

## Chapter- 1

# Relations And Functions

## Types of Relation

### Introduction:-

Relation from a set A to B:- Let A and B be two non-empty sets. Then a set R is said to be a relation from set A to set B if R is a subset of  $A \times B$ . i.e if  $R \subseteq A \times B$ .

### Example:-

Let  $R = \{1, 2, 3\}$ ,  $B = \{2, 3, 4\}$  define  $R = \{(a, b) : 2a = b, a \in A, b \in B\}$

Show that R is a relation from A to B. Also find no of possible relations from A to B.

**Relation on a set A:-** Let A be any non-empty set. Then a set R is said to be a relation on A if R is a subset of  $A \times A$ . i.e  $R \subseteq A \times A$ .

### Example:-

Let  $A = \{1, 2, 3\}$  and define  $R = \{(a, b) : 2a = b, a, b \in A\}$ . Show that R is a relation on A. What is the possible number of relations on A.

### Types of Relation:-

**(1) Empty or void relation:-** A relation R on the set A is called empty relation if no elements of A are related to any elements of A i.e  $R = \phi$ .

### Example:-

Let  $A = \{1, 2, 3\}$ , define  $R = \{(a, b) : a - b = 12\}$ . Show that R is an empty relation on set A.

**(2) Universal Relation:-** A relation R on a set A is called universal relation if each element of A is related to every element of A. i.e if  $R = A \times A$ .

**Example:-**

Let  $A = \{1, 2\}$  and define  $R = \{(a, b); a + b > 0\}$ . Show that  $r$  is a universal relation on set  $A$ .

Note:- Void and universal relations are called trivial relations.

**(3) Identity Relation:-** A relation  $R$  on set  $A$  is called identity relation if every element of  $A$  is related to itself only. i.e if  $R = \{(a, a); a \in A\}$ . The identity relation on set  $A$  is denoted by

$I_A$ .

**Example:-**

Let  $A = \{1, 2, 3\}$ , and the relation  $R$  defined by  $R = \{(a, b); a - b = 0; a, b \in A\}$ . Show that  $R$  is an identity relation.

**(4) Reflexive Relation:-** A relation  $R$  on the set  $A$  is called reflexive relation if  $a R a$  for every  $a \in A$ . i.e if  $(a, a) \in R$  for every  $a \in A$ .

**Example:-**

Let  $A = \{1, 2, 3\}$ . Define the relation  $R_1, R_2, R_3$  on  $A$  as.

(i)  $R_1 = \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3)\}$       (ii)  $R_2 = \{(1,2), (1,3), (2,3)\}$

(iii)  $R_3 = \{(1,1)\}$

Check whether  $R_1, R_2,$  and  $R_3$  are reflexive or not.

**Example:-**

The relation "equal to" in the set of Natural numbers is reflexive.

**Note:-** Identity and universal relations are reflexive, but empty relation is not reflexive. All reflexive relations are not an identity relation.

**(5) Symmetric Relation:-** A relation  $R$  on the set  $A$  is called symmetric relation if  $aRb$  implies  $bR a$ , for every  $a, b \in A$ ; if  $(a, b) \in R \Rightarrow (b, a) \in R$ ,  $a, b \in A$ .

**Example:-**

Let  $A = \{1, 2, 3\}$  define the relation  $R_1, R_2$  and  $R_3$  on  $A$  as.

$$(i) R_1 = \{(1,1), (2,2), (1,2), (2,1)\} \quad (ii) R_2 = \{(1,1), (2,2), (3,3)\}$$

$$(iii) R_3 = \{(1,1), (2,2), (1,2), (2,1), (3,1)\}$$

Check whether  $R_1, R_2$ , and  $R_3$  are symmetric or not.

**Example:-**

The relation parallel to in the set of the lines in a plane is symmetric.

**Note:-**

- Identity and universal relation are symmetric
- Empty relation is also symmetric as there is no situation in which  $(a, b) \in R$ .

**(6) Transitive Relation:-** A relation  $R$  on the set  $A$  is called transitive relation if  $aRb$  and  $bR c$  imply  $aR c$ , for every  $a, b, c \in A$ . i.e if  $(a, b) \in R \& (b, c) \in R \Rightarrow (a, c) \in R$  for every  $a, b, c \in A$ .

