a cylinder when a vacuum pump removes 1/4 of the air remaining in the cylinder at a time.

(iv) The cost of digging a well after every metre of digging, when it costs Rs 150 for the first metre and rises by Rs 50 for each subsequent metre.

(v) The amount of money in the account every year, when Rs 10000 is deposited at compound interest at 8 % per annum.

Q.2 Write first four terms of the AP, when the first term a and the common difference d are given as follows:

(i) a = 4, d = – 3 (iv) a = – 1, d = 1/2

Q.3 Which of the following are APs ? If they form an AP, find the common difference d and write three more terms.

(i) – 1.2, – 3.2, – 5.2, – 7.2, . . . (ii) $3, 3+\sqrt{2}, 3+2\sqrt{2}, 3+3\sqrt{2}, ...$

(iii) 0.2, 0.22, 0.222, 0.2222, . . . (iv) 0, – 4, – 8, –12, . . .

ANSWERS

(1) (i) Yes. 15, 23, 21, ... forms an AP as each succeeding term is obtained by adding in 8 in its preceding term.

(ii) Yes. (iii) No. Volumes are $V, \frac{3V}{4}, \left(\frac{3V}{4}\right)^2, L$ (iv) Yes. 150, 200, 250, ... form an AP.

(v) No. Amounts are $10000\left(1+\frac{8}{100}\right), 10000\left(1+\frac{8}{100}\right)^2, 10000\left(1+\frac{8}{100}\right)^3, L$

(2) (i) 4, 1, –2, –5 (ii) $-1, -\frac{1}{2}, 0, \frac{1}{2}$

(3) (i) Yes. d = –2, 9.2, –11.2, –13.2 (ii) Yes. $d = \sqrt{2}, 3+4\sqrt{2}, 3+5\sqrt{2}, 3+6\sqrt{2}$ (iii) No

(iv) Yes. d = – 4, – 16, – 20, – 24

General Term : An arithmetic sequence is a linear function. Instead of y = mx+b, we write $a_n = dn + c$ where d is the common difference and c is a constant (not the first term of the sequence, however).

A recursive definition, since each term is found by adding the common difference to the previous term is

$a_{k+1} = a_k + d$

For any term in the sequence, we've added the difference one less time than the number of the term. For example, for the first term, we haven't added the difference at all (0 times). For the second term, we've added the difference once. For the third term, we've added the difference two times.

The formula for the general term of an arithmetic sequence is: $a_n = a_1 + (n – 1) d$

Example 4 :

Find the n^{th} term of the 2, 7, 12, 17, ...

Sol. Here, first term (a) = 2

Common difference = $a_2 – a_1$ = 7 – 2 = 5 $\therefore$ nth term (a_n) = a + (n–1) d = 2 + (n – 1) (5) = 2 + 5n – 5 = 5n – 3

Example 5 :

Which term of the AP : 21, 18, 15, . . . is – 81? Also, is any term 0 ? Give reason for your answer.

Sol. Here, a = 21, d = 18 – 21 = – 3 and a_n = – 81, and we have to find n.

As a_n = a + (n – 1) d,

we have – 81 = 21 + (n – 1) (– 3)

– 81 = 24 – 3n

– 105 = – 3n. So, n = 35

Therefore, the 35th term of the given AP is – 81.

Next, we want to know if there is any n for which $a_n = 0$. If such an n is there, then

$$21 + (n - 1)(-3) = 0, \quad \text{i.e.,} \quad 3(n - 1) = 21 \quad \text{i.e., } n = 8$$

So, the eighth term is 0.

Example 6 :

Find the common difference of the given APs and write the next three terms of each.

(i) –10, –6, –2, 2, ... (ii) $-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, ...$

Sol. (i) Here, $a_2 - a_1 = -6 - (-10) = 4$

$a_3 - a_2 = (-2) - (-6) = 4$

$a_4 - a_3 = (2) - (-2) = 4$

$\therefore$ The common difference of the AP is 4.

Now, $a_5 = a + 4d = -10 + 4 \times (4) = -10 + 16 = 6$

$a_6 = a + 5d = -10 + 5 \times (4) = -10 + 20 = 10$

$a_7 = a + 6d = -10 + 6 \times (4) = -10 + 24 = 14$

Thus, the next three terms of the given AP are 6, 10 and 14.

(ii) Here, $a_2 - a_1 = -\frac{1}{2} - \left(-\frac{1}{2}\right) = 0$; $a_3 - a_2 = -\frac{1}{2} - \left(-\frac{1}{2}\right) = 0$; $a_4 - a_3 = -\frac{1}{2} - \left(-\frac{1}{2}\right) = 0$

$\therefore$ The common difference of the AP is 0.

Now, $a_5 = a + 4d = -\frac{1}{2} + 4\times = 0 = -\frac{1}{2} + 0 = -\frac{1}{2}$; $a_6 = a + 5d = -\frac{1}{2} + 5\times = 0 = -\frac{1}{2} + 0 = -\frac{1}{2}$

$a_7 = a + 6d = -\frac{1}{2} + 6\times = 0 = -\frac{1}{2} + 0 = -\frac{1}{2}$

Thus, the next three terms of the given AP are $-\frac{1}{2}$, $-\frac{1}{2}$ and $-\frac{1}{2}$.

Example 7 :

How many terms are there in the AP : 25, 50, 75, 100, ..., 1000 ?

Sol. Let there b n terms in the given AP. Then,

a_n (nth term) = a + (n – 1) d = 1000

Here, a = 25 and d = 25

$\therefore$ 25 + (n – 1) (25) = 1000 $\Rightarrow$ 25n = 1000 $\Rightarrow$ n = 40

Hence there are 40 terms in the AP.

Example 8 :

How many two-digit numbers are divisible by 3 ?

Sol. The list of two-digit numbers divisible by 3 is :

12, 15, 18, . . . , 99 Here, a = 12, d = 3, $a_n = 99$.

As $a_n = a + (n - 1) d$,

we have 99 = 12 + (n – 1) × 3

i.e., 87 = (n – 1) × 3 i.e., n – 1 = 87/3 = 29 i.e., n = 29 + 1 = 30

So, there are 30 two-digit numbers divisible by 3.

Example 9 :

Determine the AP whose 3rd term is 5 and the 7th term is 9.

