Chapter- 5

Continuity and Differentiability

INTRODUCTION

In this chapter we will discuss two very important concepts of mathematics continuity and differentiability of real functions. Also discuss the relation between them. In order to understand these concepts well one should have the knowledge of the concept of limits which was in Class – XI

Limits:

Let $a \in R$ and let 'f' be a real-valued function in real variable x defined at the points in an open interval containing 'a' except possible at 'a'. Then we say that limit of the function f(x) is a real number ℓ as x tends to 'a'. If the value of f(x) approaches ℓ as x approaches 'a'. Which is denoted by

$$\lim_{x \to a} f(x) = \ell$$

Here x can approach 'a' on a real number line in two ways, either from left or from right of a. This leads to two limits as left hand limit (LHL) and Right hand limit (RHL).

Left hand limit is the value of f(x) approaches ℓ as x approaches 'a' from the left of a. It is denoted by $\lim_{x \to a^-} f(x)$

Right hand limit is the value of f(x) approaches as x approaches 'a' from the right of 'a'. It is denoted by $\lim_{x \to a^+} f(x)$.

Existence of Limit

Whenever $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \ell$

Then $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = \ell$

LHL = $\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$

RHL = $\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$

Some Important results on limit

(a) $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

(b) $\lim\limits_{x\to 0}\dfrac{\tan x}{x}=1$

(c) $\lim\limits_{x\to 0}\dfrac{e^{x}-1}{x}=1$

(d) $\lim\limits_{x\to 0}\dfrac{a^{x}-1}{x}=\log a$

(e) $\lim\limits_{x\to 0}\dfrac{\log(1+x)}{x}=1$

(f) $\lim\limits_{x\to a}\dfrac{x^{n}-a^{n}}{x-a}=n.a^{n-1}$

(g) $\lim\limits_{x\to a}\dfrac{\sin^{-1}x}{x}=1$

(h) $\lim\limits_{x\to 0}\dfrac{\tan^{-1}x}{x}=1$

If $a\in R$ and but f, g be real valued function then

(a) $\lim\limits_{x\to a}k.f(x)=k\lim\limits_{x\to a}f(x)$ (k is constant)

(b) $\lim\limits_{x\to a}\left[f(x)\pm g(x)\right]=\lim\limits_{x\to a}f(x)\pm\lim\limits_{x\to a}g(x)$

(c) $\lim\limits_{x\to a}f(x).g(x)=\lim\limits_{x\to a}f(x).\lim\limits_{x\to a}g(x)$

(d) $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)};\lim\limits_{x\to a}g(x)\neq 0$

Intuitive Idea of continuity

Let ‘f’ be a real valued function in any interval and let y = f(x) . Then we can represent the function by a graph in xy –plane. The function ‘f’ is continuous when we try to draw the graph in one stoke, i.e without lifting pen from the plane of paper. Roughly, a function is continuous if its graph is a single unbroken curve with no holes or jumps.

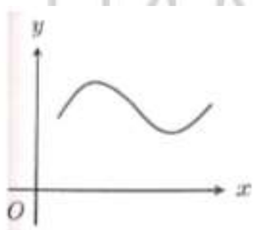


Figure 10.1

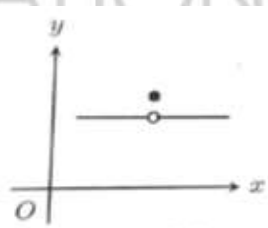


Figure 10.2

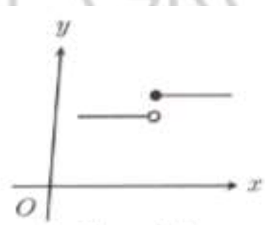


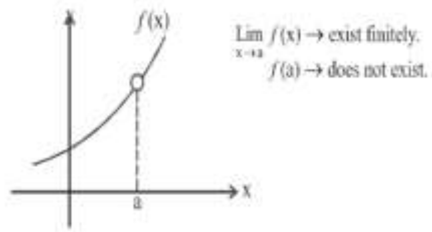
Figure 10.3

From the above idea the function shown in figure 10.1 is continuous.

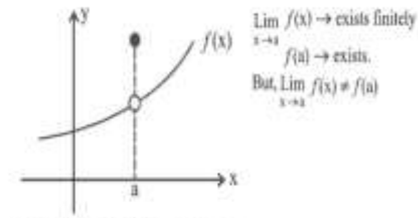
The function shown in fig 10.2 has a hole at a point and hence not continuous.

The function shown in fig. 10.3 has a jump at a point and hence is not continuous.