**Example:-**

Let  $A = \{1, 2, 3\}$ , define  $R_1, R_2, R_3, R_4$  on  $A$  as

$$(i) R_1 = \{(1,1), (1,2), (2,1), (2,2)\} \quad (ii) R_2 = \{(1,1), (2,2)\}$$

$$(iii) R_3 = \{(1,1), (1,2), (2,3)\} \quad (iv) R_4 = \{(1,2), (1,3)\}$$

Check  $R_1, R_2, R_3$ , and  $R_4$  are transitive or not.

**Example:-**

The relation greater than on  $\mathbb{R}$  is transitive.

**Note:-** If there is no situation in which  $(a, b) \in R$  and  $(b, c) \in R$ , then the relation is transitive

**Equivalence Relation and Equivalence Class**

**Equivalence Relation:-** A relation  $R$  on a set  $A$  is called equivalence relation if  $R$  is reflexive, symmetric, and transitive.

**Equivalence Class:-** Let  $R$  be an equivalence relation on set  $A$  and let  $a \in A$ . Then we define the equivalence class of 'a' as

$$[a] = \{b \in A : b \text{ is related to } a\}$$

$$= \{b \in A : (b, a) \in R\}$$

**Example:-**

$A = \{1, 2, 3\}$ . Define the relations  $R_1, R_2, R_3$ , and  $R_4$  on  $A$  as.

$$(i) R_1 = \{(1,1), (1,2), (2,1), (2,2)\} \quad (ii) R_2 = \{(1,1), (2,2), (3,3), (1,2), (2,3), (1,3)\}$$

$$(iii) R_3 = \{(1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1)\}$$

$$(iv) R_4 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$$

Check whether  $R_1, R_2, R_3$ , and  $R_4$  are equivalence relations or not, if yes, then find the equivalence classes of all elements of set  $A$ .

**Example:-**

Prove that the relation  $R$  on  $\mathbb{Z}$ , defined by  $(a, b) \in R \Leftrightarrow a - b$  is divisible by 5 is an equivalence relation on  $\mathbb{Z}$ .

**Example:-**

Show that the relation  $R$  on  $\mathbb{R}$  defined as  $R = \{(a, b) : a \leq b^2\}$  is neither reflexive nor symmetric nor transitive.

**Example:-**

Show that the relation  $R$  defined by  $(a, b)R(c, d) \Rightarrow a + b = b + c$ , in  $A \times A$  where  $A = \{1, 2, 3, \dots, 10\}$  is an equivalence relation. Hence write the equivalence class  $[(3, 4)]$ ;  $a, b, c, d \in A$ .

**Example:-**

Write the smallest and largest equivalence relation on the set  $A = \{1, 2, 3\}$

**Example:-**

For the set  $A = \{1, 2, 3\}$ , define a relation  $R$  on the set  $A$  as follows.

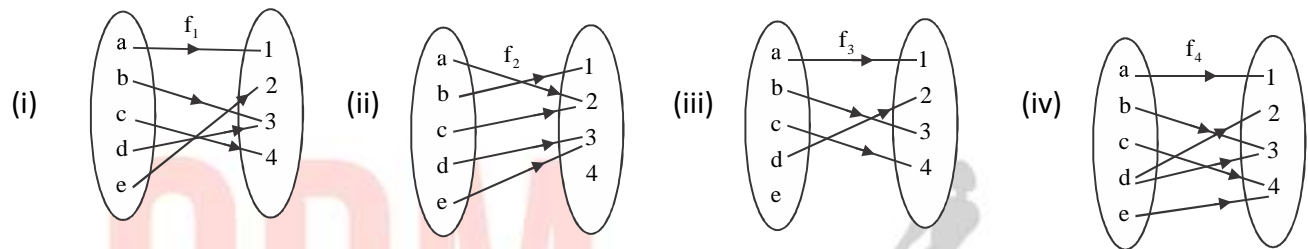
$R = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$  write the ordered pair to be added to  $R$  to make it the smallest equivalence relation.

**Types of Function**

**Function from set A to set B:-** Let A and B be two non-empty sets, then a function f from set A to set B is a rule (or map or correspondence) which associates each element of set A to exactly one element of set B. If f is a function from set A to set B, then we denote it by  $f : A \rightarrow B$ .