Sol. Let a and d respectively be the first term and common difference of the given AP.

Then, 3rd term = $a_3 = a + 2d = 5$ .... (1)

and 7th term = $a_7 = a + 6d = 9$ .... (2)

From (1) and (2), we have, $4d = 4 \Rightarrow d = 1$

From (1), we have, $a + 2(1) = 5 \Rightarrow a + 2 = 5 \Rightarrow a = 3$

Thus, the required AP is 3, 3 + 1, 3 + 2 (1), 3 + 3 (1), ... i.e., 3, 4, 5, 6, ...

Example 10 :

A sum of Rs 1000 is invested at 8% simple interest per year. Calculate the interest at the end of each year. Do these interests form an AP ? If so, find the interest at the end of 30 years making use of this fact.

Sol. The formula to calculate simple interest is given by

$$\text{Simple Interest} = \frac{P \times R \times T}{100} \quad (P = \text{Basic amount, } R = \text{Rate of interest, } T = \text{time period})$$

So, the interest at the end of the 1st year = $\text{Rs.}\frac{1000 \times 8 \times 1}{100} = \text{Rs.}80$

The interest at the end of the 2nd year = $\text{Rs.}\frac{1000 \times 8 \times 2}{100} = \text{Rs.}160$

The interest at the end of the 3rd year = $\text{Rs.}\frac{1000 \times 8 \times 3}{100} = \text{Rs.}240$

Similarly, we can obtain the interest at the end of the 4th year, 5th year, and so on.

So, the interest (in Rs) at the end of the 1st, 2nd, 3rd, . . . years, respectively are 80, 160, 240, . . .

Example 11 :

Find the value of k for which 2k + 7, 6k – 2 and 8k + 4 form three consecutive terms of AP.

Sol. We know that three terms p, q, r form consecutive terms of AP if and only if $2q = p + r$

$\therefore\ 2(6k - 2) = (2k + 7) + (8k + 4) \Rightarrow 12k - 4 = 10k + 11 \Rightarrow 2k = 15 \Rightarrow k = 15/2$

Example 12 :

Determine the AP whose 3rd term is 5 and the 7th term is 9.

Sol. We have, $a_3 = a + (3 - 1)d = a + 2d = 5$ .... (1)

and $a_7 = a + (7 - 1)d = a + 6d = 9$ .... (2)

Solving the pair of linear equations (1) and (2), we get a = 3, d = 1

Hence, the required AP is 3, 4, 5, 6, 7, . . .

Example 13 :

Which term of arithmetic progression $19, 18\frac{1}{5}, 17\frac{2}{5}, ...$ is the first negative term ?

Sol. Let a_n be the first negative term. Then $a_n < 0 \Rightarrow a + (n - 1)d < 0$

$\Rightarrow 19 + (n - 1)(-4/5) < 0$

$$\Rightarrow \frac{4}{5}(n-1) > 19 \Rightarrow n - 1 > \frac{19 \times 5}{4} = \frac{95}{4} \Rightarrow n > \frac{95}{4} + 1 = \frac{99}{4}$$

Hence, 25th term is the first negative term.

SELF CHECK

Q.1 Which term of the AP 3, 10, 17, ... will be 84 more than its 13th term ?

Q.2 If seven times the 7th term of an AP is equal to eleven times its 11th term, show that the 18th term of the AP is zero.

Q.3 If the common difference of an AP equal the first term, prove that the ratio of its mth and nth terms is m : n

Q.4 Which term of the AP : 3, 8, 13, 18, . . . , is 78?

Q.5 Check whether – 150 is a term of the AP : 11, 8, 5, 2 . . .

Q.6 An AP consists of 50 terms of which 3rd term is 12 and the last term is 106. Find the 29th term.

Q.7 The 17th term of an AP exceeds its 10th term by 7. Find the common difference.

Q.8 Two APs have the same common difference. The difference between their 100th terms is 100, what is the difference between their 1000th terms?

Q.9 For what value of n, are the nth terms of two APs: 63, 65, 67, . . . and 3, 10, 17, . . . equal ?

ANSWERS

(1) 25th term **(4)** 16th term **(5)** No **(6)** 84 **(7)** 1 **(8)** 100 **(9)** 13

SUM OF FIRST N TERMS OF AN ARITHMETIC SEQUENCE

A series is a sum of a sequence. We want to find the nth partial sum or the sum of the first n terms of the sequence. We will denote the nth partial sum as S_n.

Consider the arithmetic series $S_5 = 2 + 5 + 8 + 11 + 14$. There is an easy way to calculate the sum of an arithmetic series. $S_5 = 2 + 5 + 8 + 11 + 14$

The key is to switch the order of the terms. Addition is commutative, so changing the order doesn't change the sum.

$$S_5 = 14 + 11 + 8 + 5 + 2$$

Now, add those two equations together. $2 \times S_5 = (2 + 14) + (5 + 11) + (8 + 8) + (11 + 5) + (14 + 2)$

Notice that each of those sums on the right hand side is 16. Instead of writing 16 (the sum of the first and last terms) five times, we can write it as 5 × 16 or 5 × (2 + 14)

$$2 \times S_5 = 5 \times (2 + 14)$$

Finally, divide the whole thing by 2 to get the sum and not twice the sum $S_5 = 5/2 \times (2 + 14)$

The 2+14 is not simplified puposely so that you can see where the numbers come from.

This sum would be 5/2 × (16) = 5 (8) = 40.

Now, if we try to figure out where the different parts of that formula come from, we can conjecture about a formula for the nth partial sum. The 5 is because there were five terms, n. The 16 is the sum of the first and last terms, $a_1 + a_n$. The 2 is because we added the sum twice. Therefore, the sum of the first n terms of an arithmetic sequence is $S_n = n/2 \times (a_1 + a_n)$

There is another formula that is sometimes used for the nth partial sum of an arithmetic sequence. It is obtained by substituting the formula for the general term into the above formula and simplifying. The preferred method is to go ahead and find the nth term, and then just plug that number into the formula.

$$S_n = n/2 \times (2a_1 + (n - 1) d$$

Example 14 :

Consider the finite sequence of numbers 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31.

Sol. This sequence has the property that the difference between successive terms is constant and equal to 2.

It follows that the n-th term is obtained from the first term by adding (n – 1) × 2, and is therefore equal to 1 + 2 (n – 1).