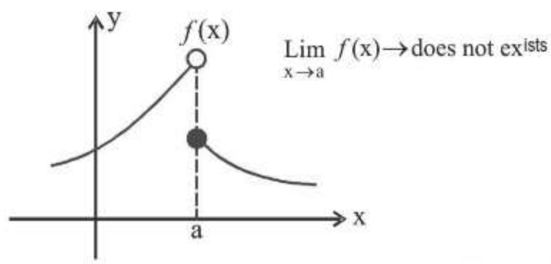
Different types of Discontinuity



missing point discontinuity at x = a



Isolated point discontinuity at x = a



non-removable discontinuity at x=a

Mathematical definition of Continuity

A function $f : D \to R$ is said to be continuous at x=c

i.e. if $\lim\limits_{x \to c^-} f(x) = \lim\limits_{x \to c^+} f(x) = f(x)$

i.e. LHL = RHL = Functional value

Otherwise the function will be discontinuous at x = c

Conclusion

As the function $f(x)$ is continuous at x=a if LHL = RHL = f(a)

But we know that when LHL=RHL= ℓ (say)

Then $\lim\limits_{x \to a} f(x)$ exists and $\lim\limits_{x \to a} f(x) = \ell$

Thus the function f(x) will be continuous at x=a if

$\lim\limits_{x \to a} f(x) = f(a)$ i.e. Limiting value = Functional value.

Example

Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$

Solution:-

given $f(x) = 2x^2 - 1$

LHL = $\lim_{x \to 3^-} f(x) = \lim_{h \to 0} f(3-h)$

$= \lim_{h \to 0}\left[2(3-h)^2 - 1\right] \qquad = 2\left[(3-0)^2 - 1\right] = 17$

RHL = $\lim_{x \to 3^+} f(x) = \lim_{h \to 0} f(3+h)$

$= \lim_{h \to 0}\left[2(3+h)^2 - 1\right] \quad = 2(3)^2 - 1 = 17$

As LHL=RHL=f(3)

So f is continuous at x =3

Example:(Exemplar)

Check the continuity of the function $f(x) = \begin{cases} 3x+5 & if\ x \geq 2 \\ x^2 & if\ x < 2 \end{cases}$ $at\ \ x = 2$

Solution:

Given that f(2)=3.2+5 =11

$$LHL = \lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} (2-h)^2 = (2-0)^2 = 4$$

$$RHL = \lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} 3(2+h)+5 \ = 3(2+0)+5 = 11$$

As $LHL \neq RHL$ so f is not continuous at x=2.

Example:

Show that the function $f(x) = \begin{cases} \dfrac{2x^2 - 3x - 2}{x-2} & if\ x \neq 2 \\ 5 & if\ x = 2 \end{cases}$ $is\ continuous\ at\ x = 2$

Solution:

Given that f(2)=5

Now $\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{2x^2 - 3x - 2}{x-2} = \lim_{x \to 2} \frac{2x^2 - 4x + x - 2}{x-2}$

$= \lim_{x \to 2} \frac{(x-2)(2x+1)}{x-2} = \lim_{x \to 2} \frac{2x+1}{1} = 2.2+1 = 5$

As $\lim_{x \to 2} f(x) = f(2)$ so f is continuous at x=2.

Example:

Discuss the continuity of $f(x)$ when $f(x) = \begin{cases} \frac{1-\cos 2x}{x^2}, & if\ x \neq 0 \\ 5 & , if\ x = 0 \end{cases}$ $at\ x = 0$

Solution:

Given that $f(o) = 5$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1-\cos 2x}{x^2} = \lim_{x \to 0} \frac{2\sin^2 x}{x^2}$

$= 2\left(\lim_{x \to 0} \frac{\sin x}{x}\right)^2 = 2(1)^2 = 2$

Since $\lim_{x \to 0} f(x) \neq f(0)$ Hence f is not continuous at x=0

Example:

Find the value of k so that the function $f(x) = \begin{cases} \frac{2^{x+2}-16}{4^x-16}, & x \neq 2 \\ k, & \text{if } x = 2 \end{cases}$ is continuous at x=2.

Solution:

Given that $f(2) = k$

$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{2^{x+2}-16}{4^x-16} = \lim_{x \to 2} \frac{2^x.2^2-16}{4^x-16} = \lim_{x \to 2} \frac{4(2^x-4)}{(2^x)^2-(4)^2} = \lim_{x \to 2} \frac{4(2^x-4)}{(2^x+4)(2^x-4)}$

$= \lim_{x \to 2} \frac{4}{2^x+4} = \frac{4}{2^2+4} = \frac{1}{2}$

As f(x) is continuous at x=2

$\lim_{x \to 2} f(x) = f(2)$

$\Rightarrow \frac{1}{2} = k$ So, $k = \frac{1}{2}$ (Ans.)

Definition :

A real function 'f' is said to be continuous if it is continuous at every point in the domain of 'f'.

Suppose 'f' is a function defined on a closed interval [a,b] , then for 'f' to be continuous, it needs to be continuous at every point of [a,b] including the end points a and b.