**Example:-**

Check whether the maps in the following diagram are functions or not.

**Domain, Co-domain, and Range of a function:-**

Let  $f : A \rightarrow B$  be function, then

(i) The set A is called the domain of function 'f'

(ii) The set B is called co-domain of 'f'

(iii) The set of all images of elements of set A under f is called range under f.

**Note:-**

(1) The range of A under f is denoted by  $f(A)$ .

(2) If  $f(a) = b$  then, b is called an image of a under f, and a is called pre-image of b.

(3) The range is always a subset of the co-domain.

(4) If  $n(A) = p, n(B) = q$ , then the number of functions from A to B is  $(q)^p$

**Types of Functions:-**

**(1) One-one function or Injective function:-** A function  $f : A \rightarrow B$  is said to be one-one if no two elements of  $A$  have the same image. i.e if  $a \neq b \Rightarrow f(a) \neq f(b)$  for all  $a, b \in A$

Or  $f(a) = f(b) \Rightarrow a = b$  for all  $a, b \in A$ .

**Note:-**

(i) If a function  $f : A \rightarrow B$  is not one-one then it is called many-one function

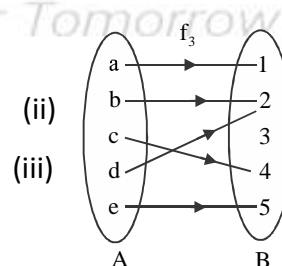
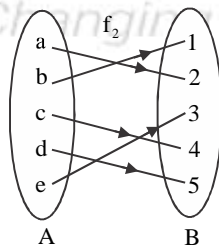
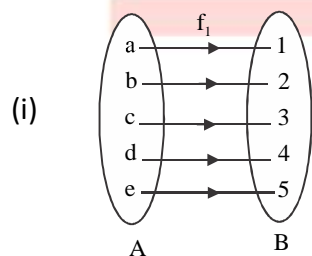
(ii) if a function  $f : A \rightarrow B$  is one-one then  $n(A) \leq n(B)$

(iii) If  $n(A) = p, n(B) = q$ , then no of one-one function from  $A$  to  $B$

$$= \begin{cases} 0, & \text{if } p > q \\ {}^qP_p = \frac{q!}{(q-p)!}, & \text{if } p \leq q \end{cases}$$

**Example:-**

Check whether the function in the diagram are one-one or not.



**On to function or subjective function:-**

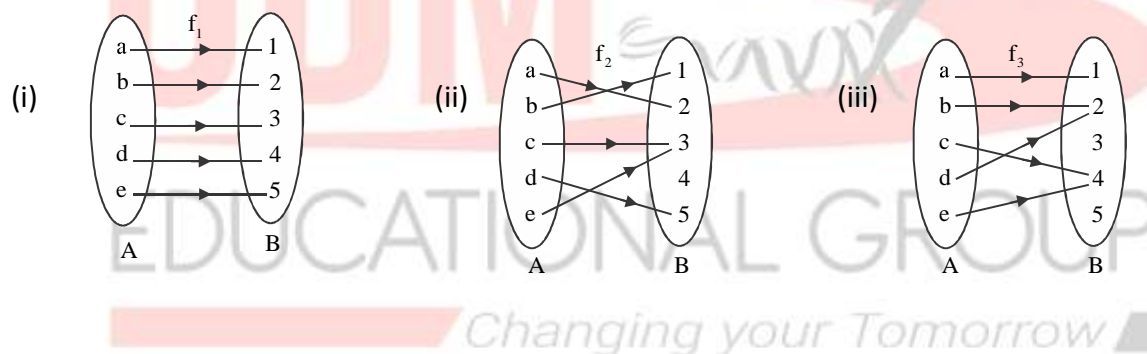
A function  $f : A \rightarrow B$  is said to be onto if, for each  $b \in B$ , there exists  $a \in A$  such that  $f(a) = b$ , we say that 'a' is pre-image of 'b'. In other words, f is onto if Range of f = co-domain of f, i.e if every element in B has a pre-image in A.

**Note:-**

(i) If a function  $f : A \rightarrow B$  is not onto then it is called into function.

(ii) If a function  $f : A \rightarrow B$  is onto then  $n(A) \geq n(B)$

(iii) Let A be any finite set  $n(A) = p$  then no of onto function from A to A is  $p!$

**Example:-**

Check whether functions in the figure are onto?