On the other hand, if we want to add all the numbers together, then we observe that

$$1 + 31 = 3 + 29 = 5 + 27 = 7 + 25 = 9 + 23 = 11 + 21 = 13 + 19 = 15 + 17,$$

so that the numbers can be paired off in such a way that the sum of the pair is always the same and equal to 32.

Note now that there are 16 numbers which form 8 pairs. It follows that

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 + 25 + 27 + 29 + 31 = 8 \times 32 = 256.$$

Example 15 :

If the sum of the first 14 terms of an AP is 1050 and its first term is 10, find the 20th term.

Sol. Here, S_{14} = 1050, n = 14, a = 10.

As $S_n = \frac{n}{2}[2a + (n-1)d]$ so, $1050 = \frac{14}{2}[20 + 3d] = 140 + 91d$

i.e., 910 = 91d or, d = 10

Therefore, a_{20} = 10 + (20 – 1) × 10 = 200, i.e. 20th term is 200.

Example 16 :

How many terms of the AP 24, 21, 18, 15, ... must be taken so that their sum is 78 ?

Sol. Here, a = 24 and d = –3

As $S_n = \frac{n}{2}[2a + (n-1)d]$. We have $78 = \frac{n}{2}[2 \times 24 + (n-1)(-3)]$

n [48 – 3 (n – 1)] = 156 ; n [51 – 3n] = 156

$51n - 3n^2 = 156$; $3n^2 - 51n + 156 = 0$

$n^2 - 17n + 52 = 0$; (n – 13) (n – 4) = 0 ; n = 13 or 4

Both the values are admissible. Hence, the number of terms is either 4 or 13.

Example 17 :

Consider the finite sequence of numbers 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32.

Sol. I^st method : This sequence has the property that the difference between successive terms is constant and equal to 3. If we want to add all the numbers together, then we observe that

2 + 32 = 5 + 29 = 8 + 26 = 11 + 23 = 14 + 20 = 2 × 17,

so that the numbers other than the middle one can be paired off in such a way that the sum of the pair is always the same and equal to 34. Note now that there are 11 numbers which form 5 pairs, as well as the number 17 which is equal to half the sum of a pair. We can therefore pretend that there are 512 pairs, each adding to 34. It follows that

2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32 = 11

2 × 34 = 187.

IInd method : $S = \frac{n}{2}(a_1 + a_n) = \frac{11}{2}(2 + 32) = 187$

Example 18 :

Find the sum of first 24 terms of the list of numbers whose nth term is given by $a_n = 3 + 2n$

Sol. As $a_n = 3 + 2n$,

so, $a_1 = 3 + 2 = 5$

$a_2 = 3 + 2 \times 2 = 7$

$a_3 = 3 + 2 \times 3 = 9$

....

List of numbers becomes 5, 7, 9, 11, . . .

Here, 7 – 5 = 9 – 7 = 11 – 9 = 2 and so on.

So, it forms an AP with common difference d = 2.

To find S_{24}, we have n = 24, a = 5, d = 2.

Therefore, $S_{24} = \frac{24}{2}[2 \times 5 + (24-1) \times 2]$ = 12 [10 + 46] = 672

So, sum of first 24 terms of the list of numbers is 672.

Example 19 :

Suppose that the 4-th and 7-th terms of an arithmetic progression are equalto 9 and –15 respectively. Find sum of first 10 terms.

Sol. $9 = a + 3d$,

$15 = a + 6d$,

So that 3d = -24. It follows that d = –8 and a = 33.

The arithmetic progression is given by 33, 25, 17, 9, 1, –7, –15, ...

The 10-th term is given by a + 9d = 33 – 72 = –39. The sum of the first 10 terms is equal to 10

2 × (33 – 39) = – 30.

Example 20 :

The houses of a row are numbered consecutively from 1 to 49. Show that there is a value of x such that the sum of the numbers of the houses preceding the house membered x is equal to the sum of the numbers of the houses following it. Find this value of x.

Sol. Here, according to the problem,

$S_{x-1} = S_{49} - S_x$ .... (1)

and a = 1, d = 1

So, $S_x = \frac{x}{2}[2 \times 1 + (x-1)(1)] = \frac{x}{2}[2 + x - 1] = \frac{x}{2}[1 + x] = \frac{x(1+x)}{2}$

$\Rightarrow S_{x-1} = \frac{x-1}{2}[x] = \frac{x(x-1)}{2} \Rightarrow S_{49} = \frac{49}{2}[50] = 1225$

From (1), we have, $\frac{x(x-1)}{2} = 1225 - \frac{x(1+x)}{2}$

$\Rightarrow x^2 - x = 2450 - x - x^2 \Rightarrow 2x^2 = 2450 \Rightarrow x^2 = 1225 \Rightarrow x = 35.$

Thus, the required value of x is 35.

SELF CHECK

Q.1 Find the sum of the following APs:

(i) 2, 7, 12, . . ., to 10 terms. (iv) $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}$,........, to 11 terms.

Q.2 In an AP :

(i) given a = 7, $a_{13} = 35$, find d and S_{13}. (ii) given $a_3 = 15$, $S_{10} = 125$, find d and a_{10}.

(iii) given a = 2, d = 8, $S_n = 90$, find n and a_n. (iv) given a = 3, n = 8, S = 192, find d.

Q.3 The first term of an AP is 5, the last term is 45 and the sum is 400. Find the number of terms and the common difference.

Q.4 Find the sum of first 22 terms of an AP in which d = 7 and 22nd term is 149.

Q.5 If the sum of the first n terms of an AP is $4n - n^2$, what is the first term (that is S_1)? What is the sum of first two terms? What is the second term? Similarly, find the 3rd, the 10th and the nth terms.

Q.6 Find the sum of the first 15 multiples of 8.

Q.7 Find the number of terms of the A.P., 54, 51, 48, ... so that their sum is 513.

Q.8 Sum of the first five and twenty-one terms of an AP are 320 and –168 respectively. Find the first term, common difference and the sum upto the 12th term of the AP.

Q.9 Find the sum of all natural numbers between 1 and 100 which are divisible by 3.

Q.10 Find the sum of natural numbers from 1 to 200, excluding those divisible by 5.