Example:- Prove that the constant function f(x) = k is continuous.

Solution:- Let 'c' be any real number

Here $f(c) = k$ for every $c \in R$

And $\lim_{x \to c} f(x) = \lim_{x \to c} k = k$

Since $\lim_{x \to c} f(x) = f(c)$ for any real number 'c' the function 'f' is continuous.

The function f is continuous at every real number.

List of some continuous functions

	Function f(x)	Interval in which f(x) is continuous
1.	constant c	$(-\infty, \infty)$
2.	x^n, n is an integer ≥ 0	$(-\infty, \infty)$
3.	x^{-n}, n is a positive integer	$(-\infty, \infty) - \{0\}$
4.	$\|x-a\|$	$(-\infty, \infty)$
5.	$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$	$(-\infty, \infty)$
6.	$\frac{p(x)}{q(x)}$, where p (x) and q (x) are polynomial in x	$(-\infty, \infty) - \{x : q(x) = 0\}$
7.	sin x	$(-\infty, \infty)$
8.	cos x	$(-\infty, \infty)$
9.	tan x	$(-\infty, \infty) - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$
10.	cot x	$(-\infty, \infty) - \{n\pi : n \in I\}$
11.	sec x	$(-\infty, \infty) - \{(2n+1)\pi/2 : n \in I\}$
12.	cosec x	$(-\infty, \infty) - \{n\pi : n \in I\}$
13.	e^x	$(-\infty, \infty)$
14.	$\log_e x$	$(0, \infty)$

Algebra of Continuous Functions:

Suppose 'f' and 'g' be two real functions at a real number 'c', then

(a) $f+g$ is continuous at $x=c$

(b) $f-g$ is continuous at $x=c$

(c) $f.g$ is continuous at $x=c$

(d) $\left(\frac{f}{g}\right)$ is continuous at $x=c$ (provided $g(c)\neq 0$)

(e) If 'f' and 'g' be two functions such that fog is defined at c and if 'f' is continuous at $g(c)$. Then (fog) is continuous at C.

Example:- Show that the function defined by $f(x)=|\cos x|$ is a continuous function.

Solution:- The function 'f' may be thought of as a composition gof of the two functions 'g' and 'h', where $g(x)=|x|$ and $h(x)=\cos x$

$$goh(x)=g(h(x))=g(\cos x)=|\cos x|=f(x)$$

Since both 'g' and 'h' are continuous functions so 'f' is continuous.

Example:

If the function $f(x)=\begin{cases} 3ax+b & \text{if } x>1 \\ 11 & \text{if } x=1 \\ 5ax-2b & \text{if } x<1 \end{cases}$ *is continuous at* $x=1$, *find the values of* a *and* b.

Continuity at $x=1$ We have, $f(1)=11$.

$$\lim_{x\to 1+}[f(x)]=\lim_{x\to 1+}[3ax+b] = \lim_{h\to 0}[3a(1+h)+b] \quad \begin{bmatrix}\text{By putting} \\ x=1+h\end{bmatrix} = 3a+b.$$

$$\lim_{x\to 1-}[f(x)]=\lim_{x\to 1-}[5ax-2b] = \lim_{h\to 0}[5a(1-h)-2b] \quad \begin{bmatrix}\text{By putting} \\ x=1-h\end{bmatrix} = 5a-2b.$$

So, f is continuous at $x=1$ if

$$\lim_{x\to 1+}[f(x)]=\lim_{x\to 1-}[f(x)]=f(1)$$

i.e., $3a+b=5a-2b=11$

i.e., $3a+b=11$ and $5a-2b=11$

i.e., $a=3, b=2.$ ∎

Example:

$$\text{Find } k, \text{ if } f(x) = \begin{cases} k \sin \frac{\pi}{2}(x+1) & \text{if } x \le 0 \\ \frac{\tan x - \sin x}{x^3} & \text{if } x > 0 \end{cases} \text{ is continuous at } x = 0.$$

Sol:

$$\text{Given } f(x) = \begin{cases} k \sin \frac{\pi}{2}(x+1) & \text{if } x \le 0 \\ \frac{\tan x - \sin x}{x^3} & \text{if } x > 0 \end{cases}.$$

Continuity at $x = 0$ We have, $f(0) = k$.