**Bijjective Function:-**

A function  $f : A \rightarrow B$  is said to be bijective if it is both one-one and onto.

**Examples:-**



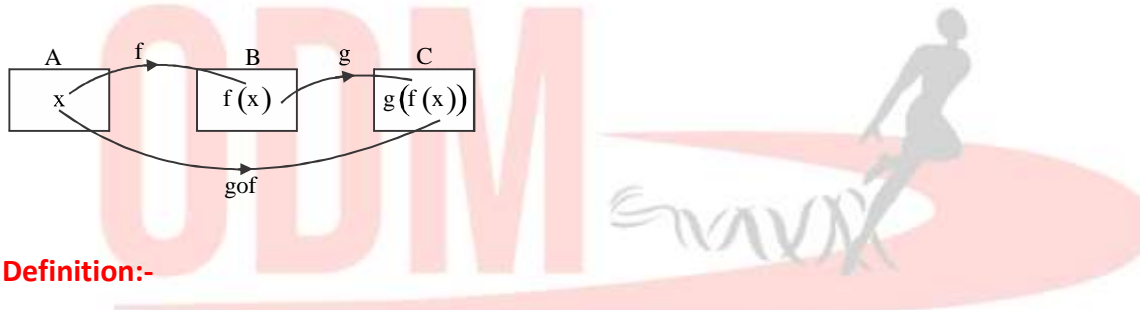
Let  $A = \{1, 2, 3, 4\}$  and  $B = \{5, 6, 7, 8\}$  let  $f : A \rightarrow B$  be defined by  $f = \{(1, 5), (2, 6), (3, 7), (4, 8)\}$ .

Show that  $f$  is one-one and onto (bijective).

### Composition of Function:-

The composition of two functions is a chain process in which the output of the first function becomes the input of the 2<sup>nd</sup> function. Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be two functions.

Exactly one element for every  $x \in A$ , there is exactly one element  $f(x) \in B$ . For  $f(x) \in B$ , there is exactly one element  $g(f(x)) \in C$ . This result is a new function from  $A$  to  $C$  as shown in the figure.



### Definition:-

Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be any two functions. Then the composition of  $f$  and  $g$  is a function  $gof : A \rightarrow C$  defined as  $(gof)(x) = g(f(x))$ .

### Examples – 01

If  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  are given by  $f(x) = \cos x$  and  $g(x) = 5x^2$ . Find  $gof$  and  $fog$  show that  $fog \neq gof$ .

### Examples-02

Let  $f : \{2, 3, 4, 5\} \rightarrow \{3, 4, 5, 9\}$  and  $g : \{3, 4, 5, 9\} \rightarrow \{7, 11, 15\}$  be functions defined by

$$f = \{(2, 3), (3, 4), (4, 5), (5, 5)\}$$

$$g = \{(3, 7), (4, 7), (5, 11), (9, 11)\}$$

Then find  $g \circ f$ . Whether  $f \circ g$  is defined or not.

**Notes:-**

- The composition  $g \circ f$  exists if the range of  $f \subseteq$  domain of  $g$ .
- The composition  $f \circ g$  exists if the range of  $g \subseteq$  domain of  $f$ .
- It may be possible  $g \circ f$  exists but  $f \circ g$  does not exist
- $g \circ f$  and  $f \circ g$  may or may not be equal.

**Properties of the composition of Functions:-**

(1) Composition of functions is associative Let  $f : A \rightarrow B, g : B \rightarrow C$  and  $h : C \rightarrow D$  then

$$(h \circ g) \circ f = h \circ (g \circ f)$$

(2) Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be two functions.

(i) If both are one-one then  $g \circ f$  is one-one

(ii) If both are onto then  $g \circ f$  is onto.

(3) Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be two functions such that  $g \circ f : A \rightarrow C$

(i) If  $g \circ f$  is onto, then  $g$  is onto

(ii) If  $g \circ f$  is one-one then  $f$  is one-one

(iii) If  $g \circ f$  is onto and  $g$  is one-one then  $f$  is onto

(iv) If  $g \circ f$  is one-one and  $f$  is onto then  $g$  is one-one

**Problems to work out:-**

(1) If the mapping  $f$  and  $g$  are given by  $f = \{(1,2), (3,5), (4,1)\}$ ,  $g = \{(2,3), (5,1), (1,3)\}$  write  $f \circ g$  and  $g \circ f$ .