<u>**ANSWERS**</u>

(1) (i) 245 **(ii)** $\frac{33}{20}$ **(2) (i)** $d = \frac{7}{3}, S_{13} = 273$ **(ii)** $d = -1, a_{10} = 8$ **(iii)** $n = 5, a_n = 34$ **(iv)** $d = 6$

(3) n = 16, d = 8/3 **(4)** Sum = 1661

(5) $S_1 = 3, S_2 = 4$; $a_2 = S_2 - S_1 = 1$; $S_3 = 3, a_3 = S_3 - S_2 = -1, a_{10} = S_{10} - S_9 = -15,$

$a_n = S_n - S_{n-1} = 5 - 2n$

(6) 960 **(7)** 18 or 19 terms **(8)** a = 82, d = – 9 a and $S_{12} = 390$

(9) 1683 **(10)** 16000

EXTRA EDGE

(i) Sum of first n natural numbers $\Rightarrow \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

(ii) Sum of first n odd natural numbers $\Rightarrow \sum_{r=1}^{n} (2r-1) = n^2$

(iii) Sum of first n even natural numbers $\Rightarrow \sum_{r=1}^{n} 2r = n(n+1)$

(iv) Sum of squares of first n natural numbers $\Rightarrow \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

(v) Sum of cubes of first n natural numbers $\Rightarrow \sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$

(vi) Sum of fourth powers of first n natural numbers

$$\Rightarrow \sum n^4 = 1^4 + 2^4 + ... + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

(vii) If r^{th} term of an A.P. , $T_r = Ar^3 + Br^2 + Cr + D$, then

sum of n term of AP is $S_n = \sum_{r=1}^{n} T_r = A\sum_{r=1}^{n} r^3 + B\sum_{r=1}^{n} r^2 + C\sum_{r=1}^{n} r + D\sum_{r=1}^{n} 1$

(viii) If for an A.P. p^{th} term is q, q^{th} term is p then m^{th} term is = p + q – m

(ix) If for an AP sum of p terms is q, sum of q terms is p, then sum of (p + q) term is - (p + q).

(x) If for an A.P. sum of p terms is equal to sum of q terms then sum of (p + q) terms is zero.

(xi) If sum of n terms S_n is given then general term $T_n = S_n - S_{n-1}$ where S_{n-1} is sum of (n –1) terms of A.P.

(xii) Common difference of AP is given by $d = S_2 - 2S_1$ where S_2 is sum of first two terms and S_1 is sum of first term or first term.

(xiii) The sum of infinite terms of an A.P. is ∞ if d > 0 and $-\infty$ if d < 0

(xiv) Sum of n terms of an A.P. is of the form $An^2 + Bn$ i.e. a quadratic expression in n, in such case the common difference is twice the coefficient of n^2. i.e. 2A

(xv) n^{th} term of an A.P. is of the form An + B i.e. a linear expression in n, in such a case the coefficient of n is the common difference of the A.P. i.e. A

(xvi) If for the different A.P.'s, $\frac{S_n}{S'_n} = \frac{f_n}{\phi_n}$ then $\frac{T_n}{T'_n} = \frac{f(2n-1)}{\phi(2n-1)}$

(xvii) If for two A.P.'s, $\frac{T_n}{T'_n} = \frac{An+B}{Cn+D}$ then $\frac{S_n}{S'_n} = \frac{A\left(\frac{n+1}{2}\right)+B}{C\left(\frac{n+1}{2}\right)+D}$

ARITHMETIC MEAN (A.M.)

If three or more than three terms are in A.P., then the numbers, lying between first and last term are known as Arithmetic Means between them. The A.M. between the two given quantities a and b is A so that a, A, b are in A.P.

i.e. $A - a = b - A \Rightarrow A = \frac{a+b}{2}$

A.M. of any n positive numbers a_1, a_2.......... a_n is $A = \frac{a_1 + a_2 + a_3 + ...a_n}{n}$

n AM's between two given numbers :

If in between two numbers 'a' and 'b' we have to insert n AM $A_1, A_2, ...A_n$ then $a_1, A_1, A_2, A_3 ... A_n$, b will be in A.P. The series consist of (n +2) terms and the last term is b and first term is a.

$\Rightarrow a + (n + 2 - 1)\, d = b \;\Rightarrow d = \frac{b-a}{n+1}$

$A_1 = a + d,\; A_2 = a + 2d, \;..... A_n = a + nd$ or $A_n = b - d$

(i) Sum of n AM's inserted between a and b is equal to n times the single AM between a and b i.e.

$\sum_{r=1}^{n} A_r = nA$ where $A = \frac{a+b}{2}$

(ii) Between two numbers $\frac{\text{sum of m AM's}}{\text{sum of n AM's}} = \frac{m}{n}$

SOME PROPERTIES OF A.P.

(i) If $t_n = an + b$, then the series so formed is an A.P.

(ii) If $S_n = an^2 + bn + c$, then series so formed is an A.P.

(iii) If each term of a given A.P. be increased, decreased, multiplied or divided by some non zero constant number then resulting series thus obtained will also be in A.P.

(iv) In an A.P. the sum of terms equidistant from the beginning and end is constant and equal to the sum of first and last term.

(v) Any term of an AP (except the first term) is equal to the half of the sum of terms equidistant from the term

i.e. $a_n = \frac{1}{2}\,(a_{n-k} + a_{n+k}),\; k < n$

(vi) If in a finite AP, the number of terms be odd, then its middle term is the AM between the first and last term and its sum is equal to the product of middle term and no. of terms

IDEA OF A GEOMETRIC PROGRESSION

So far we have learnt that an Arithmetic Progression can be formed by repeatedly adding/subtracting a constant number to/from a given first number (term). A progression can also be formed by repeatedly multiplying or dividing its preceding term by a non-zero constant.

If we consider that the starting prize in a game of 'Triple Your Money' is 3, then repeated tripling (multiplication by 3) would form the progression of possible prizes as,

3, 9, 27, 81, 243, ...(i)

If you observe each term of the above pattern/progression, you will find that there is a common ratio between two consecutive terms. In this case it is 3 : 1 as, $\frac{9}{3} = \frac{27}{9} = \frac{81}{27} = \frac{243}{81} = \frac{3}{1}$

It is usually denoted by 'r'. Such a progression is called a Geometric progression (abbreviated as G.P.)

Thus Geometric Progression is defined as a series in which ratio between any two consecutive terms is constant throughout the series. This constant ratio is called as a common ratio.