$$\lim_{x \to 0+} [f(x)]$$

$$= \lim_{x \to 0+} \left[\frac{\tan x - \sin x}{x^3}\right]$$

$$= \lim_{h \to 0} \left[\frac{\tan h - \sin h}{h^3}\right] \qquad \left[\begin{array}{l}\text{By putting} \\ x = 0 + h\end{array}\right]$$

$$= \lim_{h \to 0} \left[\frac{\tan h(1 - \cos h)}{h^3}\right]$$

$$= \lim_{h \to 0} \left[\frac{\tan h \left(2 \sin^2 \frac{h}{2}\right)}{h^3}\right]$$

$$= \lim_{h \to 0} \left[\frac{\tan h}{h} \times \frac{2 \sin^2 \frac{h}{2}}{4\left(\frac{h}{2}\right)^2}\right]$$

$$= \frac{1}{2} \lim_{h \to 0} \left[\frac{\tan h}{h}\right] \lim_{h \to 0} \left[\frac{\sin^2 \frac{h}{2}}{\left(\frac{h}{2}\right)^2}\right] = \frac{1}{2}.$$

$$\lim_{x\to 0-}[f(x)]$$
$$= \lim_{x\to 0-}\left[k\sin\frac{\pi}{2}(x+1)\right]$$
$$= \lim_{h\to 0}\left[k\sin\frac{\pi}{2}\{(0-h)+1\}\right] \quad \left[\begin{array}{l}\text{By putting}\\ x=0-h\end{array}\right]$$
$$= k\sin\frac{\pi}{2}$$
$$= k.$$

So, f is continuous at $x=0$ if

$$\lim_{x\to 0+}[f(x)] = \lim_{x\to 0-}[f(x)] = f(0)$$

i.e., $\frac{1}{2} = k = k$

i.e., $k = \frac{1}{2}.$

Example:(NCERT Ex4.1,Qno.19)

19. Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x.

Ans. Let $n \in I$

Then, $\lim_{x\to n^-}[x] = n-1$

$\therefore [x] = n-1 \ \forall\ x \in [n-1, n]$

and g(n) = n - n = 0 $[\because [n] = n \text{ because } n \in I]$

Now, $\lim_{x\to n^-} g(x) = \lim_{x\to n^-}(x-[x]) = \lim_{x\to n^-} x - \lim_{x\to n^-}[x]$

$= n-(n-1) = 1$

Also, $\lim_{x\to n^+} g(x) = \lim_{x\to n^+}(x-[x])$

$= \lim_{x\to n^+} x - \lim_{x\to n^+}[x] = n-n = 0$

Thus, $\lim_{x\to n^-} g(x) \neq \lim_{x\to n^+} g(x)$

Hence, g(x) is discontinuous at all integral points.

Example:

$$If\ \ f(x)=\begin{cases} 1 & ,if\,x\le 3 \\ ax+b & ,if\,3<x<5. \\ 7 & ,if\ \ 5\le x \end{cases}$$

Determine the values of **a** and **b** so that the function f(x) is continuous.

Sol:

The given function is continuous at each x in R so at x=3 and x=5.

At x-3

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x)=f(3)$$
$$\Rightarrow \lim_{h\to 0} f(3-h)=\lim_{h\to 0} f(3+h)=1$$
$$\Rightarrow \lim_{h\to 0} 1=\lim_{h\to 0} a(3+h)+b=1$$
$$\Rightarrow 1=3a+b=1$$
$$\Rightarrow 3a+b=1 \quad ....(1)$$

At x=5

$$\lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)=f(5)$$
$$\Rightarrow \lim_{h\to 0} f(5-h)=\lim_{h\to 0} f(5+h)=7$$
$$\Rightarrow \lim_{h\to 0} a(5-h)+b=\lim_{h\to 0} 7=7$$
$$\Rightarrow 5a+b=7=7$$
$$\Rightarrow 5a+b=7 \quad ....(2)$$

Solving equation (1) and (2) we have a = 3 and b = -8

Example:

Determine the value k so that the function f(x) is continuous at x=0 , Where

$$f(x)=\begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & if\ -1\le x<0 \\ \dfrac{2x+1}{x-1}, & if\ 0\le x\le 1 \end{cases} \quad at\ x=0.$$

Solution:

$$\therefore \quad LHL = \lim_{x\to 0^-} \frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}$$

$$= \lim_{x\to 0^-}\left(\frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}\right).\left(\frac{\sqrt{1+kx}+\sqrt{1-kx}}{\sqrt{1+kx}+\sqrt{1-kx}}\right)$$

$$= \lim_{x\to 0^-}\frac{1+kx-1+kx}{x[\sqrt{1+kx}+\sqrt{1-kx}}$$

$$= \lim_{x\to 0^-}\frac{2kx}{x\sqrt{1+kx}+\sqrt{1-kx}}$$

$$= \lim_{h\to 0}\frac{2k}{\sqrt{1+k(0-h)}+\sqrt{1-k(0-h)}}$$

$$= \lim_{h\to 0}\frac{2k}{\sqrt{1-kh}+\sqrt{1+kh}} \quad = \frac{2k}{2} = k$$

$$\text{and } f(0) = \frac{2\times 0+1}{0-1} = -1$$

$$\Rightarrow k=-1\,[\because LHL=RHL=f(0)]$$