(2) Find  $g \circ f$  and  $f \circ g$  when  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$  are defined by  $f(x) = 5x + 2$  and  $g(x) = x^2 + 6$

(3) Let  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  be two functions defined as  $f(x) = |x| + x$  and  $g(x) = |x| - x$ ,  $\forall x \in \mathbb{R}$  then find  $f \circ g(-3)$  and  $g \circ f(-2)$ .

(4) Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be signum function as  $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$ , be the greatest

integer function given by  $g(x) = [x]$ . Do  $f \circ g$  and  $g \circ f$  coincide in  $(0,1]$ ?

**Solution:-**

Let  $x \in (0,1)$  be any element

$$f \circ g(x) = f(g(x))$$

$$= f([x])$$

$$= f(0) \text{ as } x \in (0,1)$$

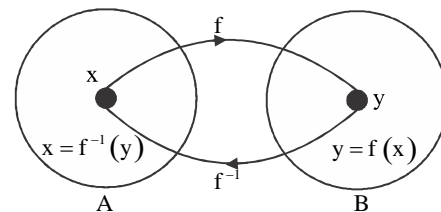
$$= 0$$

$$\text{Also } (g \circ f)(x) = g(f(x)) = g(1) = [1] = 1 \text{ as } x \in (0,1)$$

$\therefore (f \circ g)(x) \neq (g \circ f)(x)$  for every  $x \in (0,1)$ ; so  $f \circ g$  and  $g \circ f$  does not coincide in  $(0,1]$

**The inverse of a Function:-**

Let  $f$  be a one-one and on-to function from  $A$  to  $B$ . Let  $y$  be an arbitrary element of  $B$ . Then  $f$  being onto, there exists an element  $x \in A$ . Such that  $f(x) = y$ , Also  $f$  being one-one this  $x$  must be unique. Thus for each  $y \in B$ , there exists a unique element  $x \in A$  such that  $f(x) = y$ . So we may define a function denoted by  $f^{-1}$  as  $f^{-1} : B \rightarrow A$ . Such that  $f^{-1}(y) = x \Leftrightarrow f(x) = y$ .



The function  $f^{-1}$  is called the inverse of  $f$ .

**Definition (1)**

Let  $f : A \rightarrow B$  be both one-one & onto function, then  $f^{-1} : B \rightarrow A$  is a function which associates to each  $y$  of  $B$ , a unique  $x$  of  $A$  such that  $f(x) = y$  is called the inverse of function  $f$ .

$$\therefore f^{-1}(y) = x \Leftrightarrow f(x) = y$$

**Definition (2)**

Another definition of the inverse function. Let  $f : A \rightarrow B$  be one-one and onto function, then the function  $g : B \rightarrow A$  such that  $g \circ f = I_A$  and  $f \circ g = I_B$ , where  $I_A$  and  $I_B$  are identity functions on  $A$  and  $B$  respectively, is called the inverse of  $f$  i.e  $g = f^{-1}$ .

**Notes:-**

- If the inverse of a function  $f$  exists then  $f$  is called an invertible function.
- A function  $f$  is invertible if and only if  $f$  is one-one and onto.
- The two definitions of the Inverse function given above are equivalent.
- The domain of  $f^{-1} = \text{Range of } f$  and range of  $f^{-1} = \text{domain of } f$ .
- $(f^{-1} \circ f)(x) = x, \forall x \in \text{domain of } f$  i.e  $f^{-1} \circ f$  is an identity function.
- $(f^{-1})^{-1} = f$
- If  $f$  is one-one and onto then  $f^{-1}$  is also one-one and onto.

**Working Rule to find Inverse of a Function:-**

Let  $f : A \rightarrow B$  defined by  $y = f(x)$

Step – I:- Prove that  $f$  is one-one i.e take  $f(x_1) = f(x_2)$  and show that  $x_1 = x_2$

Step – II:- Prove that  $f$  is onto i.e for any  $y \in B$ , there exists  $x \in A$  s.t.  $f(x) = y$

Step – III:- Find  $x$  in terms of  $y$  from  $y = f(x)$  let  $x = g(y)$  then  $g(y) = f^{-1}(y)$

**Example -1**

Consider  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = 4x + 3$ . Show that  $f$  is invertible, find the inverse of  $f$ .