General term of a G.P. : General term (n^{th} term) of a G.P. is given by $T_n = ar^{n-1}$

(i) n^{th} term form end is given by $\frac{T_m}{r^{n-1}}$ where m stands for total no. of terms

(ii) If a_1, a_2, a_3, ... are in GP, then $r = \left(\frac{a_k}{a_p}\right)^{\frac{1}{k-p}}$

Sum of n terms of a G.P. :

The sum of first n terms of an A.P. is given by $S_n = \frac{a(1-r^n)}{1-r} = \frac{a - rT_n}{1-r}$, when r < 1

or $S_n = \frac{a(r^n - 1)}{r-1} = \frac{rT_n - a}{r-1}$, when r > 1 and $S_n = nr$, when r = 1

Sum of an infinite G.P. :

The sum of an infinite G.P. with first term a and common ratio r ($-1 < r < 1$ i.e. $|r| < 1$) is $S_\infty = \frac{a}{1-r}$

If $r \geq 1$ then $S_\infty \to \infty$

GEOMETRICAL MEAN (G. M.)

If three or more than three terms are in G.P. then all the numbers lying between first and last term are called Geometrical Means between them i.e. The G.M. between two given quantities a and b is G, so that a, G, b, are in G.P. i.e. $\frac{G}{a} = \frac{b}{G} \Rightarrow G^2 = ab \Rightarrow G = \sqrt{ab}$

(i) G.M. of any n positive numbers a_1, a_2, a_3 .... a_n is $(a_1, a_2, a_3, \ldots\ldots a_n)^{1/n}$.

(ii) If a and b are two numbers of opposite signs, then G.M. between them does not exist.

n GM's between two given numbers:

If in between two numbers 'a' and 'b', we have to insert n GM $G_1, G_2, \ldots\ldots G_n$ then $a_1, G_1, G_2, \ldots\ldots G_n$, b will be in G.P. The series consist of (n +2) terms and the last term is b and first term is a.

$$ar^{n+2-1} = b \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$G_1 = ar, G_2 = ar^2 \ldots\ldots G_n = ar^n$ or $G_n = b/r$

Note : Product of n GM's inserted between 'a' and 'b' is equal to n^{th} power of the single GM between 'a' and 'b'

i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where $G = \sqrt{ab}$

Supposition of terms in n a G. P. :

(i) When no. of terms be odd, then we take three terms as a/r, a, ar

five terms as $\frac{a}{r^2}, \frac{a}{r}$, a, ar, ar^2. Here we take middle term as 'a' and common ratio as 'r'.

(ii) When no. of terms be even then we take

4 terms as : $\frac{a}{r^3}, \frac{a}{r}$, ar, ar^3 ; 6 terms as : $\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}$, ar, ar^3, ar^5

Here we take $\frac{a}{r}$, ar as middle terms and common ratio as r^2.

Some properties of a G.P. :

(i) If each term of a G.P. be multiplied or divided by the same non zero quantity, then resulting series is also a G.P.

(ii) In an G.P. the product of two terms which are at equidistant from the first and the last term, is constant and is equal to product of first and last term

(iii) If each term of a G.P. be raised to the same power, then resulting series is also a G.P.

(iv) In a G.P. every term (except first) is GM of its two terms which are at equidistant from it.

i.e. $T_r = \sqrt{T_{r-k}T_{r+k}}$ $\quad k < r$

(v) In a finite G.P., the number of terms be odd then its middle term is the G.M. of the first and last term.

(vi) If the terms of a given G.P. are chosen at regular intervals, then the new sequence is also a G.P.

(vii) If $a_1, a_2, a_3 ..... a_n$ is a G.P. of non zero, non negative terms, then $\log a_1, \log a_2, ... \log a_n$ is an A.P. and vice-versa

(viii) If $a_1, a_2, a_3 ....$ and $b_1, b_2, b_3 ....$ are two G.P.'s then $a_1 b_1, a_2b_2, a_3b_3 ....$ is also in G.P.

ARITHMETIC GEOMETRICAL PROGRESSION (AGP)

If each term of a progression is the product of the corresponding terms of an A.P. and a G.P., then it is called arithmetic-geometric progression (A.G.P.)

e.g. a, (a + d)r, (a + 2d) r^2, ...

The general term (n^{th}term) of an A.G.P. is $T_n = [a + (n - 1)d]\, r^{n-1}$

To find the sum of n terms of an A.G.P. we suppose its sum S, multiply both sides by the common ratio of the corresponding G.P. and then subtract as in following way and we get a G.P. whose sum can be easily obtained.

$S_n = a + (a + d)\, r + (a + 2d)\, r^2 + ...\, [a + (n - 1)\, d]\, r^{n-1}$

$r\, S_n = ar + (a + d)\, r^2 + ... + [a + (n - 1)d]\, r^n$

After subtraction we get, $S_n (1 - r) = a + r.d + r^2.d\, ...\, dr^{n-1} - [a + (n - 1)d]\, r^n$

After solving, $S_n = \frac{a}{1-r} + \frac{r.d\,(1-r^{n-1})}{(1-r)^2}$ and $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

This is not a standard formula. This is only to understand the procedure for finding the sum of an A.G.P. However formula for sum of infinite terms can be used directly.

HARMONICAL PROGRESSION (H.P.)

Harmonical progression is defined as a series in which reciprocal of its terms are in A.P.

The standard from of a H.P. is $\frac{1}{a}+\frac{1}{a+d}+\frac{1}{a+2d}+.....$

a, b, c are in H.P. $\Leftrightarrow b=\frac{2ac}{a+c}$

General Term of a H.P. : General term (n^{th} term) of a H.P. is given by $T_n=\frac{1}{a+(n-1)\,d}$

(i) There is no formula and procedure for finding the sum of H.P.

(ii) If a,b,c are in H.P. then $\frac{a}{c}=\frac{a-b}{b-c}$

(iii) If a, b are first two terms of an H.P. then, $t_n=\frac{1}{\frac{1}{a}+(n-1)\left(\frac{1}{b}-\frac{1}{a}\right)}$

HARMONICAL MEAN (H.M.)

If three or more than three terms are in H.P., then all the numbers lying between first and last term are called Harmonical Means between them. i.e;

The H.M. between two given quantities a and b is H so that a, H, b, are in H.P.

i.e. $\frac{1}{a},\frac{1}{H},\frac{1}{b}$ are in A.P. ; $\frac{1}{H}-\frac{1}{a}=\frac{1}{b}-\frac{1}{H}\Rightarrow H=\frac{2ab}{a+b}$ Also, $H=\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+....+\frac{1}{a_n}}=\frac{n}{\sum_{j=1}^{n}\frac{1}{a_j}}$

The harmonic mean of n non zero numbers $a_1, a_2, a_3,..........., a_n$.

n H.M's between two given numbers :

To find n HM's between a, and b we first find n AM's between 1/a and 1/b then their reciprocals will be required HM's. If terms are given in H.P. then the terms could be picked up in the following way

For three terms : $\frac{1}{a-d},\frac{1}{a},\frac{1}{a+d}$ **For four terms :** $\frac{1}{a-3d},\frac{1}{a-d},\frac{1}{a+d},\frac{1}{a+3d}$

For five terms : $\frac{1}{a-2d},\frac{1}{a-d},\frac{1}{a},\frac{1}{a+d},\frac{1}{a+2d}$

In general, If we are to take (2r + 1) terms in H.P. we take them as

$$\frac{1}{a-rd},\frac{1}{a-(r-1)d},............,\frac{1}{a-d},\frac{1}{a},\frac{1}{a+d},.............,\frac{1}{a+rd}$$

RELATION BETWEEN A.M., G.M. & H.M.

A, G, H are AM, GM and HM respectively between two numbers 'a' and 'b' then

$A=\frac{a+b}{2}$, $G=\sqrt{ab}$, $H=\frac{2ab}{a+b}$

(i) Consider $A-G=\frac{a+b}{2}-\sqrt{ab}=\frac{(\sqrt{a}-\sqrt{b})^2}{2}\geq 0$. So $A\geq G$. In the same way $G\geq H\Rightarrow A\geq G\geq H$

(ii) Consider A.H. = $\frac{a+b}{2} \cdot \frac{2ab}{a+b}$ = ab = $G^2 \Rightarrow G^2$ = A.H.

If A, G and H are A.M., G.M. and H.M. of two positive numbers a and b, then
(a) $G^2 = AH$, (b) $A \geq G \geq H$

Note : (i) For given n positive numbers $a_1, a_2, a_3, ...a_n$, A.M.≥ G.M. ≥ H.M..
The equality holds when the numbers are equal

(ii) If sum of the given n positive numbers is constant then that their product will be maximum if numbers are equal

SOME IMPORTANT RESULTS

(i) If number of terms is an A.P./G.P./H.P. is odd then its mid term is the A.M/G.M./H.M. between the first and last number.

(ii) If the number of terms in an A.P./G.P./H.P. is even then A.M./G.M./H.M. of its two middle terms is equal to the A.M./G.M./H.M. between the first and last numbers.

(iii) a, b, c are in A.P. and H.P. ⇒ a,b,c are in G.P.

(iv) If a, b, c are in A.P. then $\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab}$ are in A.P.

(v) If a^2, b^2, c^2 are in A.P. then $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

(vi) If a,b,c are in G.P. then a^2, b^2, c^2 are in G.P.

(vii) If a,b,c,d are in G.P. then a + b, b + c, c + d are in G.P.

(viii) If a,b,c are in H.P. then $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A.P.

ADDITIONAL EXAMPLES

Example 1 :

If S_1, S_2 and S_3 denote the sum of first n_1, n_2 and n_3 terms respectively of an A.P., find

$$\frac{S_1}{n_1}(n_2 - n_3) + \frac{S_2}{n_2}(n_3 - n_1) + \frac{S_3}{n_3}(n_1 - n_2)$$

Sol. We have , $S_1 = \frac{n_1}{2}[2a + (n_1 - 1)d] \Rightarrow \frac{2S_1}{n_1} = 2a + (n_1 - 1)d$

$S_2 = \frac{n_2}{2}[2a + (n_2 - 1)d] \Rightarrow \frac{2S_2}{n_2} = 2a + (n_2 - 1)d$, $S_3 = \frac{n_3}{2}[2a + (n_3 - 1)d] \Rightarrow \frac{2S_3}{n_3} = 2a + (n_3 - 1)d$

$$\therefore \frac{2S_1}{n_1}(n_2 - n_3) + \frac{2S_2}{n_2}(n_3 - n_1) + \frac{2S_3}{n_3}(n_1 - n_2)$$

$= [2a + (n - 1)d] (n_2 - n_3) + [2a + (n_2 - 1)d] (n_3 - n_1) + [2a + (n_3 - 1) d] (n_1 - n_2) = 0$

Example 2 :

The sum of 24 terms of the following series : $\sqrt{2} + \sqrt{8} + \sqrt{18} + \sqrt{32} + ...$

Sol. We have $\sqrt{2} + \sqrt{8} + \sqrt{18} + \sqrt{32} + ...$

$= 1\sqrt{2} + 2\sqrt{2} + 3\sqrt{2} + 4\sqrt{2} + ...$ $= \sqrt{2}[1 + 2 + 3 + 4 + ...\text{upto 24 terms}]$ $= \sqrt{2} \times \frac{24 \times 25}{2} = 300\sqrt{2}$

Example 3 :

If p^{th},q^{th} and r^{th} terms of an A.P. are equal to corresponding terms of a G.P. and these are respectively x,y,z, then $x^{y-z}, y^{z-x}, z^{x-y}$ equals -

(1) 0 (2) 1 (3) 2 (4) none of these

Sol. **(2).** Let first term of an A.P. be a and c.d. be d and first term of a G.P. be A and c.r. be R, then

$a + (p-1)d = AR^{p-1} = x \Rightarrow p - 1 = (x - a)/d$ ..... (1)

$a + (q-1)d = AR^{q-1} = y \Rightarrow q - 1 = (y - a)/d$ ..... (2)

$a + (r-1)d = AR^{r-1} = z \Rightarrow r - 1 = (z - a)/d$ ..... (3)

$\therefore$ Given expression

$= (AR^{p-1})^{y-z}, (AR^{q-1})^{z-x}, (AR^{r-1})^{x-y} = A^0 R^{[(p-1)(y-z)+(q-1)(z-x)+(r-1)(x-y)]}$

$= A^0 R^{[(x-a)(y-z)+(y-a)(z-x)+(z-a)(x-y)]/d}$ [By (1), (2) and (3)]

$= A^0R^0 = 1$

Example 4 :

If n be odd or even, then find the sum of n terms of the series 1 – 2 + 3 – 4 + 5 – 6 + ...