**Example-2**

Show that  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^2 + 1$  is not invertible.

**Properties:-**

- (1) If  $f : X \rightarrow Y$   $g : Y \rightarrow Z$  are be two invertible functions. Then  $g \circ f$  is also invertible with  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ .
- (2) If  $f : X \rightarrow Y$  is invertible, then its inverse is unique.
- (3) If  $f : X \rightarrow Y$  is invertible then  $f^{-1} \circ f = I_X$  and  $f \circ f^{-1} = I_Y$
- (4) Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  be two functions such that  $g \circ f = I_X$  and  $f \circ g = I_Y$  then  $f$  and  $g$  are bijections and  $g = f^{-1}$ .

**Example-3**

If  $A = \{a, b, c, d\}$  and the function  $f = \{(a, b), (b, d), (c, a), (d, c)\}$ . Write  $f^{-1}$ .

**Example-4**

Consider  $f : \{1, 2, 3\} \rightarrow \{a, b, c\}$  given by  $f(1) = a, f(2) = b, f(3) = c$ . Show that  $(f^{-1})^{-1} = f$ .

**Example – 5**

If  $f(x) = \frac{4x+3}{6x-4}, x \neq \frac{2}{3}$  show that  $f \circ f(x) = x$  for all  $x \neq \frac{2}{3}$ . What is the inverse of  $f$ ?

**Problem – 01**

Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be defined by  $f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$  for all  $n \in \mathbb{N}$ . State whether the function is bijective justify your answer.

**Problem – 02**

Show that the modulus function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , given by  $f(x) = |x|$  is neither one-one nor onto.

**Solution:-**

**For one-one**  $f(3) = |3| = 3$        $f(-3) = |-3| = 3$

As  $f(3) = f(-3)$  but  $3 \neq -3$  so  $f$  is not one-one

**For onto** Range  $f = \mathbb{R}^+ \cup \{0\}$  Co-dom of  $f = \mathbb{R}$

As Range  $f \neq$  co-dom  $f$  so  $f$  is not onto

**Problem – 3**

Consider  $f : \mathbb{R}_+ \rightarrow [-5, \infty)$ , given by  $f(x) = 9x^2 + 6x - 5$ .

Show that  $f$  is invertible with  $f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$

**Problem – 4**

Give an example of a function

- (i) Which is one-one but not onto                      (ii) Which is not one-one but onto
- (iii) Which is neither one-one nor onto.

**Solution:-**

- (i) Let  $A = \{1, 2\}$ ,  $B = \{4, 5, 6\}$  and let  $f = \{(1, 4), (2, 5)\}$ . Since every element of A has different images in B so f is one-one. Also, the element  $6 \in B$  does not have a pre-image in A. So f is not onto
- (ii) Let  $A = \{1, 2, 3\}$ ,  $B = \{4, 5\}$  and  $g = \{(2, 4), (1, 4), (3, 5)\}$ . Since  $1, 2 \in A$  have same image 4 in B. So g is not one-one. Also, every element of B has a pre-image in A, so g is onto
- (iii)  $A = \{1, 2, 3\}$ ,  $B = \{4, 5\}$  and  $h = \{(1, 4), (2, 4), (3, 4)\}$ . Since elements  $1, 2, 3 \in A$  have the same image 4 in B. So h is not one-one. Also, the element  $5 \in B$  does not have a pre-image in A so h is not onto.

**Problem – 5**

Show that a one-one function  $f : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$  must be onto.

**Solution:-** Give that  $f : A \rightarrow B$  is a one-one function. Where  $A = B = \{1, 2, 3\}$ .

Since  $f : A \rightarrow B$  is one-one

$$\therefore n(A) \leq n(B) \dots\dots\dots(1)$$

Suppose that  $f : A \rightarrow B$  is not onto

$$\therefore n(A) < n(B) \dots\dots\dots(2)$$

So from (1) and (2), we get  $n(A) < n(B)$  which is a contradiction, as  $n(A) = n(B) = 3$ .