Sol. Given series S = 1 – 2 + 3 – 4 + 5 – 6 ....

Case I : If n is odd, say 2m + 1

In this case, the number of positive terms $= \frac{1}{2}(n+1) = \frac{1}{2}(2m+1+1) = (m+1)$

and the number of negative terms = (2m + 1) – (m + 1) = m

Then sum = [1 + 3 + 5 + ... upto (m + 1) terms] – [2 + 4 + 6 .... upto m terms]

$$= \frac{1}{2}(m+1)\,[2+(m+1)-2] - \frac{m}{2}[4+(m-1)2] = (m+1)(m+1-m) = m+1 = \frac{1}{2}(n+1)$$

Case II : If n is even, $\text{Sum} = \left(1+3+5+....\text{upto } \frac{n}{2} \text{ terms}\right) - \left(2+4+6+....\text{upto } \frac{n}{2} \text{ terms}\right)$

$$= \frac{1}{2}.\frac{n}{2}\left[2+\left(\frac{n}{2}-1\right)2\right] - \frac{1}{2}.\frac{n}{2}\left[4+\left(\frac{n}{2}-1\right)2\right] = \frac{1}{4}n\,[n-(n+2)] = -\frac{n}{2}$$

Example 5 :

The ratio of sum of m and n terms of an A.P. is $m^2 : n^2$, then the ratio of m^{th} and n^{th} term will be –

Sol. Given that : $\dfrac{\frac{m}{2}[2a+(m-1)d}{\frac{n}{2}[2a+(n-1)d} = \dfrac{m^2}{n^2} \Rightarrow \dfrac{2a+(m-1)d}{2a+(n-1)d} = \dfrac{m}{n} \Rightarrow \dfrac{a+\frac{1}{2}(m-1)d}{a+\frac{1}{2}(n-1)d} = \dfrac{m}{n}$

$$\Rightarrow an + \frac{1}{2}(m-1)nd = am + \frac{1}{2}(n-1)md \Rightarrow a(n-m) + \frac{d}{2}\,[mn - n - mn + m] = 0$$

$$\Rightarrow a(n-m) + \frac{d}{2}\,(m-n) = 0 \Rightarrow a = \frac{d}{2} \text{ or } d = 2a$$

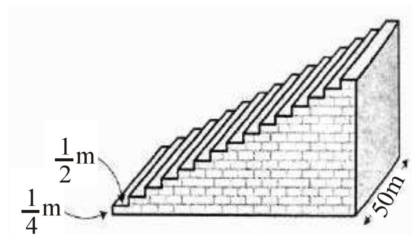
So required ratio, $\dfrac{T_m}{T_n} = \dfrac{a+(m-1)d}{a+(n-1)d} = \dfrac{a+(m-1)2a}{a+(n-1)2a} = \dfrac{1+2m-2}{1+2n-2} = \dfrac{2m-1}{2n-1}$

Example 6 :

A small terrace at a football ground comprises of 15 steps, each of which is 50m long and built o solid concrete. Each step hasa rise of (1/4)m and a thread of (1/2)m (figure). Calculate the total volume of concrete required to build the terrace.

Sol. The volume of the concrete required to build the first step

$$=\left(\frac{1}{4}\times\frac{1}{2}\times 50\right) m^3 = \frac{25}{4} m^3$$



The volume of the concrete required to build the second step

$$=\left(\frac{1}{2}\times\frac{1}{2}\times 50\right) m^3 = \frac{50}{4} m^3$$

Similarly, the volume of the concrete required to build the third, fourth, ... steps will correspondingly be

$\frac{75}{4}m^3, \frac{100}{4}m^3, ...$

We thus have an AP : $\frac{25}{4}, \frac{50}{4}, \frac{75}{4}, \frac{100}{4}, ...$ $\therefore$ $a = \frac{25}{4}, d = \frac{25}{4}$ and $n = 15$

We need to determine S_{15}. $\therefore$ $S_{15} = \frac{15}{2}\left[2\times\frac{25}{4} + (15-1)\left(\frac{25}{4}\right)\right] = \frac{15}{2}\left[\frac{25}{2} + \frac{175}{2}\right] = 750m^3$

Thus, the total volume of concrete required to build the terrace is 750 m^3

Example 7 :

The monthly salary of a person was Rs. 320 for each of the first three years. He next got annual increments of Rs. 40 p.m. for each of the following successive 12 years. His salary remained constant till reitrement and he calculated that his average monthly salary during the service period was Rs. 698. Find the period of his service.

Sol. Suppose the period of his service was n years. Then, sequence of his salary was as under :

$$\underbrace{320, 320, 320}_{3 \text{ years}}, \underbrace{360, 400, 440, ...k}_{12 \text{ years}}, \underbrace{k, k, k, ...}_{(n-15) \text{ years}}$$

where k denotes the constant salary beginning from the 16th year.

To determine the value of k, we shall find the salary at the end of 15 years, which is

$= 360 + 11 \times 40 = 360 + 440 = 800$. Thus, $k = 800$

Here, his total salary for n years = Rs. (698n) .... (1)

From the sequence, the total salary

$= (320 \times 3) + (12/2)\ [2 \times 360 + 11 \times 40] + (n - 15)\ (800)$

$= 960 + 6960 + 800n - 1200 = 800n - 4080$ .... (2)

Equating (1) and (2), we get, $698n = 800n - 4080$ $\Rightarrow$ $102n = 4080 \Rightarrow n = 40$

Hence, the period of his service was 40 years.

Example 8 :

The n^{th} term of a GP is 128 and the sum of its n terms is 255, if its common ratio is 2, find its first term

Sol. Let a be the first term. Then as given $T_n = 128$ and $S_n = 255$

But $S_n = \frac{rT_n - a}{r-1} \Rightarrow 255 = \frac{2(128) - a}{2-1} \Rightarrow a = 1$

Example 9 :

In the nth term of an A.P. be (2n – 1), find the sum of its first n terms.

Sol. Given that $T_n = 2n - 1$

First term a = 2.1 – 1 = 1, Second term = b = 2.2 – 1 = 3, Third term = c = 2.3 – 1 = 5

Therefore sequence is 1, 3, 5, ...2n – 1

Now sum of the first n terms is $S_n = \frac{n}{2}[a+1] = \frac{n}{2}[1+2n-1] = \frac{n}{2}.2n = n^2$

Example 10

If x,y,z are in G.P. and $a^x = b^y = c^z$ then -

(1) $\log_b a = \log_a c$ (2) $\log_c b = \log_a c$ (3) $\log_b a = \log_c b$ (4) none of these

Sol. **(C).** x,y,z are in G.P. $\Rightarrow y^2 = xz$ ......(i)

We have, $ax = b^y = c^z = \lambda$ (say) $\Rightarrow$ x log a = y log b = z log c = log λ

$\Rightarrow \quad x = \frac{\log\lambda}{\log a}, y = \frac{\log\lambda}{\log b}, z = \frac{\log\lambda}{\log c}$ putting x,y,z in (i), we get, $\left(\frac{\log\lambda}{\log b}\right)^2 = \frac{\log\lambda}{\log a}.\frac{\log\lambda}{\log c}$

$(\log b)^2 = \log a.\log c$ or $\log_a b = \log_b c \Rightarrow \log_b a = \log_c b$

Example 11 :

If r^{th} term of a series is $(2r+1)\,2^{-r}$, find sum of its infinite terms

Sol. Here $T_r = (2r+1)\,2^{-r}$ $\therefore$ Series is : $\frac{1}{2}\left[3+\frac{5}{2}+\frac{7}{2^2}+.....\right]$

Obviously the series in the bracket is Arithmetico-Geometrical series.

Therefore by the formula $S_\infty = \frac{a}{1-r}+\frac{r}{(1-r)^2}$. We have $S_\infty = \frac{1}{2}\left[\frac{3}{1-\frac{1}{2}}+\frac{2\left(\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)^2}\right] = 5$

Example 12 :

If between 1 and 1/31 there are n H.M.'s and ratio of 7^{th} and $(n-1)^{th}$ harmonic means is 9 : 5, find value of n.

Sol. Since there are n A.M.'s between 1 and 31 and the ratio of 7^{th} and $(n-1)^{th}$ A.M.' is 5 : 9

$$\therefore \frac{1+7\left(\frac{31-1}{n+1}\right)}{1+(n-1)\left(\frac{3n-1}{n+1}\right)} = \frac{5}{9} \Rightarrow \frac{n+211}{31n-29} = \frac{5}{9} \Rightarrow n = 14$$

Example 13 :

Find three numbers a,b,c between 2 and 18 such that-(i) Their sum is 25 (ii) The numbers 2, a, b are consecutive terms of an A.P. (iii) The numbers b, c, 18 are three consecutive terms of a G.P.

Sol. a + b + c = 25 .... (1) 2, a,b are in A.P. ⇒ a = (2+b)/2 .... (2)

and b,c, 18 are in G.P. $\Rightarrow c^2 = 18b$ .... (3)

Eliminating a and b from (1), (2) and (3), gives the following equation : $c^2 + 12c - 288 = 0$

(c – 12) (c + 24) = 0 ⇒ c =12, – 24 [Leaving c = – 24 because this is not between 2 and 18]

$\therefore$ c = 12 from (3) b = 8 and from (1) a = 5. Hence a, b, c = 5, 8, 12

CONCEPT MAP

* A succession of numbers is called a "**Sequence**" or "Sequence" means an arrangement of number in a definite order according to some rule.
* **Finite sequence :** Sequence containing finite number of terms is finite sequence
* **Infinite sequence :** Sequence containing infinite number of terms is infinite sequence.
* The sequence in which every term except the first obtained by adding fixed number (Positive or Negative) to the preceding terms is called "**Arithmetic Progression**" (A.P.).Thus any sequence a, a_2, a_3.....a_n is called an Arithmetic Progression (A.P.) if

$a_{n+1} = a_n + d,\ n \in N$; where **a** is called the **first term** and the constant number **d** is called the **common difference** of the A.P.

* **The nth term or general term of an A.P.** Let us consider an A.P. with first term a and common difference d i.e. a, a+d, a +2d , a +3d ....then

1^{st} term = a_1 = a= a + (1-1)d
2^{nd} term = a_2 = a + d = a + (2- 1)d
3^{rd} term = a_3 = a + 2d = a + (3 – 1)d

n^{th} term → a_n = a + (n -1)d

Properties of an A.P.

1. If a constant is added to each term of an A.P. the resulting sequence is also an A.P.
2. If a constant is subtracted from each term of an A.P. the resulting sequence is also an A.P.
3. If each term of an A.P. is multiplied by a constant, then the resulting sequence is also an A.P.
4. If each term of an A.P is divided by a non zero constant then the resulting sequence is also A.P.

ARITHMETIC PROGRESSION

* **The sum of n terms of an A.P. :**
Let S_n be the sum of n terms of the A.P. then

$$S_n = a + (a+d) + (a+2d) + ... + [a+(n-2)d] + [a+(n-1)d] \quad .... (1)$$

$$S_n = [a+(n-1)d] + [a+(n-2)d] ....(a+d) + a \quad .... (2)$$

On adding (1) and (2)

$$2S_n = n[a + a + (n-1)d]$$

or $S_n = \frac{n}{2}[a+\ell]$;Where $\ell = [a+(n-1)d]$

or $S_n = \frac{n}{2}[2a+(n-1)d]$

* The sequence in which every term except the first term bears a constant ratio to the term immediately preceding it, is called "**Geometric Progression**." (G.P.)
A sequence a, a_2, a_3 ....a_n is called Geometric progression (G.P.) if,

$\frac{a_{n+1}}{a_n}$ = r (constant) for every k ≥ 1

The Geometric progression is
a, ar, ar^2, ar^3..............
a is called the **first term** and the constant number **r** ≠ 0 is called the **common ratio** of the G.P.

* The sequence $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, ... \frac{1}{a_n}$ is said to be **Harmonic progression** if a_1, a_2, a_3.......a_n, is an Arithmetic progression. For instance, $\frac{1}{a}, \frac{1}{(a+d)}, \frac{1}{(a+2d)}, ...$ is Harmonic progression.