Hence  $f$  is onto

### Problem – 6

Show that an onto function  $f : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$  is always one-one

**Solution:-**

Given that  $f : A \rightarrow B$  is onto,

Where  $A = B = \{1, 2, 3\}$

Since  $f : A \rightarrow B$  is onto

$$n(A) \geq n(B) \dots\dots\dots(1)$$

Suppose that  $f : A \rightarrow B$  is not one-one

$$n(A) > n(B) \dots\dots\dots(2)$$

From (1) and (2) we get  $n(A) > n(B)$  which is a contradiction as  $n(A) = n(B) = 3$

Hence  $f$  is one-one.



**Problem - 7:-**

(a) Write total number of functions from  $\{1, 2\}$  to  $\{x, y, z\}$

**Answer:-**  $(3)^2 = 9$  as no of functions from A to B is  $[n(B)]^{n(A)}$

(b) Write total number of one-one functions from  $\{1, 2\}$  to  $\{x, y, z\}$

**Answer:-**  ${}^3P_2 = \frac{3!}{(3-2)!} = 6$

(c) Find the total number of onto functions from set  $\{x, y, z\}$  to  $\{1, 5, 25, 125\}$ .

**Answer:-** 0 As  $n(A) < n(B)$  so no of onto function is 0

(d) Find the total number of bijective functions from set  $\{x, y, z\}$  to  $\{\alpha, \beta, \gamma\}$

**Answer:-**  $3!$  As  $n(A) = n(B) = n$ , no of bijections

(e) if  $f(x) = |x|$  and  $g(x) = [x]$  find  $(f \circ g)(-1.5)$  and  $(g \circ f)(-1.5)$

**Answer:-** 2 ; 1

**Problem - 8:-**

If the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = 2x - 3$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$  by  $g(x) = x^3 + 5$ . Then find  $f \circ g$  and show that  $f \circ g$  is invertible. Also find  $(f \circ g)^{-1}$ , Hence find  $(f \circ g)^{-1}(9)$ .

**Solution:-**

Here  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f \circ g(x) = f(g(x)) = f(x^3 + 5) = 2(x^3 + 5) - 3 = 2x^3 + 7$ . Now to prove

$f \circ g$  is invertible. One-one:- Let  $x_1, x_2 \in \mathbb{R}$  and  $(f \circ g)(x_1) = (f \circ g)(x_2)$

$$\Rightarrow 2x_1^3 + 7 = 2x_2^3 + 7$$

$$\Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$$

So fog is one-one Onto:- let  $y \in \mathbb{R}$  be any element then  $\text{fog}(x) = y$

$$\Rightarrow 2x^3 + 7 = y$$

$$\Rightarrow 2x^3 = y - 7 \Rightarrow x^3 = \frac{y-7}{2}$$

$$\Rightarrow x = \sqrt[3]{\frac{y-7}{2}} \dots\dots\dots (1)$$

For every,  $y \in \mathbb{R}$  we have  $x \in \mathbb{R}$  so fog is onto.

Thus, fog is an invertible function so  $(\text{fog})^{-1} : \mathbb{R} \rightarrow \mathbb{R}$  exists and from (1)

$$(\text{fog})^{-1}(y) = \sqrt[3]{\frac{y-7}{2}}; (\text{fog})^{-1}(9) = \sqrt[3]{\frac{9-7}{2}} = 1$$

**Problem - 9:-**

If the function  $f(x) = \sqrt{2x-3}$  is invertible, then find  $f^{-1}$ . Hence prove that  $(f \circ f^{-1})(x) = x$ .

**Solution:-**

Given  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = \sqrt{2x-3}$

One-one: Let  $x_1, x_2 \in \mathbb{R}$  and  $f(x_1) = f(x_2)$

$$\Rightarrow \sqrt{2x_1-3} = \sqrt{2x_2-3}$$

$$\Rightarrow 2x_1-3 = 2x_2-3$$

$$\Rightarrow x_1 = x_2$$

So f is one-one

Onto:- Let  $y \in \mathbb{R}$  be any element then  $f(x) = y$

$$\Rightarrow \sqrt{2x-3} = y$$

$$\Rightarrow 2x-3 = y^2$$

$$\Rightarrow x = \frac{y^2+3}{2} \dots\dots\dots(1)$$

So  $f$  is onto. Thus  $f$  is an invertible function so  $f^{-1} : \mathbb{R} \rightarrow \mathbb{R}$  exists and from (1) we have

$$f^{-1}(y) = \frac{y^2+3}{2}$$

The inverse of  $f$  is given by  $f^{-1}(x) = \frac{x^2+3}{2}$

$$\text{Now } (f \circ f^{-1})(x) = f(f^{-1}(x))$$

$$= f\left(\frac{x^2+3}{2}\right) = \sqrt{2\left(\frac{x^2+3}{2}\right)} - 3$$

### Problem - 10:-

Consider  $f : \mathbb{N} \rightarrow \mathbb{N}$ ,  $g : \mathbb{N} \rightarrow \mathbb{N}$  and  $h : \mathbb{N} \rightarrow \mathbb{R}$  defined as  $f(x) = 2x$ ,  $g(y) = 3y + 4$  and  $h(x) = \sin x$  for all  $x, y, z \in \mathbb{N}$ . Show that  $h \circ (g \circ f) = (h \circ f) \circ g$

### Solution:-

Given  $f : \mathbb{N} \rightarrow \mathbb{N}$ , defined by  $f(x) = 2x$ ;  $g : \mathbb{N} \rightarrow \mathbb{N}$  defined by  $g(y) = 3y + 4$  and  $h : \mathbb{N} \rightarrow \mathbb{R}$ ,  $h(x) = \sin x$

Now  $h \circ (g \circ f) : \mathbb{N} \rightarrow \mathbb{R}$  such that  $[h \circ (g \circ f)](x) = h[g \circ f(x)]$

$$= h(g(f(x))) = h(g(2x))$$

$$=h[3(2x)+4]$$

$$=h(6x+4)$$

$$=\sin(6x+4)$$

Also  $(\text{hog})\text{of}:\mathbb{N}\rightarrow\mathbb{R}$  such that  $[(\text{hog})\text{of}](x)=(\text{hog})(f(x))$

$$=(\text{hog})(2x)=h(g(2x))$$

$$=h[3(2x)+4]$$

$$=h(6x+4)=\sin(6x+4)$$

Hence,  $[\text{ho}(\text{gof})](x)=[(\text{hog})\text{of}](x); \forall x \in \mathbb{N}$

### Example:-11

Let  $f:\mathbb{R}\rightarrow\mathbb{R}$  be the signum function defined as  $f(x)=\begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -1 & \text{if } x<0 \end{cases}$  and  $g:\mathbb{R}\rightarrow\mathbb{R}$  be the

greatest integer function given by  $g(x)=|x|$ . Do  $\text{fog}$  and  $\text{gof}$  coincide in  $(0, 1]$ ?

### Solution:-

Given  $f:\mathbb{R}\rightarrow\mathbb{R}$   $f(x)=\begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -1 & \text{if } x<0 \end{cases}$  and  $g:\mathbb{R}\rightarrow\mathbb{R}$  defined by  $g(x)=|x|$

Let  $x \in (0,1)$  be any element

Then  $(\text{fog})(x)=f(g(x))$

$$=f(|x|)$$

$$= f(0) = 0$$

$$\text{Also } (g \circ f)(x) = g(f(x))$$

$$= g(1) = [1] = 1$$

$\therefore (f \circ g)(x) \neq (g \circ f)(x)$ , for every  $x \in (0, 1]$  Hence,  $f \circ g$  and  $g \circ f$  do not coincide in  $(0, 1]$

**Example:-12**

Let  $A = \{1, 2, 3\}$  and  $B = \{5, 6, 7\}$ . Let the functions  $f: A \rightarrow B$  and  $g: A \rightarrow B$  be defined by

$f = \{(1, 5), (2, 6), (3, 7)\}$  and  $g = \{(2, 5), (1, 6), (3, 7)\}$ . Show that  $f \neq g$ .

**Solution:-**

Given  $A = \{1, 2, 3\}$  and  $B = \{5, 6, 7\}$

Also, given that  $f = \{(1, 5), (2, 6), (3, 7)\}$  and  $g = \{(2, 5), (1, 6), (3, 7)\}$

Since,  $f(1) = 5 \neq g(1)$

Therefore,  $f(x) \neq g(x)$  for some  $x \in A$ . Hence,  $f \neq g$